KYBERNETIKA — VOLUME 17 (1981), NUMBER 2

A MATHEMATICAL MODEL OF STORAGE

PETR MANDL

A method of constructing probabilistic models of storage systems is presented. It consists in assigning to each period a Gaussian random vector S_n, from which the relevant quantities like the input, the supply or the demand are obtained by means of simple transformations.

0. INTRODUCTION

Whenever in probabilistic modelling of storage the total inputs and the total demands in different periods are not assumed to be mutually independent random variables, the model can only exceptionally be investigated by traditional methods using mainly finite Markov chains. The state space of the chain is too large and its transition matrix not tractable. In this paper we present an approach which consists in replacing nonlinear functions of Gaussian vectors by equivalent Gaussian vectors. The principles introduced in [3] are here developed into a general method. To make the basic ideas apparent, we begin with an example.

1. EXAMPLE 1: THE EFFECT OF A DAM ON THE RELIABILITY OF THE WATER SUPPLY FOR IRRIGATION

The line in Figure 1a represents a river. X_n^1 denotes the flow in Profile 1 in n-th period, X_n^2 the demanded water quantity for irrigation in Profile 2 in n-th period. Let the periods in consideration be the N months of the year, when irrigation is presumable (April–October). The sequence $\{(X_n^1, X_n^2),\ n = 1, \ldots, N\}$ has the following mathematical model. We assume that it arises by transformations $f(x) = \exp\{x\}$ and $k(x) = \max(0, x)$ from a Gaussian first order autoregressive sequence.

In symbols

$$(1)\qquad X_n^1 = f(V_n^1)\,, \quad X_n^2 = k(V_n^2)\,,$$

$$V_n^1 = a_n^1 + b_n^{11} V_{n-1}^1 + b_n^{12} V_{n-1}^2 + c_n^{11} L_n^1 + c_n^{12} L_n^2 = h_n^1(V_{n-1}^1, V_{n-1}^2, L_n^1, L_n^2)\,,$$

$$(2)\qquad V_n^2 = a_n^2 + b_n^{21} V_{n-1}^1 + b_n^{22} V_{n-1}^2 + c_n^{21} L_n^1 + c_n^{22} L_n^2 =$$

$$= h_n^2(V_{n-1}^1, V_{n-1}^2, L_n^1, L_n^2)\,, \qquad n = 1, \ldots, N\,.$$

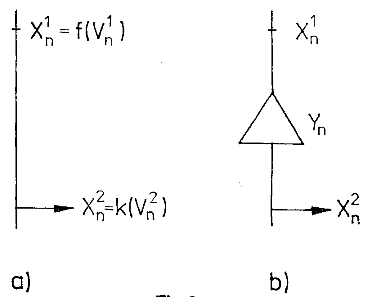

Fig. 1.

Constants a_n^1, a_n^2, b_n^{11}, ..., c_n^{22} are estimated from the observed flows, the size and the kind of irrigated fields etc. L_n^1, L_n^2, $n = 1, \ldots, N$, are mutualy independent standard normal random variables. The initial values (V_0^1, V_0^2) are assumed to have the bivariate normal distribution with known parameters. Hypothesis (1), i.e. logarithmic normal natural flows and truncated normal irrigation demands, is employed by the hydrologists as well as the autoregressive models often of higher order ([1], [2], [4]).

Our aim is to produce a mathematical model of the system after the construction of a dam between profiles 1 and 2 with a reservoir of useful capacity K (Fig. 1b). We therefore introduce variables Y_n, $n = 1, \ldots, N$, representing the water volume in the reservoir at the end of n-th month. Let the initial volume be $Y_0 = K$, the reservoir is full after the winter period. The operation rule of the dam is determined by the requirement to meet the irrigation demand X_n^2 and to maintain minimal flow q below Profile 2. This implies the relation

$$(3)\qquad \begin{aligned} Y_n &= 0 && \text{if} \quad Y_{n-1} + X_n^1 - X_n^2 - q \leqq 0\,, \\ Y_n &= Y_{n-1} + X_n^1 - X_n^2 - q && \text{if} \quad 0 < Y_{n-1} + X_n^1 - X_n^2 - q < K\,, \\ Y_n &= K && \text{if} \quad Y_{n-1} + X_n^1 - X_n^2 - q \geqq K\,. \end{aligned}$$

