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NECESSARY CONDITIONS FOR DISCRETE DYNAMICAL SYSTEMS WITH DELAYS AND GENERAL CONSTRAINTS

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A fairly general class of discrete systems with delays is studied and necessary optimality conditions (a discrete maximum principle) are derived. Both state and state-dependent control constraints are considered. After a suitable reformulation of the original problem it is possible to establish the desired necessary conditions within the framework of the existing discrete control theory. A special case having the mentioned constraints described explicitly by systems of equalities and inequalities is studied separately. An alternative possibility to obtain necessary conditions in this case is also pointed out and briefly discussed in the connection with a maximum principle formulation.

1. INTRODUCTION

At present time there exist fairly deep results dealing with necessary optimality conditions for discrete systems. Let us recall at least some of them described in references [1–4]. All these formulations admit a general structure of constraining sets which are then included into the statement of necessary conditions with the aid of a certain "conical approximation". Common feature of practically any approach to the study of discrete optimal control systems is an application of mathematical programming theory. The obvious reason for this is the fact that, after all, a discrete control problem can be regarded as a special case of a mathematical programming one, and the existing results in the theory of mathematical programming are then only worked out in the appropriate way to conform with a discrete control problem.

In many practical applications of control theory one can also encounter discrete control systems involving the so-called time delays (lags). Typical problems of this kind arise in mechanical and chemical engineering, management sciences and eco-

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nomics. This type of control problems was mentioned already by Fan and Wang in [5] when dealing with some applications of discrete control theory in chemical engineering. However, some general conclusions stated in this book were erroneous. Later, problems with delays were studied by a number of authors, e.g. see [5 – 11]. Various classes of systems with delays were thus investigated from a point of view of necessary optimality conditions. Recently the class of systems introduced in [9] was considered also by the author in [12] in more general setting on applying the results of [4]. The respective existence theory for discrete systems is given in [13] and [14]. Especially the reference [14] pertains to the case involving delays.

In [7] certain type of explictly described discrete control problems was studied and necessary optimality conditions were formulated for this special case of problems with delays. As pointed in [9] and [10], also this class of problems can be treated in a quite general way invoking the methodology of [4]. Our aim in this contribution is to give a precise derivation of necessary optimality conditions in a form of the maximum principle for this case of problems. As it is to be expected the incorporation of time delays in the formulation of a discrete control problem will cause certain difficulties when trying to derive the corresponding necessary optimality conditions. On the other hand, discrete-time control problems with delays can be ususally reformulated, at the expense of higher dimensionality, in such a way that the resulting equivalent discrete control problem does not contain any delay. Then the results of [1 – 4] are immediately applicable. Some general considerations in this respect were given in [14]. This is, however, not the case if one have to deal with continuous-time optimal control problems involving delays. Then necessarily a more sophisticated approach is to be applied, e.g. see [15] and [16].

After formulation of the studied problem in the next section, some assumptions of a technical character will be imposed to be able to apply the theory given in [4]. As this reference is generally accessible, only a minimum of definitions and results will be repeated here. For convenience, also the notation introduced in [4] is widely preserved. For the formulated problem some existence results are briefly included. Finally, the more explicit case of state and state-dependent control constraints is considered more in detail together with a discussion of some aspects regarding to the necessary conditions. In our convention, if not otherwise stated, all vectors are assumed to be column-vectors, however, as usual, the gradients of various functions are treated as row-vectors. Then more efficient matrix notation can be used.

2. DISCRETE CONTROL SYSTEMS WITH DELAYS

Discrete control systems involving various delays in state equations, objective functional and constraints were studied by Mariani and Nicoletti in [7]. Only explicit description of the constraints (inequality-type mixed state and control constraints) was assumed and necessary optimality conditions were obtained on applying a mathematical programming approach. A maximum principle formulation of necessary conditions was given only under rather restrictive linearity and convexity assumptions. As suggested by the author in [11] also this type of problems can investigated within the general framework of [4]. In this way it was possible to include also state constraints in our formulation of the problem.

The following statement of a discrete optimal control problem with delays thus stems from those given in [7] and [11]. State and state-dependent control constraints are assumed to be described implicitly as general admissible sets. In this way our formulation will include those of both mentioned references. For the sake of notational simplicity, only one delay in the state and control variable is always assumed. The extension of the further obtained results to a case of more delays is straightforward.

