KYBERNETIKA — VOLUME 17 (1981), NUMBER 4

A SHORT NOTE ON OPTIMAL REPAIR RATES OF MULTIPLE CIRCUIT TRANSMISSION LINES

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In this short note, the optimal values of repair rates of transmission lines are found via complementary geometric programming.

1. INTRODUCTION

Multiple transmission lines on a single right of way are commonly used by utilities. The phenomenon of common cause outages was first of all recognized by a Task Force from IEEE subcommittee [1]. The Task Force defined a common mode outage as "an event having a single external cause with multiple failure events, where the events are not consequences of each other". The closed form expressions for steady-state probabilities in 2- and 3-line transmission systems have been derived out by Billinton, Medicherla and Sachdev [2]. These expressions were used to study the effect of common-cause outages on state probabilities. Thus the study done by Billinton, Medicherla and Sachdev [2] is descriptive one.

This paper deals with the prescriptive behaviour of multiple circuit transmission lines. The optimal repair rates of transmission lines have been found out by complementary geometric programming. Sections 2, 3, 4, and 5 deal with the steady-state probabilities of 2-line transmission system, complementary geometric programming, optimization model, and illustration, respectively.

2. STEADY STATE PROBABILITIES OF TWO-LINE TRANSMISSION SYSTEM

The most common arrangement of two transission lines on the same tower has been shown in Fig. 1(a). Two transmission lines on the same right-of-way arrangement has been shown in Fig. 1(b). The state space diagram illustrates a set of possible transitions from each state including a common-cause failure rate. Let λ_i, μ_i and λ_c be the independent failure rate of ith line, repair rate of ith line and common mode failure rate, respectively. Let $P(j)$ be the probability of state j, $j = 1, 2, 3$ and 4. Assuming that the state residence times are exponentially distributed, the steady state probabilities can be found by using frequency balancing approach. Fig. 2 shows one possible state space model: others can be created to suit the physical failure phenomena.

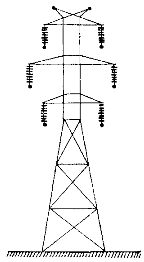

Fig. 1(a). Two transmission lines on the same transmission tower.

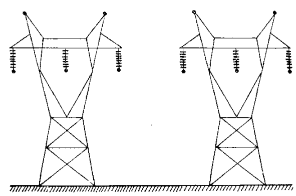

Fig. 1(b). Two transmission lines on the same right-of-way.

λ_i independent failure rate of line i
μ_i repair rate of line i
λ_c common mode failure rate
$P(j)$ probability of state j

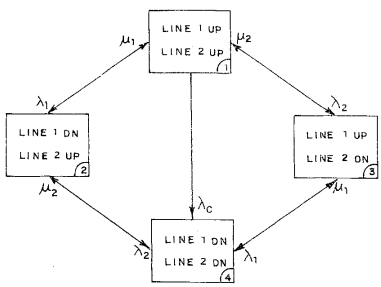

Fig. 2. State space diagram for two non-i.i.d. lines.

The steady state balance equations governing the system are given below:

(1) $$(\lambda_1 + \lambda_2 + \lambda_c)\, P(1) = \mu_1\, P(2) + \mu_2\, P(3)\,,$$

(2) $$(\lambda_2 + \mu_1)\, P(2) = \lambda_1\, P(1) + \mu_2\, P(4)\,,$$

(3) $$(\lambda_1 + \mu_2)\, P(3) = \lambda_2\, P(1) + \mu_1\, P(4)\,,$$

(4) $$(\mu_1 + \mu_2)\, P(4) = \lambda_c\, P(1) + \lambda_2\, P(2) + \lambda_1\, P(3)\,.$$

The normalized condition yields that

(5) $$\sum_{i=1}^{4} P(i) = 1 .$$

The solution of (1) through (5) is given below:

(6) $$P(1) = \mu_1\mu_2(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)/D ,$$

(7) $$P(2) = \mu_2[\lambda_1(\lambda_1 + \lambda_2 + \mu_1 + \mu_2) + \lambda_c(\lambda_1 + \mu_2)]/D ,$$

