KYBERNETIKA — VOLUME 17 (1981), NUMBER 1

ON THE SOLUTION OF OPTIMAL CONTROL PROBLEMS INVOLVING PARAMETERS AND GENERAL BOUNDARY CONDITIONS

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The paper deals with a special class of optimal control problems which involve parameters and, moreover, general (mixed) boundary conditions. For this class of problems necessary optimality conditions are presented. Two numerical approaches are explored in detail. The quasilinearization and gradient algorithms are briefly described and compared on several examples.

1. INTRODUCTION

The quite usual and often assumption in the theory of optimal control is the fixed or given initial state. Such assumption enables, at least on the primary stages, to simplify certain results. Anyhow, there exists a number of problems for which this assumption is not met. Let us mention the class of periodic problems having the direct dependence between initial and final states. Other examples of this type can be found in various areas of mechanical and chemical engineering. So it is quite reasonable to study such class of optimization problems more in detail.

Thus our formulation admits a general dependence of initial and final states (mixed boundary conditions) and also the possible parameters. From the last fact it follows that, e.g. also the case with free initial and/or final time is included. Control or parameter constraints can be alternatively present, being incorporated using the projection.

One of the first who explored optimization problems with general boundary conditions was Trojckij [1]. He used the calculus of variations to obtain necessary optimality conditions for the problem considered. Also in the book of Fan and Wang [2] the discrete analogues of such problems can be found. As it is necessary to expect the necessary optimality conditions form a nonlinear two-point boundary-value problem, for the solution of which numerical iterative methods must be applied. For the case of free initial state only, such attempt was done by Gonzales and Miele [3–5] using the sequential gradient-restoration algorithm. The use of this algorithm also for problems with general boundary conditions was demonstrated by Fidler and Doležal in [6].

In this contribution two other alternatives are described, however, with general boundary conditions given as a set of equations. For this purpose the necessary optimality conditions are derived simply applying the results of Doležal and Černý [7] which concern classical optimal control problems involving parameters. First, based directly on the necessary optimality conditions, the use of the modified quasi-linearization method (indirect approach), e.g. see [8–10], is mentioned. Then a first-order gradient algorithm is described using the obvious analogy with [7]. Several examples are presented to illustrate and to compare both methods.

2. STATEMENT OF THE PROBLEM

In this section let us formulate an optimal control problem with parameters and general boundary conditions. Matrix notation will be used for the sake of simplicity. In this connection it is assumed that all considered vectors are column-vectors except of gradients of various functions, which are always treated as row-vectors. Further defined functions φ, L, f, ψ are assumed to be at least continuously differentiable. As E^n will be denoted an n-dimensional Euclidean space.

The aim is to minimize the cost functional

$$J = \varphi(x(0), x(1), \pi) + \int_0^1 L(x, u, \pi, t)\,\mathrm{d}t \tag{1}$$

with respect to the state $x(t)$, the control $u(t)$, $0 \leqq t \leqq 1$, and the parameter π, which are subject to the differential constraints

$$\dot{x} = f(x, u, \pi, t)\,, \quad 0 \leqq t \leqq 1\,, \tag{2}$$

the boundary conditions of the mixed type

$$\psi(x(0), x(1), \pi) = 0\,, \tag{3}$$

and the control and parameter constraints

$$u(t) \in U \subset E^m\,, \quad 0 \leqq t \leqq 1\,, \tag{4}$$

$$\pi \in P \subset E^v\,. \tag{5}$$

In these equations $x(t) \in E^n$, $u(t) \in E^m$ and $\pi \in E^v$. Various functions are defined as follows:

$$f : E^n \times E^m \times E^v \times E^1 \to E^n\,, \quad \psi : E^n \times E^n \times E^v \to E^q\,,$$

$$L: E^n \times E^m \times E^v \times E^1 \to E^1\,, \quad \varphi : E^n \times E^n \times E^v \to E^1\,.$$

Obviously the condition $q \leqq 2n + v$ must hold to have the problem well-posed. The sets U and P are assumed closed and convex.

Similarly as in [3–5] the normalized time interval [0, 1] is used. To be more specific, if the actual time $t' \in [t_0, t_f]$, then the new time scale is introduced according to the formula

$$t = \frac{t' - t_0}{t_f - t_0}, \tag{6}$$

which implies $t \in [0, 1]$. The actual initial and final time, if being free, are regarded as the additional components of the parameter π to be optimized.

