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ASYMPTOTIC BEHAVIOUR OF THE LIKELIHOOD RATIO TEST UNDER PRESENCE OF DEVIATIONS OF THE MODEL

JAN ÁMOS VÍŠEK

Let us test a fixed simple hypothesis $H : \theta = \theta'$ against a fixed simple alternative $A : \theta = \theta''$ by means of the likelihood ratio test $f_n(x, \theta_1)/f_n(x, \theta_2) \gtreqless c_n$, where θ' and θ'' are assumed to belong to a neighbourhood of the points θ_1 and θ_2, respectively. In the first section of the present paper the asymptotic rate of the convergence of the error probabilities of this test is proved to be continuous in the parameter θ under a condition which is fulfilled for many commonly used types of probability distributions. In the second part of this paper bounds of the above mentioned asymptotic rate are given for the case when there are deviations from the independence or from the type of dependence between the observations.

1. INTRODUCTION

Since the theory of testing statistical hypotheses began to be built up, statisticians have realized a practical need of testing composite hypotheses against composite alternatives. Many tests had been proposed for this purpose and their properties have been studied ([6], [9], [10]). As it is well known, the optimal (minimax) tests exist for this purpose only in special cases and thus, in the other cases, there was an ambition to prove the optimality of the tests in an asymptotic sense. However this caused the need to solve many problems, starting with the measurabilty of the "statistics" and continuing with deriving asymptotic distribution under the hypotheses and the alternatives ([1], [2], [6], [7]). How complicated studying these problems is may be proved by the fact that highly effective methods had to be developed in order to solve them. Moreover, the attempt to evaluate such statistics for given data has shown, that even the numerical issue of the problem is not easy. Besides that the justification of the demand of the minimax property of the tests for the whole, usually large, composite hypothesis or alternative is under question. In fact, there are cases in which, after having obtained data and taking into account goals of testing, one is able to restrict the hypothesis and the alternative to neighbourhoods of two suitable points of the parameter space.

All these above stated reasons lead us to an idea to use the likelihood ratio of a simple hypothesis and a simple alternative as a test statistic and to study the properties of it under assumption that the true value of the parameter lies in a neighbourhood of the hypothesis or the alternative. This has been done in the first section of this paper.

Recently simulation studies ([4]) showed a bad behaviour of some statistics in the case of dependent observations. Therefore, it has been interesting to study the asymptotic behaviour of the likelihood ratio in the cases in which "slight" violations of the independence (or the type of dependence) of the observed data have occurred. The bounds of deviations of the asymptotic rate of convergence of the error probabilities for likelihood ratio test in such situations have been given in the second section of this paper.

2. NOTATIONS

Let $\mathcal{X}$, endowed by a Borel σ-algebra $\mathcal{A}$, and Θ be metric spaces and ϱ be the metric of the latter one. We shall assume to have defined for every $\theta \in \Theta$ a probability measure P_θ being absolutely continuous with respect to a σ-finite measure μ. Denote by $f(x, \theta)$ the Radon-Nikodým derivative of the measure P_θ with respect to the measure μ. Let $\mathcal{N}$ denote the set of all positive integers and R the real line. Let, for any $n \in \mathcal{N}$, $\mathcal{X}_n$ be the Cartesian product $\bigtimes_{i=1}^{n} \mathcal{X}^{(i)}$, where $\mathcal{X}^{(i)} = \mathcal{X}$ for every $i \in \mathcal{N}$ and $\mathcal{A}_n$ be the minimal σ-algebra generated by $\bigtimes_{i=1}^{n} \mathcal{A}^{(i)}$, $\mathcal{A}^{(i)} = \mathcal{A}$. Let $H_\alpha(P, Q)$ and $H(P, Q)$ denote the α-entropy and the minimal α-entropy of a probability measure P with respect to a probability measure Q, respectively, i.e.

$$H_\alpha(P, Q) = \int \left(\frac{\mathrm{d}P}{\mathrm{d}\mu}\right)^\alpha \left(\frac{\mathrm{d}Q}{\mathrm{d}\mu}\right)^{1-\alpha} \mathrm{d}\mu$$

and

$$H(P, Q) = \inf_{0<\alpha<1} H_\alpha(P, Q) .$$

Let us define $\alpha(P, Q)$ to be equal to $\alpha \in (0, 1)$ such that $H_\alpha(P, Q) = H(P, Q)$, if it exists, or let $\alpha(P, Q) = 0$ or $\alpha(P, Q) = 1$, if $H(P, Q) = \lim_{\alpha \to 0+} H_\alpha(P, Q)$ or $H(P, Q) = \lim_{\alpha \to 1-} H_\alpha(P, Q)$, respectively. Let M_n denote for any $n \in \mathcal{N}$ the product measure over $(\mathcal{X}_n, \mathcal{A}_n)$ generated by a measure M on $(\mathcal{X}, \mathcal{A})$ and put for every $n \in \mathcal{N}$ and $\gamma \in R$

$$\mathcal{K}_{n,\gamma}(P_n, Q_n) = \{x \in \mathcal{X}_n;\ \mathrm{d}P_n < \exp\{n\gamma\}\, \mathrm{d}Q_n\} .$$