(3) can be also written as

$$Y_n = g(Y_{n-1} + X_n^1 - X_n^2 - q)\,, \quad n = 1, \ldots, N\,,$$

where $g(x) = k(x) - k(x - K)$. Setting

$$V_n^3 = Y_{n-1} + X_n^1 - X_n^2 - q\,,$$

we have with regard to (1), (2)

$$Y_n = g(V_n^3)\,, \tag{4}$$

$$V_n^3 = g(V_{n-1}^3) + f(h_n^1(V_{n-1}^1, V_{n-1}^2, L_n^1, L_n^2)) - \tag{5}$$

$$- k(h_n^2(V_{n-1}^1, V_{n-1}^2, L_n^1, L_n^2)) - q = h_n^3(V_{n-1}^1, V_{n-1}^2, V_{n-1}^3, L_n^1, L_n^2)\,, \quad n = 1, \ldots, N\,.$$

(2) and (5) give a recursive representation of the random sequence

$$(V_n^1, V_n^2, V_n^3)\,, \quad n = 1, \ldots, N\,, \tag{6}$$

with $\{(L_n^1, L_n^2), n = 1, \ldots, N\}$ creating the random noise. In Case a the linearity of (2) implied the normal distribution of the sequence $\{(V_n^1, V_n^2), n = 1, \ldots, N\}$. In Case b the nonlinearity of (5) causes that the probability distribution of (6) can only be estimated on the basis of simulations. We suggest a different approach, namely to replace Model b by a related one based on a Gaussian sequence $\{(S_n^1, S_n^2, S_n^3), n = 1, \ldots, N\}$ satisfying (5) in the sense of the equality of the first two moments. Instead of (2), (5) we thus write

$$S_n^1 = h_n^1(S_{n-1}^1, S_{n-1}^2, L_n^1, L_n^2)\,,$$

$$S_n^2 = h_n^2(S_{n-1}^1, S_{n-1}^2, L_n^1, L_n^2)\,,$$

$$S_n^3 \sim h_n^3(S_{n-1}^1, S_{n-1}^2, S_{n-1}^3, L_n^1, L_n^2)\,, \quad n = 1, \ldots, N\,.$$

The reliability of the water supply is measured by the probability that the requirements will be met in all months,

$$\mathsf{P}(Y_{n-1} + X_n^1 \geqq X_n^2 + q\,,\ n = 1, \ldots, N) \sim \mathsf{P}(S_n^3 \geqq 0\,, \quad n = 1, \ldots, N)\,.$$

2. GENERAL SCHEME

Let us now formulate in general terms the construction exemplified in Section 1. It should be stressed that we do not conceive it as an approximation to the nonlinear model in consideration but as a method to build mathematical models based on transformed Gaussian sequences.

Let V_0 be an r-dimensional random vector and let $\{L_n, n = 1, \ldots, N\}$ be a sequence of mutually independent l-dimensional random vectors, independent also of V_0. Further, let us have a sequence of transformations $\{h_n, n = 1, \ldots, N\}$ mapping $(r + l)$-dimensional vectors into r-dimensional vectors. Define $V_1 = h_1(V_0, L_1)$ and assume $\mathsf{E}\|V_1\|^2 < \infty$. E means expectation. Replace V_1 by the equivalent Gaussian vector S_1, i.e. by S_1 having the same first and second moments as V_1. Next let $V_2 =$

$= h_2(S_1, L_2)$, assume $\mathsf{E}\|V_2\|^2 < \infty$, and let (S_1, S_2) be the pair of Gaussian vectors equivalent to (S_1, V_2). In subsequent steps set $V_n = h_n(S_{n-1}, L_n)$, and let $\{S_1, \ldots, S_{n-1}, S_n\}$ be the Gaussian sequence equivalent to $\{S_1, \ldots, S_{n-1}, V_n\}$ provided that $\mathsf{E}\|V_n\|^2 < \infty$. The result of the construction is a Gaussian sequence $\{S_n, n = 1, \ldots, N\}$, more precisely its probability distribution. A second sequence of transformations $\{g_n, n = 1, \ldots, N\}$ provides the state vectors of the modelled system $X_n = g_n(S_n)$, $n = 1, \ldots, N$.