This formulation includes that of [3–5], where only the separated dependence on initial and final state was assumed in (1) and (3). To avoid notational inconvenience the terms depending only on initial and/or final state are not explicitely assumed. Moreover, the formulation presented covers such a case. It would be also possible, in principal, to treat the explicitely given components of the initial state separately throughout the paper – see [3–5]. However, such an obvious manipulation can be always performed when dealing with a concrete example as shown later in this paper.

3. NECESSARY OPTIMALITY CONDITIONS

To derive necessary optimality conditions for the above stated problem let us mention two possibilities. In general, one can apply the calculus of variation to this problem analogously as in [11] or [12]. However, having in mind the existing results for optimal control problems with parameter and involving only terminal constraints, e.g. see [7], the more direct approach can be followed. In fact, also the conditions given in [3–5] can be used in this connection.

Namely, the unknown initial state, required to be given in [7], can be considered as an additional n-dimensional parameter. Then the new state variable

$$y(t) = x(t) - x(0), \quad 0 \leqq t \leqq 1, \tag{7}$$

is formally introduced implying $y(0) = 0$. Thus in the place of x always $y + x(0)$ will appear through (1)–(3), and π and $x(0)$ form now a new $(v + n)$-dimensional vector of parameters to be optimized. To such a problem the mentioned results apply.

This idea is not new, it was used to optimal control problems with free initial state in [13] or to solve similar problems for various types of differential games [14, 15]. Such interpretation of the free initial state suggests also a possible inclusion of initial state constraints of type (5) in a direct way. This remark will be more clear from further considerations.

It has no sense to bother the reader with a straightforward application of necessary optimality conditions to the transcribed problem. Only realize that

$$\frac{\mathrm{d}}{\mathrm{d}x(0)}\,\varphi(x(0), y(1) + x(0), \pi) = \varphi_{x(0)} + \varphi_{x(1)}\,, \tag{8}$$

and similarly for function ψ. Arguments on the right-hand side are omitted for brevity, lower index denotes here henceforth the corresponding partial derivative and T, as usual, will stand for the transposition.

If we omit the magnitude constraints (4)–(5), the necessary optimality conditions imply the existence of $\lambda(t) \in E^n$, $0 \leqq t \leqq 1$, and $v \in E^q$ such that

$$\dot{\lambda} = -H_x^T\,, \quad 0 \leqq t \leqq 1\,, \tag{9}$$

$$-\lambda(1) + \varphi_{x(1)}^T + \psi_{x(1)}^T v = 0\,, \tag{10}$$

$$H_u^T = 0\,, \quad 0 \leqq t \leqq 1\,, \tag{11}$$

$$\varphi_\pi^T + \psi_\pi^T v + \int_0^1 H_\pi^T\,\mathrm{d}t = 0\,, \tag{12}$$

$$\varphi_{x(0)}^T + \varphi_{x(1)}^T + \left(\psi_{x(0)}^T + \psi_{x(1)}^T\right) v + \int_0^1 H_x^T\,\mathrm{d}t = 0\,, \tag{13}$$

where

$$H(x, u, \pi, \lambda, t) = L(x, u, \pi, t) + \lambda^T f(x, u, \pi, t)\,, \quad 0 \leqq t \leqq 1\,. \tag{14}$$

On combining (9), (10), and (13) one obtains that

$$\lambda(0) + \varphi_{x(0)}^T + \psi_{x(0)}^T v = 0\,, \tag{15}$$

what is the remaining transversality condition. Thus (9)–(12) and (15) are the desired optimality conditions for problem (1)–(3).

If the constraints (4)–(5) are present, then (11) and (12) take the form (by star denoted an optimal solution)

$$\left\{\begin{aligned} &H(x^*, u^*, \pi^*, \lambda, t) = \min_{u \in U} H(x^*, u, \pi^*, \lambda, t)\,, \quad 0 \leqq t \leqq 1\,,\\ &\left(\varphi_\pi + v^T \psi_\pi + \int_0^1 H_\pi\,\mathrm{d}t\right) \delta\pi \geqq 0 \end{aligned}\right. \tag{16}$$

for any feasible parameter change, i.e. $\pi^* + \delta\pi \in P$ – see [16].

As one can see, the necessary optimality conditions (9)–(12) and (15) form a nonlinear two-point boundary-value problem. To solve it, approximate iterative methods must be applied, e.g. consult [8–10, 17]. This numerical approach, based on necessary optimality conditions, is usually denoted as an indirect one in comparison with the so-called direct methods on the minimization of the cost functional (1). Such methods are mostly of the gradient type.