Now let a discrete nonlinear dynamical system be given the behaviour of which is described by a set of equations (index k always denotes the current stage of the system)

$$x_{k+1} = f^k(x_k, x_{k-\alpha}, u_k, u_{k-\beta}), \quad k = 0, 1, \dots, K-1, \tag{1}$$

where the positive integer K denotes a given number of stages, $x_k \in E^n$ is the state, $u_k \in E^m$ is the control and $f^k : E^{2n} \times E^{2m} \to E^n$. The aim is to find a control sequence $\hat{u}_0, \hat{u}_1, \dots, \hat{u}_{K-1}$ and a corresponding trajectory $\hat{x}_0, \hat{x}_1, \dots, \hat{x}_K$, determined by (1), satisfying the state constraints

$$(x_k, x_{k-\delta}) \in A_k \subset E^{2n}, \quad k = 0, 1, \dots, K, \tag{2}$$

and the state-dependent control constraints

$$u_k \in U_k(x_k, x_{k-\eta}, u_{k-\varepsilon}) \subset E^m, \quad k = 0, 1, \dots, K-1, \tag{3}$$

such that the objective functional

$$J = \sum_{k=0}^{K-1} h^k(x_k, x_{k-\sigma}, u_k, u_{k-\theta}) \tag{4}$$

is minimized. Here $U_k : E^{2n} \times E^m \to \mathscr{P}(E^m)$ is the so-called multivalued mapping with $\mathscr{P}(E^m)$ denoting the collection of all nonempty subsets of E^m, and $h^k : E^{2n} \times E^{2m} \to E^1$. The numbers $\alpha, \beta, \delta, \eta, \varepsilon, \sigma, \theta$ are positive integers less then K representing various delays in (1)–(4). The case having these numbers negative can be handled in a quite analogous way. Finally, let the initial conditions (history) be given as

$$\begin{aligned} &\hat{x}_{-i}, \hat{x}_{-i+1}, \dots, \hat{x}_{-1}, \quad i = \max(\alpha, \delta, \eta, \sigma), \\ &\hat{u}_{-j}, \hat{u}_{-j+1}, \dots, \hat{u}_{-1}, \quad j = \max(\beta, \varepsilon, \theta). \end{aligned} \tag{5}$$

This fact is to be kept in mind when interpreting the constraints (2)–(3) in an appropriate way. Otherwise more complicated notation would be necessary. Also taking into the account a possible terminal part of J given as $g(x_K)$ would result only in additional notational difficulties without any substantial gain.

To apply the theory developed in [4] the following assumptions have to be postulated for the above problem.

(a) The functions f^k and h^k, $k = 0, 1, \ldots, K - 1$, are continuously differentiable on $E^{2n} \times E^{2m}$.

(b) For any $(x_k, x_{k-\delta}) \in A_k$, $k = 0, 1, \ldots, K$, there exists a conical approximation (of the second kind), denoted $C(x_k, x_{k-\delta}; A_k)$, to the set A_k at the point $(x_k, x_{k-\delta})$,

(c) For any x_k, $x_{k-\eta}$, u_k and $u_{k-\varepsilon}$ with $u_k \in U_k(x_k, x_{k-\eta}, u_{k-\varepsilon})$, $k = 0, 1, \ldots, K - 1$. there exists a continuously differentiable function $\omega_k : N(x_k, x_{k-\eta}, u_{k-\varepsilon}) \to E^m$ such that $\omega_k(x, \tilde{x}, u) \in U_k(x, \tilde{x}, u)$ for $(x, \tilde{x}, u) \in N(x_k, x_{k-\eta}, u_{k-\varepsilon})$ and $\omega_k(x_k, x_{k-\eta}, u_{k-\varepsilon}) = u_k$. Here as $N(x_k, x_{k-\eta}, u_{k-\varepsilon})$ denoted a neighbourhood of $(x_k, x_{k-\eta}, u_{k-\varepsilon})$ in $E^{2n} \times E^m$.

(d) Let $e_0 \in E^{1+n}$ be a vector $(-1, 0, 0, \ldots, 0)$ and define multivalued mappings $\Phi_k : E^{4n} \times E^{3m} \to \mathscr{P}(E^{1+n})$ as $\Phi_k(\cdot) = \varphi_k(x_k, x_{k-\alpha}, x_{k-\sigma}, U_k(x_k, x_{k-\eta}, u_{k-\varepsilon}), u_{k-\beta}, u_{k-\theta})$, where $\varphi_k = (h^k, f^k)$, $k = 0, 1, \ldots, K - 1$. It is assumed that the sets $\Phi_k(\cdot)$, $k = 0, 1, \ldots, K - 1$, are e_0-directionally convex for any value of the argument of Φ_k.