Let us introduce the denotation

$$\mathsf{E}S_n = m_n = (m_n^1, \ldots, m_n^r)' , \quad \mathsf{E}S_n(S_n - m_n)' = M_n = \|m_n^{ij}\|_{i,j=1}^r ,$$
$$\mathsf{E}S_k(S_n - m_n)' = M_{kn} = \|m_{kn}^{ij}\|_{i,j=1}^r , \quad k, n = 1, \ldots, N .$$

$'$ indicates the transposition of a vector or a matrix.

Theorem. $\{S_n, n = 1, \ldots, N\}$ is a Markovian sequence.

Proof. Let R_{jk} denote any regression matrix of S_k with respect to S_j. The conditional expectation $\mathsf{E}\{S_k \mid S_j\}$ is then

$$\mathsf{E}\{S_k \mid S_j\} = m_k + R'_{jk}(S_j - m_j) . \tag{7}$$

The residual

$$S_k - \mathsf{E}\{S_k \mid S_j\} \tag{8}$$

has zero covariances with S_j. Consequently,

$$M_{jk} - M_j R_{jk} = 0 . \tag{9}$$

First we prove that for $1 \leqq i \leqq j < N$ holds

$$M_{ij+1} = M_{ij} R_{jj+1} . \tag{10}$$

Next we shall prove that (10) implies the Markov property of $\{S_n, n = 1, \ldots, N\}$. For $j = i$ follows (10) from (9). Thus, assume $j > i$. Then,

$$M_{ij+1} = \mathsf{E}S_i V'_{j+1} - \mathsf{E}S_i\, \mathsf{E}V'_{j+1} = \mathsf{E}S_i h_{j+1}(S_j, L_{j+1})' - m_i m'_{j+1} =$$
$$= \mathsf{E}(\mathsf{E}\{S_i \mid S_j\}\, h_{j+1}(S_j, L_{j+1})') - m_i m'_{j+1} .$$

Using (7) we get

$$M_{ij+1} = \mathsf{E}((m_i + R'_{ji}(S_j - m_j))\, h_{j+1}(S_j, L_{j+1})') - m_i m'_{j+1} =$$
$$= \mathsf{E}(m_i + R'_{ji}(S_j - m_j))\, V'_{j+1} - m_i m'_{j+1} = \mathsf{E}R'_{ji}(S_j - m_j)\, S'_{j+1} =$$
$$= R'_{ji} M_{jj+1} = R'_{ji} M_j R_{jj+1} = M'_{ji} R_{jj+1} = M_{ij} R_{jj+1} .$$

By that (10) is proved.

From (10) follows that (9) is satisfied for

$$R_{jk} = R_{jj+1}R_{j+1j+2} \dots R_{k-1k} . \tag{11}$$

To establish the Markov property of $\{S_n, n = 1, \dots, N\}$ we assume R_{jk}, $1 \leqq j < k \leqq N$, chosen so that (11) is valid. Let us show that, for $1 \leqq j < k \leqq N$, (8) has zero covariances with S_i, $i = 1, \dots, j$. Repeated use of (10) gives

$$M_{ik} = M_{ij}R_{jj+1}R_{j+1j+2} \dots R_{k-1k} . \tag{12}$$

Hence, with regard to (11),

$$\mathsf{E}(S_i - m_i)(S_k - \mathsf{E}\{S_k \mid S_j\})' = M_{ik} - M_{ij}R_{jk} = 0 .$$

