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DISCRETE LINEAR REGULATOR REVISITED

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A new technique to calculate discrete-time linear regulators is presented. This technique is based upon transfer matrix considerations rather than employing the algebraic Riccati equation. The optimal regulator is obtained via spectral factorization and the solution of a simple equation in polynomial matrices.

This approach provides further insight and ties together the state-space and transfer-matrix methods. The resulting algorithm seems to be computationally effective.

INTRODUCTION

The discrete-time linear regulator has obtained much attention in the control literature. In fact, this problem has become standard and indispensable part of modern control theory and has found many practical applications.

The discrete-time, infinite-horizon, time-invariant, linear regulator problem can be posed as follows. Consider a reachable system

$$\begin{aligned} x_{t+1} &= Fx_t + Gu_t \\ y_t &= Hx_t + Ju_t \end{aligned} \tag{1}$$

together with the cost function

$$V = \sum_{t=0}^{\infty} y_t^T y_t \tag{2}$$

where $y_t \in R^l$, $u_t \in R^m$, $x_t \in R^n$ and F, G, H, and J are real matrices of compatible dimensions. The ^{T} denotes matrix transposition and $t = 0, 1, \ldots$ is the discrete time.

The y_t may be interpreted as an artificially defined output which serves to express a general nonnegative cost

$$V = \sum_{t=0}^{\infty} [x_t^T\ u_t^T] \begin{bmatrix} H^T H & H^T J \\ J^T H & J^T J \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}$$

in the form (2). This formulation allows for arbitrary cross-terms and singular weighting matrices in the cost function. The standard nonsingular case can be recovered by taking H and J in the partitioned form

$$H = \begin{bmatrix} \bar{H} \\ 0 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 \\ \bar{J} \end{bmatrix}$$

with $\bar{J}^T \bar{J}$ nonsingular.

The task is to find a regulator, relating u_t to x_t, which makes the closed loop system asymptotically stable and minimizes V for every x_0. It is well known, see Kalman [1] and particularly Silverman [7], that this regulator is linear and given by

$$u_t = -Kx_t \tag{3}$$

where

$$K = (G^T P G + J^T J)^{-1} (G^T P F + J^T H) \tag{4}$$

and P is the nonnegative definite solution of the algebraic Riccati equation

$$\begin{aligned} P = {} & F^T P F + H^T H - \\ & - (F^T P G + H^T J)(G^T P G + J^T J)^{-1} (G^T P F + J^T H) \end{aligned} \tag{5}$$

for which $F - GK$ has all eigenvalues in magnitude less than 1.

As shown by Silverman [7], the matrix $G^T P G + J^T J$ is invertible if and only if the system (1) is left invertible. If this is not the case, the pseudoinverse operation must replace the ordinary inverse in (4) and (5) and the optimal regulator K is no longer unique.

Ever since Kalman's original paper on the subject, the importance of the Riccati equations in linear regulator problems has been widely emphasized. In a recent paper [4], however, the author developed an alternative technique of solving the linear regulator problem for continuous-time systems. This technique is based upon transfer matrix considerations and makes use of polynomial algebra. Throughout this decade we have witnessed the growing presence of algebra and the comeback of transfer matrix methods in system and control theory. They have proved useful in providing further insight, see Rosenbrock [5], as well as in obtaining efficient design procedures, see Kučera [3]. Our aim here is to apply this new technique to solve the discrete-time linear regulator problem. The basic idea is to use suitable matrix fractions to describe the system; the calculation of the optimal regulator is then reduced to spectral factorization and the solution of a simple equation in polynomial matrices.

PRELIMINARIES

Let p be a real polynomial in an indeterminate d. By a root of p we mean any (possibly complex) number α such that $d - \alpha$ is a divisor of p.

Now let P be a real polynomial matrix in d,

$$P = P_0 + P_1 d + \dots + P_k d^k .$$

If $P_k \neq 0$ then k is the degree of P, denoted by deg P. The P is said to be *causal* if it is square and its determinant has no zero root (that is, P_0 is invertible). Further we define P to be *regular* if it is square and its determinant has no root in magnitude equal to 1, to be *Hurwitz* if it is square and its determinant has no root with magnitude less than 1, and to be *stable* if it is square and its determinant has no root whose magnitude is less than or equal to 1. Finally P is *unimodular* if it is square and its determinant has no root at all (that is, P^{-1} is again polynomial).

Two polynomial matrices P and Q having the same number of rows are said to be left coprime if every square common left divisor of theirs is unimodular, and two polynomial matrices $\bar{P}$ and $\bar{Q}$ having the same number of columns are right coprime if every square common right divisor of theirs is unimodular.