(8) $$P(3) = \mu_1[\lambda_2(\lambda_1 + \lambda_2 + \mu_1 + \mu_2) + \lambda_c(\lambda_2 + \mu_1)]/D$$

and

(9) $$P(4) = [\lambda_1\lambda_2(\lambda_1 + \lambda_2 + \mu_1 + \mu_2) + \lambda_c(\lambda_1 + \mu_2)(\lambda_2 + \mu_1)]/D ,$$

where

(10) $$D = (\lambda_1 + \mu_1)(\lambda_2 + \mu_2)(\lambda_1 + \lambda_2 + \mu_1 + \mu_2) + \\ + \lambda_c[(\lambda_1 + \mu_1)(\lambda_2 + \mu_1 + \mu_2) + \mu_2(\lambda_2 + \mu_2)] .$$

3. COMPLEMENTARY GEOMETRIC PROGRAMMING

Avriel and Williams [3] defined the complementary geometric program as follows:

PROGRAM 1. Minimize $R_0(\mathbf{x})$
subject to

$$R_k(\mathbf{x}) \leqq 1 , \quad k = 1, 2, \ldots, K , \quad \text{and} \quad \mathbf{x} > \mathbf{0} ,$$

where

$$\mathbf{x} = (x_1, x_2, \ldots, x_m) , \quad \mathbf{0} = (0, 0, \ldots, 0)$$

and

(11) $$R_k(\mathbf{x}) = [A(\mathbf{x}) - B(\mathbf{x})]/[C(\mathbf{x}) - D(\mathbf{x})] , \quad k = 0, 1, \ldots, K ,$$

with $A(\mathbf{x})$, $B(\mathbf{x})$, $C(\mathbf{x})$ and $D(\mathbf{x})$ as posynomials such that some of them may be absent.

Introducing a new variable x_0, constrained to satisfy $x_0 > 0$ and $x_0 \geqq R_0(\mathbf{x})$, we find the following standard complementary geometric program:

PROGRAM 2. Minimize x_0

subject to

$$[P_k(\mathbf{x})/Q_k(\mathbf{x})] \leqq 1 , \quad k = 0, 1, \ldots, K , \quad \text{and} \quad \mathbf{x} > 0 ,$$

where

(12) $$P_k(\mathbf{x}) = \sum_j c_{jk} \prod_{i=0}^{m} x_i^{a_{ijk}} , \quad k = 0, 1, \ldots, K ,$$

and

(13) $$Q_k(\mathbf{x}) = \sum_j d_{jk} \prod_{i=0}^{m} x_i^{b_{ijk}}, \quad k = 0, 1, \ldots, K,$$

with $\mathbf{x} = (x_0, x_1, \ldots, x_m)$.

To solve *PROGRAM* 2, we start to condense $Q_k(\mathbf{x})$ at some feasible point $\mathbf{x}^{(1)}$, and replace $Q_k(\mathbf{x})$ by its condensed value $Q_k(\mathbf{x}, \mathbf{x}^{(1)})$ and then solve the resulting geometric program to obtain the next point of condensation. In this way, we generate a sequence $\{\mathbf{x}^{(\alpha)}\}$, where $\mathbf{x}^{(\alpha+1)}$ is the solution of an ordinary geometric program given below:

PROGRAM 3. Minimize x_0

subject to

$$[P_k(\mathbf{x})/Q_k(\mathbf{x}, \mathbf{x}^{(\alpha)})] \leqq 1, \quad k = 0, 1, \ldots, K, \quad \text{and} \quad \mathbf{x} > \mathbf{0}.$$

In the terminology of geometric programming, the constraints $[P_k(\mathbf{x})/Q_k(\mathbf{x})] \leqq 1$, $\forall k$, are called the forced constraints whereas the constraints $x_i > 0$, $\forall i$, are called the natural constraints.

An alternate method to solve complementary geometric program has been developed in Swarup and Verma [4]. Another method to solve complementary geometric program has been given in Beightler and Phillips [5, p. 342]. It may be worth-mentioning that the algorithms given in Beightler and Phillips [5, p. 342] and Avriel and Williams [3] are identical. The convergence criteria used by Avriel and Williams [3], Beightler and Phillips [5, p. 342] and Dembo [6] are specific, but not universal. We shall solve our optimization problem by using the technique of double (total) condensation. The advantage of this technique is that it readily gives us an ordinary geometric program with zero degree of difficulty.

4. OPTIMIZATION MODEL

In this section, we shall formulate and solve a mathematical program pertaining to the maximization of the probability that the system is *totally up* subject to repair-cost and processing constraints which are defined below:

4.1. Repair-Cost Constraint

Define:

c_i = repair cost of ith line when the repair rate is one line per unit time.