For any $A_n \in \mathscr{A}_n$, A_n^c will denote the complement of A_n with respect to $\mathscr{X}_n$. Now let us write for $A_n \in \mathscr{A}_n$ and for measures P and Q

$$e_n(P_n, A_n) = \int_{A_n} \mathrm{d}P_n$$

and

$$e_n(P_n, Q_n, A_n) = e_n(P_n, A_n) + e_n(Q_n, A_n^c) .$$

In the case when $\gamma = 0$ we shall write $\mathscr{K}_n(P_n, Q_n)$ instead of $\mathscr{K}_{n,0}(P_n, Q_n)$. Moreover if $P = P_{\theta_1}$ and $Q = P_{\theta_2}$ for some $\theta_1, \theta_2 \in \Theta$, we shall write $\mathscr{K}_{n,\gamma}(\theta_1, \theta_2)$ instead of $\mathscr{K}_{n,\gamma}((P_{\theta_1})_n, (P_{\theta_2})_n)$ and analogously $H_\alpha(\theta_1, \theta_2)$ and $H(\theta_1, \theta_2)$ instead of $H_\alpha(P_{\theta_1}, P_{\theta_2})$ and $H(P_{\theta_1}, P_{\theta_2})$, respectively. Similar notation will be used for $e_n(P_n, Q_n, A_n)$. Finally, we shall say that we use a test A_n $(A_n \in \mathscr{A}_n)$, if we use A_n as a critical region (in the $\mathscr{X}_n$).

3. RESULTS

1. DEVIATION FROM ASSUMED VALUE OF PARAMETERS

Lemma 1. Let $\theta_1, \theta_2 \in \Theta$ be such that $P_{\theta_1} \neq P_{\theta_2}$ and $\gamma \in R$. Then

$$(1) \qquad \log H(\theta_1, \theta_2) \leqq \liminf_{n\to\infty} n^{-1} \log e_n(\theta_1, \theta_2, \mathscr{K}_{n,\gamma}(\theta_1, \theta_2)) \leqq$$
$$\leqq \limsup_{n\to\infty} n^{-1} \log e_n(\theta_1, \theta_2, \mathscr{K}_{n,\gamma}(\theta_1, \theta_2)) \leqq \log H(\theta_1, \theta_2) + |\gamma| .$$

Proof. It is well known that

$$e_n(\theta_1, \theta_2, \mathscr{K}_n(\theta_1, \theta_2)) = \min_{A \in \mathscr{A}_n} e_n(\theta_1, \theta_2, A)$$

and

$$\lim_{n\to\infty} n^{-1} \log e_n(\theta_1, \theta_2, \mathscr{K}_n(\theta_1, \theta_2)) = \log H(\theta_1, \theta_2)$$

(see [3]).

Therefore we have to prove only the last inequality in (1). Let $0 < \alpha < 1$. Then

$$(2) \qquad \limsup_{n\to\infty} n^{-1} \log e_n(\theta_1, \theta_2, \mathscr{K}_{n,\gamma}(\theta_1, \theta_2)) =$$
$$= \limsup_{n\to\infty} n^{-1} \log \left\{ \int_{\mathscr{K}_{n,\gamma}(\theta_1,\theta_2)} \prod_{i=1}^n f(x_i, \theta_1)\, \mathrm{d}\mu + \int_{\mathscr{K}_{n,\gamma}^c(\theta_1,\theta_2)} \prod_{i=1}^n f(x_i, \theta_2)\, \mathrm{d}\mu \right\} \leqq$$
$$\leqq \limsup_{n\to\infty} n^{-1} \log \left\{ \mathrm{e}^{n\alpha.\gamma} \int_{\mathscr{K}_{n,\gamma}(\theta_1,\theta_2)} \prod_{i=1}^n f^{1-\alpha}(x_i, \theta_1) f^\alpha(x_i, \theta_2)\, \mathrm{d}\mu + \right.$$
$$\left. + \mathrm{e}^{n(\alpha-1)\gamma} \int_{\mathscr{K}_{n,\gamma}^c(\theta_1,\theta_2)} \prod_{i=1}^n f^{1-\alpha}(x_i, \theta_1) f^\alpha(x_i, \theta_2)\, \mathrm{d}\mu \right\} \leqq$$
$$\leqq \limsup_{n\to\infty} n^{-1} \log \mathrm{e}^{n|\gamma|} \left\{ \int \prod_{i=1}^n f^{1-\alpha}(x_i, \theta_1) f^\alpha(x_i, \theta_2)\, \mathrm{d}\mu \right\} = \log H_{1-\alpha}(\theta_1, \theta_2) + |\gamma| .$$

As (2) holds for any $\alpha \in (0, 1)$, (1) is proved. □

Theorem 1. Let $\theta_1, \theta_2 \in \Theta$ and $P_{\theta_1} \neq P_{\theta_2}$. Let us assume, that there exist $\gamma_0 \leqq 0$, $\gamma_1 \geqq 0$, $|\gamma_0| + |\gamma_1| > 0$ such that for any $\gamma \in (\gamma_0, \gamma_1)$ the exists $\delta_\gamma > 0$ such that for any $\theta \in \Theta$, $\varrho(\theta, \theta_1) < \delta_\gamma$ and $n \in \mathscr{N}$ we have

$$\mathscr{K}_n(\theta_1, \theta_2) \subset \mathscr{K}_{n,\gamma}(\theta, \theta_2)\,. \tag{3}$$

Let, moreover, $H(\theta, \theta_2)$ be a continuous function of θ at the point θ_1. Then for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $\theta \in \Theta$, $\varrho(\theta, \theta_1) < \delta_\varepsilon$ we have

$$\log H(\theta_1, \theta_2) - \varepsilon \leqq \liminf_{n\to\infty} n^{-1} \log e_n(\theta, \theta_2, \mathscr{K}_n(\theta, \theta_2)) \leqq$$

$$\leqq \limsup_{n\to\infty} n^{-1} \log e_n(\theta, \theta_2, \mathscr{K}_n(\theta_1, \theta_2)) \leqq \log H(\theta_1, \theta_2) + \varepsilon\,.$$

Remark 2. Conditions under which $H(\theta, \theta_2)$ is a continuous function of θ at the point θ_1 have been given in [8]. The sufficient condition for this continuity occurred to be, e.g., the continuity of the density $f(x, \theta)$ (as a function of θ) in the mean (with respect to the measure μ) at the point θ_1.