Any real rational matrix R in the indeterminate d can be written in terms of the matrix fractions

$$R = P^{-1}Q = \bar{Q}\bar{P}^{-1}$$

where P and Q are left coprime polynomial matrices in d while $\bar{P}$ and $\bar{Q}$ are right coprime polynomial matrices in d. An alternative expression for R is

$$R = R_k d^k + R_{k+1} d^{k+1} + \dots$$

where k is an integer (either negative, or positive, or zero). It is convenient to define

$$R_* = R_k^T d^{-k} + R_{k+1}^T d^{-(k+1)} + \dots$$

and to write

$$\langle R \rangle = R_0$$

for the constant term of R. Finally the symbol I_n is reserved for the $n \times n$ identity matrix.

LINEAR REGULATOR

The discrete-time linear regulator problem formulated in the Introduction will now be solved by applying transfer matrix methods. To this effect, we define the *delay* operator d by $dx_{t+1} = x_t$ for any sequence $x = (x_t)$, $t = 0, 1, \dots$. The system equations (1) and the regulator equation (3) can then be written in the form

$$(6) \qquad \begin{aligned} (I_n - dF)\,x &= dGu + x_0 \\ y &= Hx + Ju \\ u &= -Kx \end{aligned}$$

and let

$$(7) \qquad (I_n - dF)^{-1}\, dG = BA^{-1}$$

where A, B is any right coprime matrix fraction representation of $(I_n - dF)^{-1}$ dG. Clearly, A is $m \times m$ and causal whereas B is $n \times m$ with $\langle B \rangle = 0$.

Further let the $l \times m$ matrix $HB + JA$ have rank m and construct a Hurwitz polynomial matrix C satisfying

$$(8) \qquad (HB + JA)_* (HB + JA) = C_* C\,.$$

This operation is known as the spectral factorization; the spectral factor C always exists and is unique up to a constant orthogonal multiplier on the left, see Youla [9]. Note that

$$\begin{aligned} \operatorname{rank}(HB + JA) &= \operatorname{rank}(HBA^{-1} + J) \\ &= \operatorname{rank}\left[H(I_n - dF)^{-1} G + J\right]. \end{aligned}$$

Thus our rank condition is equivalent to left invertibility of system (1).

Now we are prepared to state our major result.

Theorem. Let $HB + JA$ have rank m. Then the linear regulator problem is solvable if and only if every square right divisor of $HB + JA$ is regular. The optimal regulator K, if it exists, is unique and given by

$$(9) \qquad K = X^{-1} Y$$

where X and Y is the constant solution of the equation

$$(10) \qquad XA + YB = C\,.$$

Proof. The proof will be divided into two parts. First we shall construct the optimal regulator provided it exists and then we shall discuss its existence and uniqueness.

Regarding the first part, let a regulator K exist which minimizes the cost (2) while making the closed loop system asymptotically stable. This cost can be expressed as

$$(11) \qquad V = \langle y_* y \rangle$$

and the underlying philosophy of the proof is to express V as a sum of terms with only one depending on K so that the optimum is obtained by setting this term to zero.

Define rational matrices M, N and $\overline{M}$, $\overline{N}$ by the relations

$$(12) \qquad \begin{aligned} MA + NB &= I_m \\ (I_n - dF)\,\overline{M} + dG\overline{N} &= I_n \end{aligned}$$

and

$$(13) \qquad K = M^{-1}N = \overline{N}\,\overline{M}^{-1}\,.$$

It then follows from equations (6) that

$$u = -(I_m + KBA^{-1})\,K(I_n - dF)^{-1}\,x_0$$
$$= -AN(I_n - dF)^{-1}\,x_0$$

and

$$x = -BN(I_n - dF)^{-1}\,x_0 + (I_n - dF)^{-1}\,x_0$$

so that

$$y = H(I_n - dF)^{-1}\,x_0 - (HB + JA)\,N(I_n - dF)^{-1}\,x_0\,.$$

Therefore

$$y_*y = x_0^T V_0 x_0$$

where

$$(I_n - dF)_*\,V_0(I_n - dF) = H^TH - H^T(HB + JA)\,N -$$
$$- N_*(HB + JA)_*\,H + N_*(HB + JA)_*\,(HB + JA)\,N =$$
$$= H^TH - H^T(HB + JA)\,N - N_*(HB + JA)_*\,H + N_*C_*CN$$