Then the per unit time total repair cost is $(c_1\mu_1 + c_2\mu_2)$. If the per unit time total repair cost is constrained not to exceed a fixed amount c (say) available for repair purposes, the repair-cost constraint is given below:

(14) $$c^{-1}(c_j\mu_1 + c_2\mu_2) \leqq 1.$$

4.2. Processing Constraints

Since we need to define the prescriptive behaviour of the system when it is actually processing, hence it is assumed that $\mu_1 > 0$ and $\mu_2 > 0$ for given λ_1, λ_2 and λ_c such that $0 < \lambda_1, \lambda_2, \lambda_c < \infty$. The constraints $\mu_1 > 0$ and $\mu_2 > 0$ are called processing constraints. These fulfil the condition of the strict positivity for the decision variables as required in geometric programming. These are called the natural constraints in the terminology of geometric programming.

4.3. Formulation of Mathematical Program and its Solution

The mathematical program pertaining to the maximization of $P(1)$, the steady-state probability that the system is *totally up*, subject to repair-cost and processing constraints is given below:

PROGRAM 4.*) Minimize $f_1(\mu_1, \mu_2)/f_2(\mu_1, \mu_2)$

subject to

$$(14) \quad \text{and} \quad \mu_i > 0\,, \quad i = 1, 2\,,$$

where

$$f_1(\mu_1, \mu_2) = [\{\lambda_1\lambda_2(\lambda_1 + \lambda_2) + \lambda_1\lambda_2\lambda_c\} + \{\lambda_1\lambda_2 + (\lambda_2 + \lambda_c)(\lambda_1 + \lambda_2)\}\,\mu_1 + \{\lambda_1\lambda_2 + (\lambda_1 + \lambda_c)(\lambda_1 + \lambda_2)\}\,\mu_2 + (\lambda_2 + \lambda_c)\,\mu_1^2 + (\lambda_1 + \lambda_c)\,\mu_2^2 + \mu_1^2\mu_2 + \mu_1\mu_2^2 + (2\lambda_1 + 2\lambda_2 + \lambda_c)\,\mu_1\mu_2] \tag{15}$$

and

$$f_2(\mu_1, \mu_2) = \mu_1\mu_2(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)\,. \tag{16}$$

The above program is a non-standard complementary geometric program in the sense of Avriel and Williams [3]. The standard form of the above program is given below:

PROGRAM 5. Minimize μ_0
subject to

$$[\mu_0^{-1} f_1(\mu_1, \mu_2)/f_2(\mu_1, \mu_2)] \leqq 1\,, \quad (14) \quad \text{and} \quad \mu_i > 0\,, \quad i = 0, 1, 2\,.$$

Now, we can solve the above program by condensation method. Let $(\bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2)$ be the point of condensation of the above program. Then the condensation of $f_2(\mu_1, \mu_2)$ about this point of condensation gives rise to the following program.

PROGRAM 6. Minimize μ_0
subject to

$$g(\mu_0, \mu_1, \mu_2) \leqq 1\,, \quad (14) \text{ and } \quad \mu_i > 0\,, \quad i = 0, 1, 2\,,$$

*) Since the maximization of $P(1)$ is equivalent to the minimization of $1/P(1)$.

where

$$g(\mu_0, \mu_1, \mu_2) = A\ \bar{\mu}_0^{-1}\bar{\mu}_1^{-S_1}\bar{\mu}_2^{-S_2} f_1(\mu_1, \mu_2)\,, \tag{17}$$

with

$$A = \prod_{i=1}^{2} \bar{\mu}_i^{S_i} / f_2(\bar{\mu}_1, \bar{\mu}_2)\,, \tag{18}$$

$$S_1 = (\lambda_1 + \lambda_2 + 2\bar{\mu}_1 + \bar{\mu}_2)/S\,, \tag{19}$$

$$S_2 = (\lambda_1 + \lambda_2 + \bar{\mu}_1 + 2\bar{\mu}_2)/S \tag{20}$$

and

$$S = (\lambda_1 + \lambda_2 + \bar{\mu}_1 + \bar{\mu}_2)\,. \tag{21}$$

Here $f_2(\bar{\mu}_1, \bar{\mu}_2)$ is the value of $f_2(\mu_1, \mu_2)$ at $(\bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2)$.

The above program is an ordinary geometric program with seven degrees of difficulty. If we condense $g(\mu_0, \mu_1, \mu_2)$ about the same point of condensation, we get the following ordinary geometric program with zero degree of difficulty. The degree of an ordinary geometric program is defined as [number of terms – (number of variables + 1)].