4. FIRST-ORDER GRADIENT ALGORITHM

The previous discussion enables also to formulate a first-order gradient algorithm for the studied problem. Also now the results known for the case with parameters only [7], yield such an algorithm in a simple way. A possible way of its derivation is sketched in [18]. If the constraints (4) and (5) are present the idea of the so-called projection is applied. This means that for $\gamma_0 \in E^n$ and $Q \subset E^n$ we define

$$\text{proj}\,[\gamma \mid Q] = \arg\min\,[\|\gamma - \gamma_0\| \mid \gamma \in Q]\,, \tag{17}$$

i.e. under the projection of the point γ_0 we understand its nearest point $\tilde{\gamma} \in Q$. For a brief discussion of this approach see [18].

Therefore the details of derivation of the algorithm are omitted. The resulting algorithm then consists of the following steps.

STEP 1. Select a nominal feasible solution estimate, i.e. a function $u(t)$, $0 \leqq t \leqq 1$, and a parameter π satisfying (4)–(5), and also an initial state $x(0)$.

STEP 2. Using this estimate integrate the system equations (2) in the sense of the increasing time (forward run). Record the time-histories $x(t)$ and $u(t)$ and the corresponding values $\varphi_{x(0)}$, $\varphi_{x(1)}$, φ_π, $\psi_{x(0)}$, $\psi_{x(1)}$ and ψ_π.

STEP 3. Integrating in the sense of the decreasing time (backward run) determine the n-dimensional function $p(t)$ and $(n \times q)$-dimensional function $R(t)$, $0 \leqq t \leqq 1$, according to the formulas

$$\dot{p} = -f_x^T p - L_x^T\,, \qquad p(1) = \varphi_{x(1)}^T\,, \tag{18}$$

$$\dot{R} = -f_x^T R\,, \qquad R(1) = \psi_{x(1)}^T\,. \tag{19}$$

STEP 4. Compute the following expressions (dimensions are obvious from the above considerations):

$$\left\{\begin{aligned} I_{\psi\psi} &= \int_0^1 R^T f_u W_u f_u^T R \,\mathrm{d}t\,, \quad & I_{J\psi} &= \int_0^1 (p^T f_u + L_u)\, W_u f_u^T R \,\mathrm{d}t\,, \\ I_{\psi\pi} &= \int_0^1 R^T f_\pi \,\mathrm{d}t\,, & I_{J\pi} &= \int_0^1 (p^T f_\pi + L_\pi)\,\mathrm{d}t\,, \end{aligned}\right. \tag{20}$$

where $W_u(t)$, $0 \leqq t \leqq 1$, is a positive definite $(m \times m)$-dimensional matrix (stepsize).

STEP 5. Select $\delta\psi$ to achieve the better satisfaction of the constraints, e.g. put

$$\delta\psi = -\varepsilon\psi(x(0), x(1), \pi)\,, \quad 0 \leqq \varepsilon \leqq 1\,. \tag{21}$$

Then compute the q-vector

$$v = -A_{\psi\psi}^{-1}(\delta\psi + A_{J\psi}^T) \tag{22}$$

assuming that the indicated inversion exists, and using the notation

$$(23)\begin{cases} A_{\psi\psi} = I_{\psi\psi} + (\psi_\pi + I_{\psi\pi})\, W_\pi(\psi_\pi + I_{\psi\pi})^T + (\psi_{x(0)} + R^T(0))\, W_x(\psi^T_{x(0)} + R(0))\,, \\ A_{J\psi} = I_{J\psi} + (\varphi_\pi + I_{J\pi})\, W_\pi(\psi_\pi + I_{\psi\pi})^T + (\varphi_{x(0)} + p^T(0))\, W_x(\psi^T_{x(0)} + R(0))\,. \end{cases}$$

Here W_π and W_x are $(v \times v)$-dimensional and $(n \times n)$-dimensional positive definite matrices, respectively.

STEP 6. Update the existing solution estimates $u(t)$, π and $x(0)$ by adding the corrections

$$(24)\quad \begin{cases} \delta u(t) = -W_u[L_u + (p + Rv)^T f_u]^T\,, \quad 0 \leqq t \leqq 1\,, \\ \delta\pi \quad = -W_\pi[\varphi_\pi + I_{J\pi} + v^T(\psi_\pi + I_{\psi\pi})]^T\,, \\ \delta x(0) = -W_x[\varphi_{x(0)} + p(0) + v^T(\psi_{x(0)} + R(0))]^T\,, \end{cases}$$

and check, if the new solution estimates satisfy the constraints (4)–(5). This not being the case, perform the projection as indicated by the relation (17).