More details concerning various concepts used to formulate these assumptions can the interested reader find in [4]. Assumption (c) is alternatively called as the existence of "local sections" in [2–3] or of the "programs" in [16]. Anyway, this type of assumption cannot be avoided when dealing with state-dependent regions of admissible controls. The question of a directional convexity and that of a conical approximation was discussed in [1] in detail. The above assumptions make it possible to apply the basic result of [4] to the following equivalent problem in a straightforward way.

3. EQUIVALENT PROBLEM

Define variables $y^1, \ldots, y^i \in E^n$ and $v^1, \ldots, v^j \in E^m$ by the equations

$$\begin{array}{llll} y^1_{k+1} = x_k\,, & y^1_0 = \hat{x}_{-1}\,, & v^1_{k+1} = u_k\,, & v^1_0 = \hat{u}_{-1}\,, \\ y^2_{k+1} = y^1_k\,, & y^2_0 = \hat{x}_{-2}\,, & v^2_{k+1} = v^1_k\,, & v^2_0 = \hat{u}_{-2}\,, \\ \vdots & & \vdots & \\ y^i_{k+1} = y^{i-1}_k\,, & y^i_0 = \hat{x}_{-i}\,, & v^j_{k+1} = v^{j-1}_k\,, & v^j_0 = \hat{u}_{-j}\,, \end{array} \tag{6}$$

where $k = 0, 1, \ldots, K - 1$. Then introducing an augmented state variable $z = (x, y^1, \ldots, y^i, v^1, \ldots, v^j)$ belonging to $E^{(n+1)i+mj} = \tilde{E}$, and denoting $F^k(z_k) = (f^k(x_k, x_{k-\alpha}, u_k, u_{k-\beta}),\ x_k, y^1_k, \ldots, y^{i-1}_k, u_k, v^1_k, \ldots, v^{j-1}_k)$, $\tilde{h}^k(z_k) = h^k(x_k, x_{k-\sigma}, u_k, u_{k-\vartheta})$, it is possible to write

$$z_{k+1} = F^k(z_k, u_k)\,, \quad k = 0, 1, \ldots, K - 1\,, \tag{7}$$

$$z_k \in \tilde{A}_k \subset \tilde{E}\,, \qquad k = 0, 1, \ldots, K\,, \tag{8}$$

$$u_k \in \tilde{U}_k(z_k) \subset E^m\,, \quad k = 0, 1, \ldots, K - 1\,, \tag{9}$$

$$(10) \qquad J = \sum_{k=0}^{K-1} \tilde{h}^k(z_k, u_k) \, .$$

This form of a discrete optimal control problem was studied in [4]. The postulated assumptions guarantee the application of the basic result of [4, Theorem 3] to (7)–(10). It has to be realized that thanks to a rather special structure of this problem some simplifications of general conditions are to be expected. For example, the conditions $z_k \in \tilde{A}_k$ only means that $(x_k, x_{k-\delta}) = (x_k, y_k^\delta) \in A_k$, and other components of z_k are not bounded by any constraint. In a similar way also the constraints (9) should be understand. So one can finally conclude that by the indicated construction the problems (1)–(5) and (7)–(10) are equivalent.

Moreover, the problem (7)–(10) suggests a straightforward application of [13] to deal also with an existence problem. On the other hand, the existence theory of [14] enables to treat directly the original problem (1)–(5). Let us only summarize the final result.

Proposition 1. Let the discrete optimal control problem (1)–(5) satisfy the following assumptions.

(a) The functions f^k and h^k, $k = 0, 1, \ldots, K - 1$, are continuous on $E^{2n} \times E^{2m}$.

(b) The initial set A_0 is compact and the sets A_k, $k = 1, \ldots, K$, are closed.

(c) The multivalued mappings U_k, $k = 0, 1, \ldots, K - 1$, are compact-valued and upper semicontinuous on $E^{2n} \times E^m$.

(d) There exists at least one admissible control process $\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{K-1}$ and $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_K$ in the given system.

Then the discrete optimal control problem with delays (1)–(5) has a solution.

4. NECESSARY OPTIMALITY CONDITIONS

The mentioned theorem of reference [4] requires that the sets $\tilde{A}_k$, and therefore also A_k, are such that they admit the so-called conical approximation of the first kind [1, 4]. However, if one compares the given derivation of this result it is easily seen that this restriction was evidently superfluous. Thus a more general character of possible state constraints is assumed here, see the assumption (b) earlier.