Since noncorrelated Gaussian variables are independent, we conclude that (8) is independent of $S_1, \dots, S_j$. Consequently, the conditional distribution of S_k given $S_1, \dots, S_j$ depends only on S_j. This is in fact the Markovian property. □

Let us deduce some implications of the Theorem. Denote

$$D_n^* = S_n - \mathsf{E}\{S_n \mid S_{n-1}\} , \quad n = 2, \dots, N .$$

In the above proof we saw that D_n^* is independent of $S_1, \dots, S_{n-1}$ and therefore of $D_2^*, \dots, D_{n-1}^*$ as well. Thus, we can write

$$S_n = m_n + R'_{n-1n}(S_{n-1} - m_{n-1}) + D_n^* , \quad n = 2, \dots, N , \tag{13}$$

where $\{D_2^*, \dots, D_N^*\}$ is a sequence of mutually independent Gaussian vectors. The covariance matrix of D_n^* is

$$\mathsf{E}D_n^*D_n^{*\prime} = M_n - R'_{n-1n}M_{n-1}R_{n-1n} .$$

Recall that

$$M_{n-1}R_{n-1n} = M_{n-1n} , \tag{14}$$

$$m_n = \mathsf{E}h_n(S_{n-1}, L_n) , \tag{15}$$

$$M_n = \mathsf{E}(h_n(S_{n-1}, L_n) - \mathsf{E}h_n(S_{n-1}, L_n))\, h_n(S_{n-1}, L_n)' , \tag{16}$$

$$M_{n-1n} = \mathsf{E}(S_{n-1} - m_{n-1})\, h_n(S_{n-1}, L_n)' . \tag{17}$$

The transformation h_n is given as well as the probability distribution of L_n. Moreover, L_n is independent of S_{n-1}. Hence, from (14)–(17) follows that in autoregressive relation (13) the coefficients and the covariance matrix of D_n^* are functions of (m_{n-1}, M_{n-1}), since the parameters (m_{n-1}, M_{n-1}) specify fully the normal distribution of S_{n-1}. We can therefore write

$$S_n = a_n(m_{n-1}, M_{n-1}) + B_n(m_{n-1}, M_{n-1})\, S_{n-1} + C_n(m_{n-1}, M_{n-1})\, D_n , \tag{18}$$

$$n = 2, \dots, N .$$

In (18), D_n, $n = 2, \ldots, N$, are mutually independent standard normal random vectors of dimension $q \leqq r$, and

$$a_n(m, M)\,, \quad B_n(m, M)\,, \quad C_n(m, M)\,, \quad n = 2, \ldots, N\,,$$

are matrices having dimensions $r \times 1$, $r \times r$, and $r \times q$, respectively. The matrices depend on the parameters (m, M) of the r-dimensional normal distribution.

3. NUMERICAL METHODS

To compute the parameters of the model one calculates the basic moments from (15)–(17). From (14), R_{n-1n}, $n = 2, \ldots, N$, can be obtained, and hence, using (12), other covariance matrices or the coefficients in (18) etc. To facilitate the evaluation of the right hand sides in (15)–(17) we present formulae applicable to models of the kind considered in Section 1, where L_n, $n = 1, \ldots, N$, are Gaussian, and h_n, $n = 1, \ldots$ $\ldots, N$, are linear combination of three basic functions:

$$i(x) = x\,, \quad f(x) = \exp\{x\}\,, \quad k(x) = \max(0, x)\,.$$

Let (S^1, S^2) denote a random vector having the bivariate normal distribution with mean m and covariance matrix M. Further, let

$$\varphi(x) = \frac{1}{\sqrt{(2\pi)}} \exp\{-\tfrac{1}{2}x^2\}\,, \quad \Phi(x) = \int_{-\infty}^{x} \varphi(y)\,\mathrm{d}y\,.$$