PROGRAM 7. Minimize μ_0
subject to

$$B\mu_0^{-1} \prod_{i=1}^{2} \mu_i^{T_i} \leqq 1\,, \quad (14) \text{ and } \quad \mu_i > 0\,, \quad i = 0, 1, 2\,,$$

where

$$B = g(\bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2)\, \bar{\mu}_0 \prod_{i=1}^{2} \bar{\mu}_i^{-T_i}\,, \tag{22}$$

$$T_1 = [\{\lambda_1\lambda_2 + (\lambda_2 + \lambda_c)(\lambda_1 + \lambda_2)\}\, \bar{\mu}_1 + 2(\lambda_2 + \lambda_c)\, \bar{\mu}_1^2 + {} \tag{23}$$
$$+ 2\bar{\mu}_1^2\bar{\mu}_2 + \bar{\mu}_1\bar{\mu}_2^2 + (2\lambda_1 + 2\lambda_2 + \lambda_c)\, \bar{\mu}_1\bar{\mu}_2] / f_1(\bar{\mu}_1, \bar{\mu}_2) - S_1$$

and

$$T_2 = [\{\lambda_1\lambda_2 + (\lambda_1 + \lambda_c)(\lambda_1 + \lambda_2)\}\, \bar{\mu}_2 + 2(\lambda_1 + \lambda_c)\, \bar{\mu}_2^2 + {} \tag{24}$$
$$+ \bar{\mu}_1^2\bar{\mu}_2 + 2\bar{\mu}_1\bar{\mu}_2^2 + (2\lambda_1 + 2\lambda_2 + \lambda_c)\, \bar{\mu}_1\bar{\mu}_2] / f_1(\bar{\mu}_1, \bar{\mu}_2) - S_2$$

with $f_1(\bar{\mu}_1, \bar{\mu}_2)$ and $g(\bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2)$ as the respective values of $f_1(\mu_1, \mu_2)$ and $g(\mu_0, \mu_1, \mu_2)$ at $(\bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2)$.

The dual of the above program is given below:

PROGRAM 8. Maximize V
subject to

$$\lambda_{01} = 1\,, \quad \lambda_{01} - \lambda_{11} = 0\,,$$
$$T_i\lambda_{11} + \lambda_{2i} = 0\,, \quad i = 1, 2\,, \quad \text{and} \quad \lambda_{01}, \lambda_{11}, \lambda_{21}, \lambda_{22} \geqq 0$$

where

$$V = B^{\lambda_{11}} \prod_{i=1}^{2} [\lambda_{2i}^{-1}(\lambda_{21} + \lambda_{22})\, c_i/c]^{\lambda_{2i}}\,. \tag{25}$$

The optimal solution of the above program is given below:

$$\lambda_{01}^{0} = 1 , \tag{26}$$

$$\lambda_{11}^{0} = 1 \tag{27}$$

and

$$\lambda_{2i}^{0} = \begin{cases} -T_i > 0 & \text{if} \quad T_i < 0 \\ 0 & \text{if} \quad T_i \geqq 0 . \end{cases}$$

If one or both of λ_{2}^{0}, and λ_{22}^{0} are zero at optimality, repair-cost constraint becomes slack. In this case, the method of Duffin and Peterson [9] can be used to find out the optimal solution vector of *PROGRAM* 7. If $\lambda_{2i}^{0} > 0$, $\forall i = 1, 2$; the primal-dual relationship gives us the following solution vector of *PROGRAM* 7

$$(\mu_0^0, \mu_1^0, \mu_2^0) ,$$

where

$$\mu_0^0 = V^0 \tag{29}$$

and

$$\mu_i^0 = (c/c_i)\left[T_i/(T_1 + T_2)\right] , \quad i = 1, 2 , \tag{30}$$

with V^0 as the value of V evaluated at λ_{11}^{0}, λ_{01}^{0} and λ_{2i}^{0}.

Now, the optimal solution vector of *PROGRAM* 7 is called the current solution vector and following steps are followed.

Step 1: If the following conditions are satisfied, the current solution vector is the optimal vector of *PROGRAM* 5 which, in turn, gives the optimal solution vector of *PROGRAM* 4. Otherwise go to Step 2.