Remark 3. The assertion of Theorem 1 based on the continuity of $H(\theta, \theta_2)$ at θ_1 enables us to claim the following statement: Having used the likelihood ratio test $\mathscr{K}_n(\theta_1, \theta_2)$ under the assumption that the true value of parameter θ lies in a neighbourhood of θ_1 (resp. θ_2) the asymptotic rate of convergence of the error probabilities will be only a little worse than in the case of using the best test $\mathscr{K}_n(\theta, \theta_2)$.

Remark 4. It may be seen from the following proof of Theorem 1 that it is easy in practice to evaluate numerical relations between δ_ε and ε. In fact, verifying (3) one finds out δ_γ as a function of γ (see Examples 1 and 2) and using $H(\theta, \theta_2)$ (as a function of θ) one can find out $\delta_\gamma = \delta_\gamma(\varepsilon/2)$ and $\delta_{\theta_1} = \delta_{\theta_1}(\varepsilon/2)$ for given $\varepsilon > 0$ (see the proof of Theorem 1) and then $\delta_\varepsilon = \min\{\delta_\gamma(\varepsilon/2); \delta_{\theta_1}(\varepsilon/2)\}$.

Proof of Theorem 1. Let $\varepsilon > 0$. Take $\gamma \in (\gamma_0, \gamma_1)$, $|\gamma| \leqq \varepsilon/2$ and find δ_γ according to the assumption of Theorem 1. Let $\delta_{\theta_1} > 0$ be chosen so that for all $\theta \in \Theta$ $\varrho(\theta, \theta_1) < \delta_{\theta_1}$ we have

$$|\log H(\theta, \theta_2) - \log H(\theta_1, \theta_2)| < \varepsilon/2\,. \tag{4}$$

and denote by $\delta_\varepsilon = \min\{\delta_{\theta_1}, \delta_\gamma\}$. Now let $\theta' \in \Theta$ be such that $\varrho(\theta', \theta_1) < \delta_\varepsilon$. From Lemma 1A we have

$$\limsup_{n\to\infty} n^{-1} \log e_n(\theta', \theta_2, \mathscr{K}_n(\theta_1, \theta_2)) =$$

$$= \limsup_{n\to\infty} n^{-1} \log \max\{e_n(\theta', \mathscr{K}_n(\theta_1, \theta_2)), e_n(\theta_2, \mathscr{K}_n^c(\theta_1, \theta_2))\}\,.$$

Let us split the set of all positive integers into two subsets as follows:

$$\mathscr{N}_1 = \{n \in \mathscr{N} : e_n(\theta', \mathscr{K}_n(\theta_1, \theta_2)) > e_n(\theta_2, \mathscr{K}_n^c(\theta_1, \theta_2))\}$$

and $\mathcal{N}_2 = \mathcal{N} \setminus \mathcal{N}_1$. Then let us denote $\{n_l\}_{l=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$ the increasing sequences of positive integers contained in $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively (i.e. $\{n_l\}_{l=1}^{\infty} \cup \{n_k\}_{k=1}^{\infty} = \mathcal{N}$).

Then

$$\limsup_{n\to\infty} n^{-1} \log \max \{e_n(\theta', \mathcal{K}_n(\theta_1, \theta_2)), e_n(\theta_2, \mathcal{K}_n^c(\theta_1, \theta_2))\} =$$

$$= \max \{\limsup_{l\to\infty} n_l^{-1} \log \max \{e_{n_l}(\theta', \mathcal{K}_{n_l}(\theta_1, \theta_2)), e_{n_l}(\theta_2, \mathcal{K}_{n_l}^c(\theta_1, \theta_2))\} ,$$

$$\limsup_{k\to\infty} n_k^{-1} \log \max \{e_{n_k}(\theta', \mathcal{K}_{n_k}(\theta_1, \theta_2)), e_{n_k}(\theta_2, \mathcal{K}_{n_k}^c(\theta_1, \theta_2))\}\} .$$

But

$$\limsup_{k\to\infty} n_k^{-1} \log \max \{e_{n_k}(\theta', \mathcal{K}_{n_k}(\theta_1, \theta_2)), e_{n_k}(\theta_2, \mathcal{K}_{n_k}^c(\theta_1, \theta_2))\} =$$

$$= \limsup_{k\to\infty} n_k^{-1} \log e_{n_k}(\theta_2, \mathcal{K}_n^c(\theta_1, \theta_2)) = \log H(\theta_1, \theta_2)$$

and

$$\limsup_{l\to\infty} n_l^{-1} \log \max \{e_{n_l}(\theta', \mathcal{K}_{n_l}(\theta_1, \theta_2)), e_{n_l}(\theta_2, \mathcal{K}_{n_l}^c(\theta_1, \theta_2))\} =$$

$$= \limsup_{l\to\infty} n_l^{-1} \log e_{n_l}(\theta', \mathcal{K}_{n_l}(\theta_1, \theta_2)) = \limsup_{l\to\infty} n_l^{-1} \log \int_{\mathcal{K}_{n_l}(\theta_1,\theta_2)} \prod_{i=1}^{n} f(x_i, \theta')\, \mathrm{d}\mu \leqq$$