STEP 7. Using the projected values compute the feasible changes $\delta\bar{u}(t)$, $0 \leqq t \leqq 1$, and $\delta\bar{\pi}$, and evaluate the expression

$$(25)\qquad \mathscr{E} = \delta x^T(0) W_x^{-1}\, \delta x(0) + \delta\bar{\pi}^T W_\pi^{-1}\, \delta\bar{\pi} + \int_0^1 \delta\bar{u}(t)\, W_u^{-1}(t)\, \delta\bar{u}(t)\, \mathrm{d}t\,.$$

If $\mathscr{E} < \varkappa$ and $|\psi| < \delta$, where $\varkappa$ and δ are the permitted errors in optimality conditions and boundary constraints, respectively, then stop the computations; else go to Step 2.

The same remarks as in [18] apply also here as the choice of various weighting matrices and stopping conditions is concerned.

5. ILLUSTRATIVE EXAMPLES

Let us illustrate the practical importance of the presented algorithm when solving several optimal control problems. Only the case of initial or mixed boundary conditions is studied. Parameter optimization is treated in [7] and will not be considered here. All examples were solved with the aid of the SIMFOR simulation program [19] in the connection with EAI PACER 600 hybrid system (digital part). This program enables a simple realization of various numerical algorithms suggested for optimization problems.

Through this section all variables let be scalars. As stopping criterion the values $\varkappa = 10^{-10}$ and $\delta = 10^{-6}$ for $\varepsilon = 1$ (if necessary) are used. Figures are the direct prints of the computer display using the Hard Copy Unit. All integrations were performed using the 3rd order variable-step Runge-Kutta method with the overall permitted error $e_{max} = 10^{-4}$. Definite integrals were evaluated using the Simpson's rule. The stepsize $W_u(t)$ is chosen constant for $0 \leqq t \leqq 1$. The nominal estimates were always $x_1(0) = 0$, $u(t) = 0$, $0 \leqq t \leqq 1$.

The most important data for all examples are collected in Table 1. The meaning of all symbols is obvious, only N stands for the number of iterations needed for the convergence and J^* denotes the optimal costs. Also some computational details are included and a brief comparison with the quasilinearization method is performed for which the value of a quadratic change less then 10^{-20} is applied to stop the computations. The quasilinearization algorithm was realized in PL/1 on IBM 370/135 computer.

Example 1. Consider the system dynamics representing the Van der Pol oscillator

(26) $$\dot{x}_1 = x_2 ,$$
$$\dot{x}_2 = -x_1 + u + (1 - x_1^2)\, x_2 ,$$

subject to the boundary (initial) condition

(27) $$x_2(0) = 1 ,$$

and the cost functional

(28) $$J = \tfrac{1}{2} \int_0^1 (x_1^2 + x_2^2 + u^2)\, \mathrm{d}t .$$

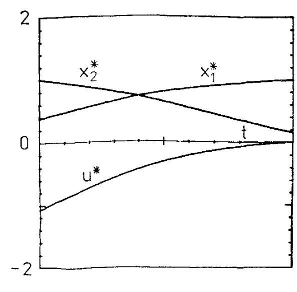

Fig 1. Optimal solution of Example 1.

The results are given in Table 1 and in Fig. 1. The fact that $x_2(0)$ is given is used to avoid unnecessary computations. One can easily see that to compute optimal $x_1(0)$ the correction $\delta x_1(0) = -W_x\, p_1(0)$ is used – see (24).

Applying the quasilinearization approach the following two-point boundary-value problem is to be solved:

$$(29)\quad \begin{cases} \dot{x}_1 = x_2 , \\ \dot{x}_2 = -x_1 - \lambda_2 + (1 - x_1^2)\, x_2 , \\ \dot{\lambda}_1 = -x_1 + 2x_1x_2\lambda_2 + \lambda_2 , \\ \dot{\lambda}_2 = -x_2 - \lambda_1 - (1 - x_1^2)\, \lambda_2 , \end{cases}$$

subject to the boundary conditions

$$(30)\qquad x_2(0) = 1 , \quad \lambda_1(0) = 0 , \quad \lambda_1(1) = 0 , \quad \lambda_2(1) = 0 .$$

Table 1. Summary of the results.

	J^*	$x_1^*(0)$	v	N	W_u	W_x
Example 1	0·65062	0·37920	—	16	0·3	0·3
Example 2	0·66936	0·43414	—	14	0·5	0·5
Example 3	0·77931	0·78427	−0·43439	15	0·5	0·5
Example 4	1·0187	0·94889	−1·3283	10	1·0	0·7

Using zero nominal estimate the solution was obtained in 5 iterations yielding $J^* = 0{\cdot}65601$ and $x_1^*(0) = 0{\cdot}39182$, being thus in a good agreement with the above results.