The following relations hold:

$$\mathsf{E} f(S^1) = \exp\{m^1 + \tfrac{1}{2}m^{11}\}\,;$$

$$\mathsf{E}\, k(S^1) = k(m^1) \quad \text{if} \quad m^{11} = 0\,,$$

$$= m^1\, \Phi\left(\frac{m^1}{\sqrt{m^{11}}}\right) + \sqrt{(m^{11})}\, \varphi\left(\frac{m^1}{\sqrt{m^{11}}}\right) \quad \text{if} \quad m^{11} > 0\,;$$

$$\mathsf{E}\, S^1 f(S^2) = (m^1 + m^{12}) \exp\{m^2 + \tfrac{1}{2}m^{22}\}\,;$$

$$\mathsf{E}\, S^1 k(S^2) = m^1 k(m^2) \quad \text{if} \quad m^{22} = 0\,,$$

$$= \left(m^1 - \frac{m^{12}}{m^{22}}\, m^2\right) \mathsf{E}\, k(S^2) + \frac{m^{12}}{m^{22}}\, \mathsf{E}\, k(S^2)^2 \quad \text{if} \quad m^{22} > 0\,;$$

$$\mathsf{E} f(S^1) f(S^2) = \exp\{m^1 + \tfrac{1}{2}(m^{11} + 2m^{12} + m^{22}) + m^2\}\,,$$

$$\mathsf{E} f(S^1)\, k(S^2) = \exp\{m^1 + \tfrac{1}{2}m^{11}\}\, \mathsf{E}\, k(S^2 + m^{12})\,.$$

When computing $\mathsf{E}\, k(S^1)\, k(S^2)$, three cases are to be distinguished:

1. $$m^{12} = 0 : \mathsf{E}\, k(S^1)\, k(S^2) = \mathsf{E}\, k(S^1)\, \mathsf{E}\, k(S^2)\,.$$

2. $$0 < (m^{12})^2 = m^{11}m^{22} : \mathsf{E}\, k(S^1)\, k(S^2) =$$

$$= (m^1 m^2 + \sqrt{m^{11}}\sqrt{m^{22}})\, \Phi(d) + \frac{m^1 m^2}{d}\, \varphi(d)\,,$$

where $d = \min(m^1/\sqrt{m^{11}}, m^2/\sqrt{m^{22}})$.

3. $$0 < (m^{12})^2 < m^{11}m^{22} : \mathsf{E}\, k(S^1)\, k(S^2) = \mathsf{E}\, k(S^1)\, \mathsf{E}\, k(S^2) +$$

(19) $$+ \sqrt{m^{11}}\sqrt{m^{22}} \sum_{j=0}^{\infty} \frac{r^{j+1}}{(j+1)!}\, \varphi^{(j-1)}\left(\frac{m^1}{\sqrt{m^{11}}}\right) \varphi^{(j-1)}\left(\frac{m^2}{\sqrt{m^{22}}}\right),$$

where $r = m^{12}/\sqrt{(m^{11}m^{22})}$, $\varphi^{(j-1)}(x) = d^j\, \Phi(x)/\mathrm{d}x^j$, $j = 0,1, \ldots$. Expansion (19) follows from Mahler's formula for the bivariate normal density.

For illustration we shall consider an example.

4. EXAMPLE 2

A waste product of a factory is to be processed by a plant able to handle q tons per day. Let the quantity of the waste produced during the n-th day be $X_n = f(S_n^1)$ tons, where

(20) $$S_n^1 = bS_{n-1}^1 + \sqrt{(1 - b^2)}\, D_n\,, \quad n = 1, \ldots, N\,.$$

b is a constant, $|b| < 1$, and D_n, $n = 1, \ldots N$, are mutually independent standard normal random variables. Sequence $\{S_n^1, n = 1, \ldots, N\}$ is assumed to be stationary. Hence, $\mathsf{E}\, S_n^1 = 0$, $\mathsf{E}\,(S_n^1)^2 = 1$. The nonprocessed waste is to be stored. Let Y_n denote the quantity in storage at the end of n-th day. We have then

$$Y_n = k(Y_{n-1} + f(S_n^1) - q)\,, \quad n = 1, \ldots, N\,.$$

Replacing $Y_{n-1} + f(S_n^1) - q$ by the equivalent Gaussian variable S_n^2 we arrive at the relation