Now let $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{K-1}$ and a corresponding $\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_K$ be an optimal solution to (7)–(10). Then, according to [4], there exists a number $\mu \leqq 0$, row-vector multipliers $\Lambda_k \in \tilde{E}$, $k = 1, \ldots, K$, and row-vectors $\tilde{b}_k \in C'(\hat{z}_k; \tilde{A}_k)$ – the dual cone to $C(\hat{z}_k; \tilde{A}_k)$ in $\tilde{E}$, such that the following conditions are satisfied.

(i) If $\mu = 0$, then at least one of the vectors Λ_k is nonzero.

(ii) The row-vectors Λ_k satisfy the adjoint equation

$$(11) \qquad \Lambda_k = \hat{\mathscr{H}}_z^{k+1}(\hat{z}_k) - \tilde{b}_k \, , \quad k = 0, 1, \ldots, K \, ,$$

where we formally define $\Lambda_0 = 0$ and $\hat{\mathscr{H}}^{K+1} \equiv 0$ to obtain the respective boundary conditions. Here

$$(12) \qquad \hat{\mathscr{H}}^{k+1}(z_k) = \mathscr{H}^{k+1}(z_k, \hat{\omega}_k(x_k, y_k^\eta, v_k^\varepsilon)), \quad k = 0, 1, \ldots, K-1,$$

with

$$(13) \qquad \mathscr{H}^{k+1}(z_k, u_k) = \mu\tilde{h}^k(z_k, u_k) + \Lambda_{k+1}F^k(z_k, u_k), \quad k = 0, 1, \ldots, K-1,$$

and $\hat{\omega}_k$, $k = 0, 1, \ldots, K-1$, being the functions the existence of which is guaranteed by assumption (c) and which correspond to the optimal solution.

(iii) The maximum condition

$$(14) \qquad \mathscr{H}^{k+1}(\hat{z}_k, \hat{u}_k) = \max_{u \in \hat{U}_k(\hat{z}_k)} \mathscr{H}^{k+1}(\hat{z}_k, u), \quad k = 0, 1, \ldots, K-1,$$

is satisfied along the optimal solution.

Now one has to realize that the vectors $\tilde{b}_k$ have nonzero components only on the places corresponding to x_k, y_k^δ, and let us denote these parts of $\tilde{b}_k$ as $b_k' \in E^n$ and $b_k'' \in E^n$. Further, if $\Lambda_k = (\lambda_k, \lambda_k^1, \ldots, \lambda_k^i, \tilde{\lambda}_k^1, \ldots, \tilde{\lambda}_k^j)$, with $\lambda_k, \lambda_k^1, \ldots, \lambda_k^i \in E^n$, $\tilde{\lambda}_k^1, \ldots$ $\ldots, \tilde{\lambda}_k^j \in E^m$, one can easily write the corresponding adjoint equations for all of the indicated constituents of Λ_k. Because of a special structure of this problem finally only equation for λ_k is deduced. These calculations are rather straightforward, however somewhat lengthy, so let us omit them for short. Then (11) implies the equation

$$(15) \qquad \lambda_k = \hat{H}_{x_k}^{k+1} + \hat{H}_{x_k}^{k+1+\alpha} + \hat{H}_{x_k}^{k+1+\sigma} + \hat{H}_{x_k}^{k+1+\eta} - b_k' - b_{k+\delta}'', \quad k = 0, 1, \ldots, K-1,$$

with boundary conditions $\lambda_0 = 0$ and $\lambda_K = -b_K'$. Here

$$(16) \qquad \hat{H}^{k+1}(z_k) = H^{k+1}(x_k, x_{k-\alpha}, x_{k-\sigma}, \hat{\omega}_k(x_k, x_{k-\eta}, u_{k-\varepsilon}), u_{k-\beta}, u_{k-\theta}), \quad k = 0, 1, \ldots, K-1,$$

with

$$(17) \qquad H^{k+1}(x_k, x_{k-\alpha}, x_{k-\sigma}, u_k, u_{k-\beta}, u_{k-\theta}) = \mu h^k(x_k, x_{k-\sigma}, u_k, u_{k-\theta}) + \lambda_{k+1} f^k(x_k, x_{k-\alpha}, u_k, u_{k-\beta}), \quad k = 0, 1, \ldots, K-1.$$

Implicitly it is assumed that in the above expressions $\hat{H}^\varkappa = 0$, $b_\varkappa'' = 0$ for any $\varkappa > K$. This convention will be maintained throughout the paper. It should be also pointed out that in a case of some delays being equal, certain precautions are to be observed to include the appropriate number of members on the right-hand side of (15).