$$\leqq \limsup_{l\to\infty} n_l^{-1} \log \int_{\mathcal{K}_{n_l,\gamma}(\theta',\theta_2)} \prod_{i=1}^{n} f(x_i, \theta')\, \mathrm{d}\mu \leqq \lim_{n\to\infty} n^{-1} \log e_n(\theta', \theta_2, \mathcal{K}_{n,\gamma}(\theta', \theta_2)) \leqq$$

$$\leqq \log H(\theta', \theta_2) + |\gamma| \leqq \log H(\theta_1, \theta_2) + \varepsilon ,$$

where we used Lemma 1 and (4). On the other hand

$$\lim_{n\to\infty} n^{-1} \log e_n(\theta', \theta_2, \mathcal{K}_n(\theta', \theta_2)) = \log H(\theta', \theta_2) \geqq \log H(\theta_1, \theta_2) - \varepsilon/2 .$$

□

Example 1. Let

$$\Theta = R , \quad \gamma > 0 , \quad f(x, \Theta) = (2\pi)^{-1} \exp\{-\tfrac{1}{2}(x - \theta)^2\}$$

and $0 < \theta_2 < \theta_1$. Then for $\theta > \theta_2$

$$\mathcal{K}_n(\theta_1, \theta_2) = \left\{x \in \mathcal{X}_n : \sum_{i=1}^{n} x_i < \frac{n}{2} (\theta_1 + \theta_2)\right\}$$

and

$$\mathcal{K}_{n,\gamma}(\theta, \theta_2) = \left\{x \in \mathcal{X}_n : \sum_{i=1}^{n} x_i < \frac{n}{2} \frac{2\gamma + \theta^2 - \theta_2^2}{\theta - \theta_2}\right\} .$$

At first let $\theta > \theta_1$. Then for any $n \in \mathcal{N}$ and $x \in \mathcal{K}_n(\theta_1, \theta_2)$ we have

$$\sum_{i=1}^{n} x_i < \frac{n}{2} (\theta_1 + \theta_2) < \frac{n}{2} \frac{2\gamma + \theta^2 - \theta_2^2}{\theta - \theta_2} ,$$

i.e. $x \in \mathscr{K}_{n,\gamma}(\theta, \theta_2)$. Now let $\theta_1 \geqq \theta > \max\{\theta_1 - 2\gamma/\theta_2, \theta_2\}$. Then

$$0 < 2\gamma + \theta(\theta - \theta_1)$$

and from it

$$\theta(\theta_1 + \theta_2) - \theta_1\theta_2 - \theta_2^2 < 2\gamma + \theta^2 - \theta_2^2 ,$$

i.e.

$$\theta_1 + \theta_2 < \frac{2\gamma + \theta^2 - \theta_2^2}{\theta - \theta_2}$$

and again from $x \in \mathscr{K}_n(\theta_1, \theta_2)$ follows $x \in \mathscr{K}_{n,\gamma}(\theta, \theta_2)$. So we have proved that it suffices to put

$$\delta_\gamma = \min\left\{\frac{2\gamma}{\theta_2},\ \theta_1 - \theta_2\right\}.$$

Example 2. Let

$$\Theta = R\,, \quad \gamma > 0\,, \quad f(x, \theta) = \frac{1}{\pi}\,\frac{1}{1 + (x - \theta)^2}\,.$$

Put

$$\delta_\gamma^2 = \frac{1 - \exp\{-\gamma\}}{\exp\{-\gamma\}}\,.$$

Then for any $\theta_1, \theta_2, \theta \in \Theta$ such that $|\theta - \theta_1| < \delta_\gamma$ and any $n \in \mathscr{N}$ we shall exhibit that

(5) $$\mathscr{K}_n(\theta_1, \theta_2) \subset \mathscr{K}_{n,\gamma}(\theta, \theta_2)\,.$$

Let

$$h(x) = \log \frac{1 + (x - \theta)^2}{1 + (x - \theta_1)^2}.$$

To prove (5) we shall show at first that for any $x \in R$

(6) $$h(x) > -\gamma\,.$$

It is easy to verify that $\lim_{|x| \to \infty} h(x) = 0 > -\gamma$. So we can find $K \in R$ such that for any $x \in R$, $|x| > K$ we have $h(x) > -\gamma$. Now, putting the derivative equal zero, one can find the maximum and minimum of the continuous function $h(x)$ on the closed interval $[-K, K]$. The minimum is attained for $x = \theta$ and it is equal to

$$\log \frac{1}{1 + (\theta - \theta_1)^2} > \log \frac{1}{1 + \delta_\gamma^2} = -\gamma\,.$$

So (6) is proved. Then however

$$\sum_{i=1}^{n} \log \frac{1 + (x_i - \theta)^2}{1 + (x_i - \theta_1)^2} > -n\gamma\,.$$

Let $x \in \mathscr{K}_n(\theta_1, \theta_2)$, i.e.

$$\sum_{i=p}^{n} \log \frac{1 + (x_i - \theta_1)^2}{1 + (x_i - \theta_2)^2} > 0 .$$

Then

$$\sum_{i=1}^{n} \log \{1 + (x_i - \theta)^2\} > -n\gamma + \sum_{i=1}^{n} \log \{1 + (x_i - \theta_2)^2\} ,$$

i.e., (5) is proved, too.