(21) $$S_n^2 \sim k(S_{n-1}^2) + f(bS_{n-1}^2 + \sqrt{(1 - b^2)}\, D_n) - q\,, \quad n = 1, \ldots, N\,.$$

The parameters of the distribution of $S_n = (S_n^1, S_n^2)$ are

$$m_n = \begin{bmatrix} 0 \\ m_n^2 \end{bmatrix}, \quad M_n = \begin{bmatrix} 1, & m_n^{12} \\ m_n^{12}, & m_n^{22} \end{bmatrix}, \quad n = 1, \ldots, N\,.$$

Introduce the function

$$\Psi(x, y) = x\, \Phi(x/\sqrt{y}) + \sqrt{(y)}\, \varphi(x/\sqrt{y})\,,$$

and abbreviated denotations

$$\Phi_n = \Phi(m_n^2/\sqrt{m_n^{22}})\,, \quad \Psi_n = \Psi(m_n^2, m_n^{22})\,.$$

From (20), (21) using the formulae of Section 3 one obtains the following recurrent relations for the parameters:

$$m_n^2 = \Psi_{n-1} + \sqrt{(e)} - q\,,$$

$$m_n^{22} = m_{n-1}^2 \Psi_{n-1} + m_{n-1}^{22} \Phi_{n-1} + 2\sqrt{(e)}\,\Psi(m_{n-1}^2 + bm_{n-1}^{12}, m_{n-1}^{22}) + $$
$$+ e^2 - (\Psi_{n-1} + \sqrt{e})^2\,,$$

$$m_n^{12} = bm_{n-1}^{12} \Phi_{n-1} + \sqrt{e}\,, \quad n = 1, \ldots, N\,.$$

The covariance matrix for adjacent days is

$$M_{n-1n} = \begin{bmatrix} b, & m_{n-1}^{12} \Phi_{n-1} + b\sqrt{e} \\ bm_{n-1}^{12}, & m_{n-1}^{22} \Phi_{n-1} + bm_{n-1}^{12}\sqrt{e} \end{bmatrix}.$$

Let $b = 0, 6.q = 2$, and let the initial quantity of stored waste be $Y_0 = 1$. Autoregressive relation (18) for $\{S_n, n = 2, \ldots, 5\}$ is then

(22) $$S_n^1 = 0.6 S_{n-1}^1 + 0.8 D_n^1\,,$$

$$S_n^2 = a_n^2 + 0.989 S_{n-1}^1 + b_n^{22} S_{n-1}^2 + 1.319 D_n^1 + c_n^{22} D_n^2\,.$$

The parameters of S_1 and the coefficients in (22) have the following numerical values:

$$m_1^2 = 0{\cdot}649\,, \quad m_1^{22} = 4{\cdot}671\,, \quad m_1^{12} = 0{\cdot}649\,;$$

n	2	3	4	5
a_n^2	0·473	0·701	0·876	0·968
b_n^{22}	0·618	0·624	0·647	0·677
c_n^{22}	1·547	1·696	1·786	1·831

(Received August 19, 1980.)

REFERENCES

[1] Z. Kos: Ekonomická výše odběrů pro závlahy ve vodohospodářských soustavách (Economical Size of Irrigation Draws in Water Resource Systems). Thesis. ČVUT, Praha 1968.

[2] Z. Kos, V. Zeman: Vodohospodářské soustavy ve Směrném vodohospodářském plánu (Water Resource Systems in General Water Resources Plan). Státní zemědělské nakladatelství, Praha 1976.

[3] P. Mandl: Water storage models based on transformed normal distribution. Ekonomicko-matematický obzor *15* (1979), 455—467.

[4] M. Schramm: Zur mathematischen Darstellung und Simulation des natürlichen Durchflussprozesses. Acta Hydrophysica *19* (1975), 77—191.

RNDr. Petr Mandl, DrSc., matematicko-fyzikální fakulta Univerzity Karlovy (Faculty of Mathematics and Physics — Charles University), Sokolovská 83, 186 00 Praha 8. Czechoslovakia.