In a quite analogical way also the maximum condition (14) can be brought to a more familiar form. Apparently only those terms should be taken into the account which explicitely depend on u_k, i.e. schematically denoted

$$(18) \qquad \mathscr{H}^{k+1} = H^{k+1} + \tilde{\lambda}_{k+1}^1 u_k + W_k, \quad k = 0, 1, \ldots, K-1,$$

where W_k does not depend on u_k and where

(19) $$\tilde{\lambda}^1_{k+1} = \hat{H}^{k+1+\beta}_{u_k} + \hat{H}^{k+1+\theta}_{u_k} + \hat{H}^{k+1+\varepsilon}_{u_k}\,, \quad k = 0, 1, \ldots, K-1\,,$$

with the right-hand side again evaluated along the optimal solution. Then for an optimal control sequence $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{K-1}$ one has that

(20) $$\hat{u}_k = \arg \max_{u_k \in U_k(\hat{x}_k, \hat{x}_{k-\eta}, \hat{u}_{k-\varepsilon})} \left[H^{k+1}(x_k, x_{k-\alpha}, x_{k-\sigma}, u_k, u_{k-\beta}, u_{k-\theta}) + \right.$$
$$\left. + \left(\hat{H}^{k+1+\beta}_{u_k} + \hat{H}^{k+1+\theta}_{u_k} + \hat{H}^{k+1+\varepsilon}_{u_k}\right) u_k\right], \quad k = 0, 1, \ldots, K-1\,.$$

It is not difficult to realize that either in (15) or in (20), $\hat{H}$ can be sometimes replaced simply by H, as far as the indicated differentiation does not influence function $\hat{\omega}_k$, e.g. $\hat{H}^{k+1+\alpha}_{x_k} = H^{k+1+\alpha}_{x_k}$, etc. Anyhow, to preserve a uniform notation only the symbol $\hat{H}$ was used in the mentioned expressions. These considerations are summarized in the following theorem.

Theorem 1. Consider a discrete optimal control problem with delays (1)–(5) and let the assumptions (a)–(d) be satisfied. If $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{K-1}$ is an optimal control sequence and $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K$ a corresponding trajectory, then there exists a number $\mu \leqq 0$, row-vector multipliers $\lambda_k \in E^n$, $k = 0, 1, \ldots, K$, and row vectors $b'_k, b''_k \in E^n$, with $(b'_k, b''_k) \in C'(\hat{x}_k, \hat{x}_{k-\delta}; A_k)$ – the dual cone to $C(\hat{x}_k, \hat{x}_{k-\delta}; A_k)$ in E^{2n}, such that (all expressions are to be evaluated along the optimal solution and set to zero if the stage index exceeds the indicated range)

(i) if $\mu = 0$, then at least one of vectors λ_k is nonzero;

(ii) the row-vectors λ_k satisfy the adjoint equation (15) with the indicated boundary conditions;

(iii) the maximum condition (20) holds along the optimal solution.

5. CASE OF EXPLICIT CONSTRAINTS

Let us assume a more concrete form of (2) and (3). Namely, let these sets be given as systems of equalities and inequalities

(21) $$A_k = \{(x_k, x_{k-\delta}) \in E^{2n} \mid S^k(x_k, x_{k-\delta}) = 0\,,\ s^k(x_k, x_{k-\delta}) \leqq 0\}\,,$$
$$k = 0, 1, \ldots, K\,,$$

(22) $$U_k(\cdot) = \{(x_k, x_{k-\eta}, u_k, u_{k-\varepsilon}) \in E^{2n} \times E^{2m} \mid Q^k(x_k, x_{k-\eta}, u_k, u_{k-\varepsilon}) = 0\,,$$
$$q^k(x_k, x_{k-\eta}, u_k, u_{k-\varepsilon}) \leqq 0\}\,, \quad k = 0, 1, \ldots, K-1\,,$$

where $S^k : E^{2n} \to E^{\varrho}$, $s^k : E^{2n} \to E^{\pi}$, $Q^k : E^{2n} \times E^{2m} \to E^{\gamma}$ and $q^k : E^{2n} \times E^{2m} \to E^{\tau}$. The inequality sign for vectors is to be taken componentwise. In the same way as in

[4, Theorems 4 and 5] one can handle also this discrete control problem with delays and explicit constraints. Some results in this respect appeared in [11].