Remark 5. As in both examples the parameter of the model was the location one, it is easy to verify that the condition of the continuity of the minimal α-entropy $H(\theta, \theta_2)$ is fulfilled at any point θ_1 (see [8]). Therefore, having used the test $\mathscr{K}_n(\theta_1, \theta_2)$ for testing the composite hypothesis $H : \theta \in \mathcal{O}(\theta_1, \delta_1)$ $(=\{\theta \in \Theta, \varrho(\theta, \theta_1) < \delta_1\})$ against the composite alternative $A : \theta \in \mathcal{O}(\theta_2, \delta_2)$, the asymptotic rate of convergence of the error probabilities will be only a little less, depending on δ_1 (resp. δ_2), than the maximal possible rate.

Remark 6. By a straightforward computation one can show that (3) may be fulfilled for many other families of distributions, e.g. for lognormal, Maxwell's, Rayleigh's, Pareto's, gamma (when only one parameter varies (i.e., e.g., exponential distribution)), Poisson's, binomial, geometric, etc.

2. DEVIATION FROM INDEPENDENCE OR A TYPE OF DEPENDENCE

Lemma 2. Let P_n and Q_n be the product probability measures introduced above and P_n^* a probability measure defined on the measurable space $(\mathscr{X}_n, \mathscr{A}_n)$. Let for any $n \in \mathscr{N}$ the measure P_n^* be absolutely continuous with respect to the measure P_n. Then for any $\varepsilon > 0$ for which the inequality

$$\text{(7)} \qquad \limsup_{n\to\infty} n^{-1} \log \int \left[\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}\right]^{\alpha(P_1, Q_1)} \mathrm{d}P_n^* \leqq \log H(P_1, Q_1) + \varepsilon$$

holds, we have

$$\limsup_{n\to\infty} n^{-1} \log e_n(P_n^*, Q_n, \mathscr{K}_n(P_n, Q_n)) \leqq \log H(P_1, Q_1) + \varepsilon .$$

Remark 7. If $\alpha(P_1, Q_1) = 0$ or $\alpha(P_1, Q_1) = 1$, then in (7) and everywhere in the proof of Lemma 2

$$\lim_{\alpha\to 0+} \int \left[\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}\right]^{\alpha} \mathrm{d}P_n^* \quad \text{or} \quad \lim_{\alpha\to 1-} \int \left[\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}\right]^{\alpha} \mathrm{d}P_n^*$$

must be written instead of

$$\int \left[\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}\right]^{\alpha(P_1,Q_1)} \mathrm{d}P_n^* .$$

Proof of Lemma 2. A simple computation gives

$$e_n(P_n^*, Q_n, \mathscr{K}_n(P_n, Q_n)) = \int_{\mathscr{K}_n(P_n,Q_n)} \mathrm{d}P_n^* + \int_{\mathscr{K}_n^c(P_n,Q_n)} \mathrm{d}Q_n \leqq$$

$$\leqq \int_{\mathscr{K}_n(P_n,Q_n)} \left[\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}\right]^{\alpha(P_1,Q_1)} \mathrm{d}P_n^* + \int_{\mathscr{K}_n^c(P_n,Q_n)} \left[\frac{\mathrm{d}P_n}{\mathrm{d}Q_n}\right]^{1-\alpha(P_1,Q_1)} \mathrm{d}Q_n \leqq$$

$$\leqq \int \left[\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}\right]^{\alpha(P_1,Q_1)} \mathrm{d}P_n^* - \int \left[\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}\right]^{\alpha(P_1,Q_1)} \mathrm{d}P_n +$$

$$+ \int \left[\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}\right]^{\alpha(P_1,Q_1)} \mathrm{d}P_n + \int \left[\frac{\mathrm{d}P_n}{\mathrm{d}Q_n}\right]^{1-\alpha(P_1,Q_1)} \mathrm{d}Q_n .$$

Let us denote

$$\tau_n = \left\{\int \left[\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}\right]^{\alpha(P_1,Q_1)} \mathrm{d}P_n^*\right\}^{1/n}$$

and

$$v_n = \left\{\int \left[\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}\right]^{\alpha(P_1,Q_1)} \mathrm{d}P_n\right\}^{1/n} .$$

Then

(8) $$e_n(P_n^*, Q_n, \mathscr{K}_n(P_n, Q_n)) \leqq \tau_n^n - v_n^n +$$

$$+ 2\int \left[\frac{\mathrm{d}P_n}{\mathrm{d}\mu_n}\right]^{\alpha(P_1,Q_1)} \left[\frac{\mathrm{d}Q_n}{\mathrm{d}\mu_n}\right]^{1-\alpha(P_1,Q_1)} \mathrm{d}\mu_n .$$

Let us define $\{n_k\}_{k=1}^{\infty} = \{n \in \mathscr{N} : \tau_n \leqq v_n\}$ and $\{n_l\}_{l=1}^{\infty} = \mathscr{N} \setminus \{n_k\}_{k=1}^{\infty}$. Then we have

(9) $$\limsup_{k\to\infty} n_k^{-1} \log e_{n_k}(P_{n_k}^*, Q_{n_k}, \mathscr{K}_{n_k}(P_{n_k}, Q_{n_k})) \leqq$$

$$\leqq \lim_{k\to\infty} n_k^{-1} \log \int \left[\frac{\mathrm{d}P_{n_k}}{\mathrm{d}\mu_{n_k}}\right]^{\alpha(P_1,Q_1)} \left[\frac{\mathrm{d}Q_{n_k}}{\mathrm{d}\mu_{n_k}}\right]^{1-\alpha(P_1,Q_1)} \mathrm{d}\mu_n = \log H(P_1, Q_1) .$$