Example 2. If now additionaly the constraint

$$(31)\qquad |u(t)| \leqq 0{\cdot}5 , \quad 0 \leqq t \leqq 1 ,$$

are assumed, the solution can be obtained using the gradient algorithm without any

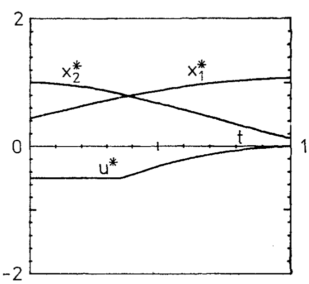

Fig. 2. Optimal solution of Example 2.

troubles – see Table 1 and Fig. 2. However, the quasilinearization method cannot be applied in a straightforward way to take the account of (31).

Example 3. Assume the same problem as in Example 1 with the additional terminal condition

(32) $$x_1(1) - x_2(1) - 1{\cdot}5 = 0 .$$

The results obtained by the gradient algorithm are given in Table 1 and Fig. 3.

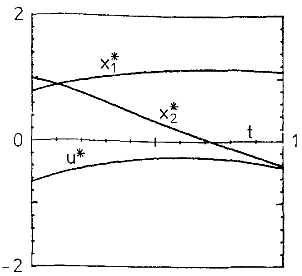

Fig. 3. Optimal solution of Example 3.

Again only $x_1(0)$ is corrected using the given value $x_2(0) = 1$. The quasilinearization method requires to solve (29) subject to the boundary conditions

(33) $$x_2(0) = 1 , \quad \lambda_1(0) = 0 , \quad x_1(1) - x_2(1) - 1{\cdot}5 = 0 , \quad \lambda_1(1) + \lambda_2(1) = 0 .$$

The solution was achieved in 4 iterations with $J^* = 0{\cdot}78123$ and $x_1^*(0) = 0{\cdot}77997$.

Example 4. For the system (26) and cost functional (28) consider the boundary conditions of the general (mixed) type

(34) $$x_1(0) = x_1(1) = 0 , \quad x_2(0) = 1 .$$

The solution is described in Table 1 and depicted in Fig. 4. Also now the given value of $x_2(0)$ is used during the computations. To apply the quasilinearization algorithm the boundary conditions take the form

(35) $$x_1(0) - x_1(1) = 0 , \quad x_2(0) = 1 , \quad \lambda_1(0) - \lambda_1(0) = 0 , \quad \lambda_2(1) = 0 .$$

The optimal solution was reached in 5 iterations yielding $J^* = 1{\cdot}0368$ and $x_1^*(0) = 0{\cdot}95882$.

In all cases various initial estimates were tested with the same results. The ~mall differences in the results are due to various integration methods applied in gradient algorithm and quasilinearization. In the latter case only a simple Euler method was used to suit the principally discrete schema in [9]. As a rule, the optimal costs

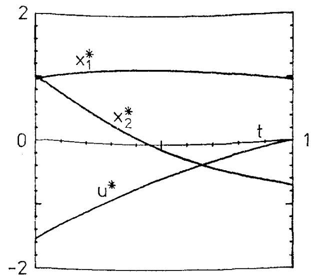

Fig. 4. Optimal solution of Example 4.

J^* were obtained in a smaller number of iterations than needed for convergence. This fact indicates rather flat minima in the studied cases. Similar results were also achieved for penalizing only the terminal state in (28), i.e. taking

$$J = \tfrac{1}{2}(x_1^2(1) + x_2^2(1)) + \int_0^1 u^2 \, dt \,. \tag{36}$$

If control constraints analogous as in Example 2 are assumed in last two examples having constraints on the final state, then certain troubles must to be expected due to the „hard" control constraints. Namely, it is no more possible to satisfy the final or general constraints with a negligible error and the stopping condition (constant δ) have to be increased. The value of $\delta = 10^{-2}$ has shown to be acceptable. Such approximate solution cannot be simply obtained through the quasilinearization algorithm which, on the other hand, exhibits very good convergence properties in cases without control constraints.

6. CONCLUSIONS

A first-order gradient algorithm was described to solve optimal control problems with parameters and general boundary conditions. It was demonstrated that the unknown initial state can be treated like a parameter. Then it was rather simple matter to derive necessary optimality conditions and the gradient algorithm.

Several examples were solved to illustrate practical importance of the studied subject. A comparison with a quasilinearization method (second-order method) was also performed.

(Received January 25, 1980.)

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