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ON THE "DICKE-FIX" DETECTION OF FINITE BINARY PHASE KEYED SEQUENCES

LUDVÍK PROUZA

A formula for the detection probability of binary sequences in presence of ideal limiting is investigated.

1. INTRODUCTION

Let a finite binary phase keyed sequence corrupted by noise be received. Let the phase of the received signal with respect to a reference be arbitrary. Then the quadrature channel reception is natural ([3], 20–21).

Let x_i $(i = 1, \ldots, n)$ denote the cosinus ("real" quadrature channel) and y_i the sinus ("imaginary" quadrature channel) components (both signal plus noise) of the sequence terms.

Without loss of generality, one may suppose that the terms of the pure signal sequence lie on the real axis (i.e. their phase with respect to the reference is 0 or π), thus their magnitudes are

$$Q_i = \pm Q \,, \quad Q > 0 \,, \quad (i = 1, \ldots, n) \,. \tag{1}$$

The x_i, y_i noise components of the received signal are supposed white, mutually independent Gaussian $N(0, 1)$ according to the Rice model of the noise.

Before being splitted in both quadrature channels with the aid of phase detectors (supposed ideal, i.e. producing the "real" and "imaginary" components of the signal plus noise vector), the signal is "ideally limited", i.e. transformed to the absolute magnitude 1 and unchanged phase, so that

$$\xi_i = \cos \varphi_i \,, \quad \eta_i = \sin \varphi_i \,, \quad (i = 1, \ldots, n) \tag{2}$$

are inputs of the respective "real" and "imaginary" quadrature channels (after the phase detectors), φ_i being the argument (random phase angle with respect to the reference) of the i-th term of the received sequence.

In each channel a matched filter is inserted with the weighting sequence

$$\begin{aligned} a_{n-i} &= 1 \quad \text{for} \quad Q_i > 0\,, \\ &= -1 \quad \text{for} \quad Q_i < 0\,. \end{aligned} \tag{3}$$

On the matched filter outputs, one gets resp. $\sum_{i=1}^{n} \xi_i$, $\sum_{i=1}^{n} \eta_i$. Then, a sum of squares

$$\left(\sum_{i=1}^{n} \xi_i\right)^2 + \left(\sum_{i=1}^{n} \eta_i\right)^2 = S \tag{4}$$

is formed and the sequence is detected if $S > Z$, Z being a threshold.

From (1), (2), (3), (4) there is clear that the described detection method is a modification of the known Dicke-Fix detection [1].

2. AN APPROXIMATE FORMULA FOR THE DETECTION PROBABILITY

Following the description and supposition in the Introduction, the probability distribution of the random phase angle $\varphi = \varphi_i$ $(i = 1, \ldots, n)$ is almost immediately seen to be

$$P(\varphi \in \langle 0, \Phi \rangle) = \frac{1}{\sqrt{(2\pi)}} \int_0^\infty e^{-(x-Q)^2/2} \left(\frac{1}{\sqrt{(2\pi)}} \int_0^{x \operatorname{tg} \Phi} e^{-y^2/2} \, dy \right) dx \tag{5}$$

for Φ from the first quadrant, with obvious modifications for the remaining quadrants. No attempt has been made to simplify this expression, although it would be certainly possible [4].

Instead, the distribution of the "real" component $\xi = \xi_i$ $(i = 1, \ldots, n)$ has been derived from (5). After some computational labor, one obtains an exact probability density formula for $\xi = \xi_i$ with $Q_i = \pm Q$, $Q > 0$ and $\sigma_{x_i} = \sigma_{y_i} = 1$ $(i = 1, \ldots, n)$. There holds for $\xi \in (-1, 1)$

$$f(\xi) = \frac{1}{\pi} \cdot \frac{1}{\sqrt{(1 - \xi^2)}} \cdot e^{-Q^2/2} \int_0^\infty e^{-\frac{1}{2}x(x - 2Q\xi)} \, . \, x \, dx \, . \tag{6}$$

Ones sees that at both ends of the interval $(-1, 1)$, the density grows as resp. $1/\sqrt{(1 + \xi)}$, $1/\sqrt{(1 - \xi)}$.

For $Q = 0$, that is for the noise alone, it follows from (6)

$$f(\xi) = \frac{1}{\pi} \cdot \frac{1}{\sqrt{(1 - \xi^2)}} \, . \tag{7}$$

The respective moments of the noise are

$$\text{(8)} \qquad \mu_{1\xi} = 0\,, \quad \mu_{2\xi} = \tfrac{1}{2}$$
$$\mu_{1\eta} = 0\,, \quad \mu_{2\eta} = \tfrac{1}{2}\,.$$

Generally,

$$\text{(9)} \qquad \mu_{1\eta} = 0\,,$$

$$\text{(10)} \qquad \mu_{2\eta} = \int_{-1}^{1} (1 - \xi^2) f(\xi)\, \mathrm{d}\xi = 1 - \mu_{2\xi}\,.$$