Further for $l \in \mathscr{N}$ let us write

$$\tau_{n_l}^{n_l} - v_{n_l}^{n_l} = (\tau_{n_l} - v_{n_l}) \sum_{j=1}^{n_l} \tau_{n_l}^{n_l-j} v_{n_l}^{j-1} \leqq (\tau_{n_l} - v_{n_l})\, n_l \tau_{n_l}^{n_l-1} .$$

From it we derive

$$(10) \qquad \limsup_{l\to\infty} n_l^{-1} \log \{\tau_{nl}^{n_l} - v_{nl}^{n_l}\} \leqq \limsup_{l\to\infty} (n_l^{-1} + 1) \log \tau_{nl} \leqq$$
$$\leqq \log H(P_1, Q_1) + \varepsilon$$

as it follows from (7). Making use of (8), (9) and (10) one may easily conclude the proof of the Lemma 2. □

Remark 8. Lemma 2 holds for P_n and Q_n not necessarily of the product type. It is sufficient to fulfil the conditions GC of [5]. The proof remains essentially the same. For the sake of simplicity of the notation used in the formulation of Lemma 2 and in its proof the author of the present paper decided to state Lemma 2 in the simpler form.

Lemma 3. Let P_n and Q_n be the product probability measures as above and P_n^* a probability measure defined on the measurable space $(\mathscr{X}_n, \mathscr{A}_n)$ dominating probability measures P_n and Q_n for every $n \in \mathscr{N}$. Let us denote by f_n, g_n and f_n^*. the densities of probability measures P_n, Q_n and P_n^* with respect to the measure μ_n, respectively. Let us assume that there is a positive ε, $\varepsilon < -\log H(P_1, Q_1)$ such that

$$(11) \qquad \limsup_{n\to\infty} n^{-1} \int \left[\log \frac{f_n^{\alpha(P_1,Q_1)}\, g_n^{1-\alpha(P_1,Q_1)}}{f_n^*}\right] \frac{f_n^{\alpha(P_1,Q_1)}\, g_n^{1-\alpha(P_1,Q_1)}}{[H(P_1, Q_1)]^n} \,\mathrm{d}\mu_n \leqq \varepsilon\,.$$

Then

$$\liminf_{n\to\infty} n^{-1} \log e_n(P_n^*, Q_n, \mathscr{K}_n(P_n^*, Q_n)) \geqq \log H(P_1, Q_1) - \varepsilon\,.$$

Remark 9. It is again necessary to carry out the proper substitution in the cases having been described in Remark 7.

To be able to prove Lemma 3 we need the following assertion (see [5], (1.7 p. 622)).

Assertion 1. Let T_n, S_n, R_n be probability measures and for every $n \in \mathscr{N}$ let R_n be absolutely continuous with respect to T_n as well as to S_n. Let us denote

$$h(R, T) = \limsup_{n\to\infty} n^{-1} \int \log \frac{\mathrm{d}R_n}{\mathrm{d}T_n}\,\mathrm{d}R_n$$

and

$$h(R, S) = \limsup_{n\to\infty} n^{-1} \int \log \frac{\mathrm{d}R_n}{\mathrm{d}S_n}\,\mathrm{d}R_n\,.$$

Then

$$\liminf_{n\to\infty} n^{-1} \log e_n(T_n, S_n, \mathscr{K}_n(T_n, S_n)) \geqq \min\{-h(R, T), -h(R, S)\}\,.$$

Proof of Lemma 3. Let us define for any $A \in \mathscr{A}_1 \ (= \mathscr{A})$

$$R_1(A) = \frac{1}{H(P_1, Q_1)} \int_A f_1^{\alpha(P_1,Q_1)} g_1^{1-\alpha(P_1,Q_1)} \, d\mu$$

and let R_n be the product probability measure generated by R_1. It is easy to verify that R_n is absolutely continuous with respect to P_n as well as to Q_n and moreover one may find out that

$$\log H(P_1, Q_1) = \min \{ -h(R, P), -h(R, Q) \}$$

(see [5], Perez 1972, proof of Theorem 2.1). From it we have

$$-h(R, Q) \geqq \log H(P_1, Q_1) \, .$$

A reformulation of the assumption of Lemma 3 gives

$$-h(R, P^*) \geqq \log H(P_1, Q_1) - \varepsilon$$

and from the assumption of the absolute continuity of P_n and Q_n with respect to P_n^* we deduce that R_n is absolutely continuous with respect to P_n^*. Using Assertion 1 we conclude

$$\liminf_{n \to \infty} n^{-1} \log e_n(P_n^*, Q_n, \mathscr{K}_n(P_n^*, Q_n)) \geqq$$
$$\geqq \min \{ -h(R, P^*), -h(R, Q) \} \geqq \log H(P_1, Q_1) - \varepsilon \, . \qquad \square$$

Remark 10. Lemma 3 holds (analogously as Lemma 2) for P_n and Q_n not necessarily of the product type, too. The reader, who would like to see it, should consult [5] to become familiar with the techniques used there.