Also now let the assumptions (a) and (d) be satisfied. For (c) to hold it is sufficient that gradients of active constraints in (22) with respect to the vector $(x_k, x_{k-\eta}, u_{k-\varepsilon})$ are linearly independent. This easily follows when comparing the equivalent problem (7)–(10) with a general scheme of [4]. It is not necessary to assume anything more as the state constraints (21) are concerned. However, to avoid a trivial satisfaction of the conditions presented further it is sometimes convenient that also gradients of active constraint in (21) with respect to the vector $(x_k, x_{k-\delta})$ are linearly independent Let us formulate only the final result which is not not very difficult to deduce on combining the mentioned theorems of [4].

Theorem 2. Consider a discrete optimal control problem with delays, where the constraints are given as in (21)–(22), and let the above indicated assumptions be satisfied. If $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{K-1}$ is an optimal control sequence and $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K$ a corresponding trajectory, then there exists a number $\mu \leqq 0$, and row-vector multipliers

$$\lambda_k \in E^n\,, \quad \psi_k \in E^\varrho\,, \quad v_k \in E^\pi\,, \quad k = 0, 1, \ldots, K\,,$$

$$\zeta_k \in E^\gamma\,, \quad \xi_k \in E^\tau\,, \quad k = 0, 1, \ldots, K-1\,,$$

such that (all expressions are to be evaluated along the optimal solution and set to zero if the stage index exceeds the indicated range)

(i) if $\mu = 0$, then at least one of vectors λ_k and ψ_k is nonzero;

(ii) the row-vectors λ_k satisfy the adjoint equation

$$\lambda_k = H^{k+1} + H^{k+1+\alpha}_{x_k} + H^{k+1+\sigma}_{x_k} + \zeta_k Q^k_{x_k} + \xi_k q^k_{x_k} + \zeta_{k+\eta} Q^{k+\eta}_{x_k} +$$

$$+ \xi_{k+\eta} q^{k+\eta}_{x_k} + \psi_k S^k_{x_k} + v_k s^k_{x_k} + \psi_{k+\delta} S^{k+\delta}_{x_k} + v_{k+\delta} s^{k+\delta}_{x_k}\,, \quad k = 0, 1, \ldots, K-1\,,$$

where H^{k+1} is defined by (17), with boundary conditions

$$\lambda_0 = 0\,, \quad \lambda_K = \psi_K S^K_{x_K} + v_K s^K_{x_K}\,;$$

(iii) the maximum condition (20) with $U_k(\cdot)$ given by (22) holds along the optimal solution;

(iv) $v_k \leqq 0$, $v_k s^k = 0$, $k = 0, 1, \ldots, K$;

(v) $\xi_k \leqq 0$, $\xi_k q^k = 0$, $k = 0, 1, \ldots, K-1$.

One can see that trying to preserve the maximum principle formulation not too much is gained by this explicit formulation of the constraints. If instead of (20), while taking into the account (22), one writes down a necessary condition for the indicated maximum, one obtains that

$$H^{k+1}_{u_k} + H^{k+1+\beta}_{u_k} + H^{k+1+\theta}_{u_k} + \zeta_k Q^k_{u_k} + \xi_k q^k_{u_k} + \zeta_{k+\varepsilon} Q^{k+\varepsilon}_{u_k} + \xi_{k+\varepsilon} q^{k+\varepsilon}_{u_k} = 0\,, \tag{23}$$

$$k = 0, 1, \ldots, K-1\,.$$

However, this condition can be derived using only a mathematical programming approach, e.g. to the equivalent problem (7)–(10), and no assumption regarding to the required directional convexity is then needed. On the other hand, if the aim is to formulate the maximum principle type necessary conditions, this assumption cannot be avoided. As also observed in [1], it is no important difference of these two alternative ways of formulation of necessary conditions when applying them to a solution of practical problems.

6. CONCLUSIONS

For a general class of discrete optimal control systems with delays a set of necessary optimality conditions (the discrete maximum principle) was derived on applying the results in this area, being of sufficient generality. Thus some practical problems of this type studied in [8–9] can be now treated in a more general setting involving the considered type of state and state-dependent control constraints.

On the other hand, one has to aware of possible numerical difficulties during practical applications as the dimensionality of a respective equivalent problem can be prohibitive from a computational point of view. Especially in this respect the obtained conditions can be of reasonable benefit as the higher dimensionality of the equivalent problem resulted only in more complicated relations, while the number of equations to be solved remained the same as in a case without delays.

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