The variable η depends on ξ so that

$$\text{(11)} \qquad \begin{aligned} P(\eta \mid \xi) &= \tfrac{1}{2} \quad \text{for} \quad \eta = +\sqrt{(1 - \xi^2)}\,, \\ &= 0 \quad \text{for} \quad \eta \neq \sqrt{(1 - \xi^2)}\,, \\ &= \tfrac{1}{2} \quad \text{for} \quad \eta = -\sqrt{(1 - \xi^2)}\,. \end{aligned}$$

From (11), there follows

$$\text{(12)} \qquad \mathrm{E}(\xi\eta) = 0$$

(E denotes the expectation). Thus ξ_i, η_i with the same subscripts are not correlated. Further

$$\text{(13)} \qquad \mathrm{E}\Big(\sum_{i=1}^{n} \xi_i \,.\, \sum_{i=1}^{n} \eta_i\Big) =$$
$$= \mathrm{E}\Big(\sum_{i=1}^{n} \xi_i \eta_i\Big) + \mathrm{E}\Big(\xi_1 \sum_{i \neq 1} \eta_i + \ldots + \xi_n \sum_{i \neq n} \eta_i\Big) = 0\,.$$

The first term on the right is 0, as follows from (12). For the second one, one knows that ξ_i, ξ_j with $i \neq j$ are independent, thus also ξ_i, η_j with $i \neq j$ are independent. With (9) the second term is also 0.

According the central limit theorem, the sums in (4) are asymptotically normal for $n \to \infty$. Since they are uncorrelated for each n according to (13), there follows that they are asymptotically independent for $n \to \infty$.

Thus for sufficiently great n the first sum in (4) is approximately $N(n\mu_{1\xi}, \sqrt{(n \,.\, (\mu_{2\xi} - \mu_{1\xi}^2))})$, and the second sum in (4) is approximately $N(0, \sqrt{(n(1 - \mu_{2\xi}))})$.

There follows that the probability of the inequality in (4) is approximately

$$\text{(14)} \qquad P_{da} = 1 - \frac{1}{\sqrt{(2\pi)}} \int_a^b \exp\left(-\zeta^2/2\right) \left(1 - \frac{2}{\sqrt{(2\pi)}} \int_c^\infty \exp\left(-\eta^2/2\right) \mathrm{d}\eta\right) \mathrm{d}\zeta\,,$$

where

$$\text{(15)} \qquad a = \frac{-\sqrt{Z} - n\mu_{1\xi}}{\sqrt{(n(\mu_{2\xi} - \mu_{1\xi}^2))}}\,, \quad b = \frac{\sqrt{Z} - n\mu_{1\xi}}{\sqrt{(n(\mu_{2\xi} - \mu_{1\xi}^2))}}\,,$$
$$c = \sqrt{\left(\frac{Z - x^2}{n(1 - \mu_{2\xi})}\right)}\,, \quad x = \zeta \sqrt{(n(\mu_{2\xi} - \mu_{1\xi}^2))} + n\mu_{1\xi}\,.$$

For $Q = 0$, there follows from (14), (8)

$$P_{fa} = 1 - P_{\chi_2^2}\left(\psi < \frac{2Z}{n}\right), \tag{16}$$

where ψ is distributed as χ^2 with 2 degrees of freedom.

Thus, the threshold Z can be determined from (16) given P_{fa}, and then P_{da} with this Z is computed from (14), (15).

One sees that only $\mu_{1\xi}$, $\mu_{2\xi}$ of the distribution (6) are needed in (14), (15).

From [2], one gets after some computing, remembering the definitions of the Bessel functions and denoting $(Q/2)^2 = \beta$

$$\mu_{1\xi} = \sqrt{\frac{\pi\beta}{2}} \,.\, e^{-\beta} \,.\, (J_0(\beta) + J_1(\beta)), \tag{17}$$

$$\mu_{2\xi} = \tfrac{1}{2}(1 + e^{-\beta}((1 + 1/\beta)\sin\beta - \cos\beta)). \tag{18}$$

There is also clear that

$$Q^2/2 = S/N \tag{19}$$

is the signal/noise ratio in a single term of the input sequence before limiting.

3. VERIFICATION BY MONTE-CARLO SIMULATION

Since (14) holds asymptotically for $n \to \infty$, its usefulness for moderate n is not clear at the first sight. Therefore it has been checked by Monte Carlo simulations outgoing from the exact distribution (6) and using (4). 2000 samples have been made for $n = 13$, and the observations of Z have been grouped in intervals of the width 10.

The simulations have been executed in the statistical laboratory of the Institute of Information and Automation Theory of the Czechoslovak Academy of Sciences.

Here, only a short extract of the results will be given in the following Table:

$P_{fa} = 0{\cdot}01$						
n	13		26		52	
Z	60		120		240	
Q	1	0·5	1	0·5	1	0·5
P_{da} (14)	0·43	0·09	0·89	0·24	0·99	0·56
P_{da} (simul.)	0·48	0·07	0·91	0·21	1·00	0·55

There follows that the formula (14) can be used for moderate n.

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RNDr. Ludvík Prouza, DrSc., Tesla — Ústav pro výzkum radiotechniky (Institute of Radioengineering), Opočínek, 533 31 p. Lány na Důlku. Czechoslovakia.