Theorem 2. Under the assumptions of Lemma 2 and Lemma 3 we have the following bounds for the asymptotic rate of convergence of the error probabilities

$$\log H(P_1, Q_1) - \varepsilon \leqq \liminf_{n \to \infty} n^{-1} \log e_n(P_n^*, Q_n, \mathscr{K}_n(P_n^*, Q_n)) \leqq$$
$$\leqq \limsup_{n \to \infty} n^{-1} \log e_n(P_n^*, Q_n, \mathscr{K}_n(P_n, Q_n)) \leqq \log H(P_1, Q_1) + \varepsilon \, .$$

Proof. The middle inequality follows from the fact that $\mathscr{K}_n(P_n^*, Q_n)$ minimizes the sum of the error probabilities when testing P_n^* against Q_n. The other ones have been proved in Lemma 2 and Lemma 3, respectively. $\square$

In the rest of the paper we are going to give an example of a non-product type probability measure satisfying both conditions of Theorem 2.

Example 3. Let $X_1, X_2, X_3, \ldots$ be a sequence of i.i.d. random variables. Let us define P_n and Q_n as probability measures generated by the random vector $X_n' =$

$= (X_1, X_2, \ldots, X_n)$ when X_1 is distributed as $N(0, \sigma^2)$ and $N(\mu, \sigma^2)$, respectively. Let A be the following $(n \times n)$ matrix

$$A = \begin{bmatrix} 1, & 0, & 0, & \ldots, 0 \\ -\lambda, & 1, & 0, & \ldots, 0 \\ \lambda^2, & -\lambda, & 1, & \ldots, 0 \\ \vdots & \vdots & \vdots & \\ (-\lambda)^{n-1}, & (-\lambda)^{n-2}, & (-\lambda)^{n-3}, & \ldots, 1 \end{bmatrix}$$

Now let P_n^* be a probability measure generated by the random vector Y_n, where $Y_n = A \,.\, X_n$, when X_n is distributed according to P_n. As $\det A = 1$ and the inverse matrix of A has the form

$$A^{-1} = \begin{bmatrix} 1, & 0, & 0, & \ldots, & 0 \\ \lambda, & 1, & 0, & \ldots, & 0 \\ 0, & \lambda, & 1, & \ldots, & 0 \\ \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & 0 \\ 0, & 0, & 0, & \ldots, & 1 \end{bmatrix}$$

we may find out that

$$f_n^*(x_n) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\left[(1 + \lambda^2)\sum_{i=1}^{n} x_i^2 + 2\lambda \sum_{i=1}^{n-1} x_i x_{i+1}\right]\right\}.$$

It is easy to show that $\alpha(P_1, Q_1) = \frac{1}{2}$ (see [8], proof of Lemma 5). Then the straightforward computation gives $H(P_1, Q_1) = \exp\{-\mu^2/8\sigma^2\}$. Substituting

$$f_n(x_n) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right\},$$

$$g_n(x_n) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} (x_i - \mu)^2\right\}$$

and $f_n^*(x_n)$ into the left hand side of the inequality (11) one obtains

$$(12) \quad \limsup_{n\to\infty} \frac{1}{4n\sigma^2} (2\pi\sigma^2)^{-n/2} \int \left[2\lambda^2 \sum_{i=1}^{n} x_i^2 + 2\mu \sum_{i=1}^{n} x_i + 4\lambda \sum_{i=1}^{n-1} x_i x_{i+1} - n\mu^2\right] .$$

$$. \exp\left\{-\frac{\sum_{i=1}^{n} (x_i - \frac{1}{2}\mu)^2}{2\sigma^2}\right\} \mathrm{d}x_n =$$

$$= \limsup_{n\to\infty} \frac{1}{4n\sigma^2} \left[2\lambda^2 n(\sigma^2 + \tfrac{1}{4}\mu^2) + n\mu^2 + \lambda(n-1)\mu^2 - n\mu^2\right] =$$

$$= \frac{1}{4\sigma^2}\left[2\lambda^2(\sigma^2 + \tfrac{1}{4}\mu^2) + \lambda\mu^2\right].$$

Let $\varepsilon > 0$. Having solved the quadratic equation $h_\varepsilon(\lambda) = 0$, where

(13) $$h_\varepsilon(\lambda) = \lambda^2(\sigma^2 + \tfrac{1}{4}\mu^2) + \tfrac{1}{2}\lambda\mu^2 - 2\varepsilon\sigma^2\,,$$

we get

$$\lambda_1 = \frac{-\mu^2 - \sqrt{(\mu^4 + 32\varepsilon\sigma^2(\sigma^2 + \frac{1}{4}\mu^2))}}{4(\sigma^2 + \frac{1}{4}\mu^2)} \quad \text{and} \quad \lambda_2 = \frac{-\mu^2 + \sqrt{(\mu^4 + 32\varepsilon\sigma^2(\sigma^2 + \frac{1}{4}\mu^2))}}{4(\sigma^2 + \frac{1}{4}\mu^2)}.$$

Hence, owing to the convexity of the function $h_\varepsilon(\lambda)$, we have for any $\varepsilon > 0$ and λ, $\lambda_1 \leqq \lambda \leqq \lambda_2$, $h_\varepsilon(\lambda) \leqq 0$ and it implies (see (12) and (13)) that (11) is fulfilled (for P_n, Q_n and P_n^* as defined above).