(i) Feasibility if $|F_i^0 - 1| \leqq \varepsilon_1$, $i = 1, 2$,

(ii) optimality if (i) held and $|(\mu_0^0 - v_0)/v_0| \leqq \varepsilon_2$, where ε_1 and ε_2 are predetermined smallest quantities,

$$F_1 = (\mu_0^0)^{-1} f_1(\mu_1^0, \mu_2^0)/f_2(\mu_1^0, \mu_2^0) \tag{31}$$

and

$$F_2^0 = c^{-1}(c_1\mu_1^0 + c_2\mu_2^0) . \tag{32}$$

Step 2: Find out "new current solution vector" by using the"current solution vector" as a point of condensation and go to Step 3.

Step 3: Call "new current solution vector" the "current solution vector" and go to Step 1.

5. ILLUSTRATION

Let $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_c = 3$, $c_1 =$ Rs 5·00, $c_2 =$ Rs 8·00, $c =$ Rs 18·00, $\varepsilon_1 = 10^{-3}$ and $\varepsilon_2 = 10^{-2}$. Then the standard complementary geometric program is given below:

PROGRAM 9. Minimize μ_0
subject to

$$(\Phi_1/\Phi_2) \leqq 1\,,$$

$$0{\cdot}2777777\,\mu_1 + 0{\cdot}4444444\,\mu_2 \leqq 1$$

$$\text{and} \quad \mu_0, \mu_1, \mu_2 > 0\,,$$

where

$$\Phi_1 = \mu_0^{-1}(12 + 17\mu_1 + 14\mu_2 + 5\mu_1^2 + 4\mu_2^2 + \mu_1^2\mu_2 + \\ + \mu_1\mu_2^2 + 9\mu_1\mu_2) \tag{33}$$

and

$$\Phi_2 = 3\mu_1\mu_2 + \mu_1^2\mu_2 + \mu_1\mu_2^2\,. \tag{34}$$

Let $(\bar{\mu}_0^0, \bar{\mu}_1^0, \bar{\mu}_2^0) = (1, 1, 1)$ be the point of condensation of *PROGRAM* 9. Then the total condensation of (Φ_1/Φ_2) about $(1, 1, 1)$ gives rise to the following ordinary geometric program with zero degree of difficulty.

PROGRAM 10. Minimize μ_0
subject to

$$12{\cdot}6\,\mu_0^{-1}\,\mu_1^{-0.5609524}\,\mu_2^{-0.6603175} \leqq 1\,,$$

$$0{\cdot}2777777\,\mu_1 + 0{\cdot}4444444\,\mu_2 \leqq 1$$

$$\text{and} \quad \mu_0, \mu_1, \mu_2 > 0\,.$$

The optimal solution vector of the above program is given by $(\mu_0^0, \mu_1^0, \mu_2^0) =$ $= (8{\cdot}2640, 1{\cdot}6844, 1{\cdot}1969)$. But the convergence criteria are not satisfied; hence this current solution vector is taken as a point of condensation to find out new current

Table 1

Iteration	μ_0^0	μ_1^0	μ_2^0	Optimal value of $P(1)$	Optimal value of $1/P(1)$
0	1·0000	1·0000	1·0000	0·07936	12·6
1	8·2640	1·6844	1·1969	0·11711	8·538
2	8·5076	1·6236	1·2353	0·11755	8·5071
3	8·5075	1·6236	1·2353	0·11755	8·5071

solution vector given at iteration no. 2 in Table 1. The optimal value of repair rate of first line is $\mu_1^0 = 1{\cdot}6236$ and that of second line is $\mu_2^0 = 1{\cdot}2353$. The optimal (maximal) value of the probability that the system is *totally up* is 0·11755.

CONCLUSIONS

Billinton, Medicherla and Sachdev [2] studied the descriptive behaviour of multiple circuit transmission lines system whereas this paper deals with the prescriptive behaviour of multiple circuit transmission lines system via complementary geometric programming. The salient feature of this study is that it advances the state-of-art of applications of geometric programming (confirm Duffin, Peterson and Zener [7], Beightler and Phillips [5] and Rijckaert and Martènes [8]). The convergence criteria used here is what have been proposed by Dembo [6].

ACKNOWLEDGEMENTS

My sincere thanks go to the principal of J. V. Jain College, Saharanpur, who allowed me to pursue my doctoral work in the department of Mathematics of this College. I am grateful to my supervisor Prof. Kanti Swarup of I. I. P. A., New Delhi, for his guidance. Thanks are also due to Dr. P. K. Gupta for his encouragement. I am thankful to the referee and the Editor for their fruitful suggestions which went into reshaping this paper.

(Received April 10, 1979.)

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