To find an analogous interval for λ the inequality (7) holds in, one has to evaluate the limit

(14)

$$\limsup_{n\to\infty} n^{-1} \log \int \exp\left\{-\frac{1}{2\sigma^2}\left[(1+\lambda^2)\sum_{i=1}^{n} x_i^2 - \mu\sum_{i=1}^{n} x_i + 2\lambda\sum_{i=1}^{n-1} x_i x_{i+1} + \tfrac{1}{2}n\mu^2\right]\right\} \mathrm{d}\boldsymbol{x}_n.$$

Let us denote $\boldsymbol{\Sigma} = \boldsymbol{A} \,.\, \boldsymbol{A}'$ and $\boldsymbol{\mu}' = (\mu, \mu, \ldots, \mu)$ and put

$$\boldsymbol{t} = \tfrac{1}{2}\boldsymbol{\Sigma} \,.\, \boldsymbol{\mu}\,.$$

Then we can rewrite (14) into the form

$$\limsup_{n\to\infty} n^{-1} \log \int \exp\left\{-\frac{1}{2\sigma^2}\left[(\boldsymbol{x}_n - \boldsymbol{t}_n)' \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}_n - \boldsymbol{t}_n) - \boldsymbol{t}_n' \boldsymbol{\Sigma}^{-1}\boldsymbol{t}_n + \tfrac{1}{2}n\mu^2\right]\right\} \mathrm{d}\boldsymbol{x}_n =$$

$$= \limsup_{n\to\infty} \frac{1}{2n\sigma^2}\left(\boldsymbol{t}_n' \boldsymbol{\Sigma}^{-1}\boldsymbol{t}_n - \tfrac{1}{2}n\mu^2\right) =$$

$$= \limsup_{n\to\infty} \frac{1}{2n\sigma^2}\left(\tfrac{1}{4}\boldsymbol{\mu}' \boldsymbol{\Sigma}\boldsymbol{\mu} - \tfrac{1}{2}n\mu^2\right) = \limsup_{n\to\infty} \frac{1}{2n\sigma^2}\left[\tfrac{1}{4}(\boldsymbol{A}\boldsymbol{\mu})' \boldsymbol{A}\boldsymbol{\mu} - \tfrac{1}{2}n\mu^2\right].$$

One may see that

$$(\boldsymbol{A}\boldsymbol{\mu})_j = \left[1 - \lambda + \lambda^2 - \lambda^3 + \ldots + (-\lambda)^{j-1}\right]\mu = \frac{1 - (-\lambda)^j}{1 + \lambda}\,\mu$$

and

$$(\boldsymbol{A}\boldsymbol{\mu})' \boldsymbol{A}\boldsymbol{\mu} = \sum_{j=1}^{n}\left(\frac{1 - (-\lambda)^j}{1 + \lambda}\,\mu\right)^2 = \frac{n - 2\lambda\,\dfrac{1 - (-\lambda)^n}{1 + \lambda} + \lambda^2\,\dfrac{1 - \lambda^{2n}}{1 - \lambda^2}}{(1 + \lambda)^2}\,\mu^2\,.$$

So for λ, $|\lambda| \leqq 1$ we have found that (14) is equal

$$\frac{\mu^2}{2\sigma^2}\left[\frac{1}{4(1 + \lambda)^2} - \tfrac{1}{2}\right].$$

Now (7) may be written as follows:

$$\frac{\mu^2}{2\sigma^2}\left[\frac{1}{4(1+\lambda)^2} - \frac{1}{2}\right] \leqq -\frac{1}{8}\frac{\mu^2}{\sigma^2} + \varepsilon\,.$$

This inequality holds for $\lambda \geqq \sqrt{(1 + 8\varepsilon(\sigma^2/\mu^2))^{-1}} - 1$, if $0 < \varepsilon < \frac{1}{2}(\mu^2/\sigma^2)$ and for any λ, otherwise. (Let us remind that $\log H(P_1, Q_1) = -\frac{1}{8}(\mu^2/\sigma^2)$ and hence the latter case is not interesting.) Finally, let $0 < \varepsilon < \frac{1}{2}(\mu^2/\sigma^2)$. Then for any λ,

$$\lambda \in [\max(\lambda_1, \sqrt{(1 + 8\varepsilon(\sigma^2/\mu^2))^{-1}} - 1),\ \min(\lambda_2, 1)]$$

both the conditions of the Theorem 2 hold and so the asymptotic rates of the convergence of the error probabilities lie within the bounds given by this theorem.

4. APPENDIX

Lemma 1A. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers and for every $n \in \mathcal{N}$ we have $\max\{a_n, b_n\} > 0$. Then

$$\limsup_{n\to\infty} n^{-1} \log(a_n + b_n) = \limsup_{n\to\infty} n^{-1} \log \max\{a_n, b_n\}$$

and

$$\liminf_{n\to\infty} n^{-1} \log(a_n + b_n) = \liminf_{n\to\infty} n^{-1} \log \max\{a_n, b_n\}\,.$$

Proof. From

$$\max\{a_n, b_n\} \leqq a_n + b_n \leqq 2 \max\{a_n, b_n\}\,,$$

it follows that

$$n^{-1} \log \max\{a_n, b_n\} \leqq n^{-1} \log(a_n + b_n) \leqq n^{-1}(\log 2 + \log \max\{a_n, b_n\})$$

and finally

$$\limsup_{n\to\infty} n^{-1} \log \max\{a_n, b_n\} \leqq \limsup n^{-1} \log(a_n + b_n)\,,$$

$$\limsup_{n\to\infty} n^{-1} \log(a_n + b_n) \leqq \limsup n^{-1} \log \max\{a_n, b_n\}\,.$$

Similarly we can find

$$\liminf_{n\to\infty} n^{-1} \log \max\{a_n, b_n\} \leqq \liminf n^{-1} \log(a_n + b_n)$$

and

$$\liminf_{n\to\infty} n^{-1} \log(a_n + b_n) \leqq \liminf n^{-1} \log \max\{a_n, b_n\}\,. \qquad \square$$

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RNDr. Jan Ámos Víšek, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.