with C defined in (8). Completing the squares, we get

$$V_0 = V_1 + W_{1*}W_1$$

where

$$(I_n - dF)_*\,V_1(I_n - dF) =$$
$$= H^TH - H^T(HB + JA)\,C^{-1}C_*^{-1}(HB + JA)_*\,H$$

and

$$W_1 = C_*^{-1}(HB + JA)_*\,H(I_n - dF)^{-1} - CN(I_n - dF)^{-1}\,.$$

Now decompose the first term of W_1 as

$$C_*^{-1}(HB + JA)_*\,H(I_n - dF)^{-1} = C_*^{-1}Z_* + Y(I_n - dF)^{-1}$$

where Y and Z are polynomial matrices such that $\langle Z \rangle = 0$. In fact, Y is a constant matrix. Then

$$W_1 = C_*^{-1}Z_* + W$$

where

$$W = Y(I_n - dF)^{-1} - CN(I_n - dF)^{-1} =$$
$$= Y(I_n - dF)^{-1}\,[(I_n - dF)\,\overline{M} + dG\overline{N}] - CN(I_n - dF)^{-1} =$$
$$= Y\overline{M} + YBA^{-1}\overline{N} - CA^{-1}\overline{N} =$$
$$= Y\overline{M} - (C - YB)\,A^{-1}\overline{N} =$$
$$= Y\overline{M} - X\overline{N}$$

on employing (12) and (10).

The cost function (11) can now be written as

$$V = \langle x_0^T[V_1 + (C_*^{-1}Z_* + W)_* (C_*^{-1}Z_* + W)]\, x_0 \rangle \, .$$

By construction of Z, the cross-terms above vanish:

$$\langle ZC^{-1}W \rangle = \langle W_* C_*^{-1} Z_* \rangle = 0$$

and hence V can finally be given the form

$$V = \langle x_0^T (V_1 + ZC^{-1}C_*^{-1}Z_*)\, x_0 \rangle + \langle x_0^T W_* W x_0 \rangle$$

in which the first term does not depend on K. Thus V is minimized by setting $Wx_0 = 0$. Since x_0 is arbitrary, this calls for

(14) $$Y\bar{M} = X\bar{N} \, .$$

Any solution $\bar{X}$, $\bar{Y}$ of equation (10) is related to X, Y by

(15) $$[\bar{X}\ \bar{Y}] = [X\ Y] + T[-dG\ I_n - dF]$$

for some polynomial matrix T. As a result, the $[X\ Y]$ is a remainder after dividing $[-dG\ I_n - dF]$ into $[\bar{X}\ \bar{Y}]$. Hence $\deg [X\ Y] < \deg [-dG\ I_n - dF]$, that is, both X and Y are constant matrices. Moreover, the X is invertible since $X\langle A \rangle = \langle C \rangle$ and both A and C are causal. Thus the optimal regulator (9) follows from (13) and (14) combined.

As to the second part of the proof, observe that the existence of K hinges on that of C. Equation (10) tells us that the determinant of C is the characteristic polynomial of the closed loop system, see Kučera [3]. Thus C must be stable, not merely Hurwitz, and this property depends on $HB + JA$ having only regular square right divisors.

To prove the uniqueness of K, note that for any C the constant solution X, Y of (10) is unique due to (15). Since any two spectral factors C are left orthogonal multiples of each other, the same must be true of the corresponding solution matrices $[X\ Y]$. Therefore X and Y depend upon C but K is unique. □

EXAMPLES

To illustrate the preceding theory, consider system (1) with the matrices

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$H = [-1\ \ 0] \qquad J = [1]$$

and find the regulator K which minimizes the cost (2). Denoting ξ_1 and ξ_2 the state variables, this is equivalent to minimizing

$$V = \sum_{t=0}^{\infty} y_t^2$$

subject to

$$\xi_{1\,t+1} = \xi_{1t} + u_t, \quad \xi_{2\,t+1} = u_t$$

and

$$y_t = u_t - \xi_{1t}.$$

The first step is to obtain the polynomial decomposition (7). Since

$$(I_n - dF)^{-1}\,dG = \begin{bmatrix} \dfrac{d}{1-d} \\ d \end{bmatrix}$$

we have

$$B = \begin{bmatrix} d \\ d - d^2 \end{bmatrix}, \qquad A = [1 - d].$$

Next we have to calculate the spectral factor C defined in (8). Seeing that

$$HB + JA = 1 - 2d$$

we have

$$(1 - 2d^{-1})(1 - 2d) = (2 - d^{-1})(2 - d)$$

so that

$$C = 2 - d.$$

Finally the equation (10) becomes

$$X[1 - d] + Y\begin{bmatrix} d \\ d - d^2 \end{bmatrix} = 2 - d$$

and we easily find the constant solution

$$X = [2], \quad Y = [1 \;\; 0].$$

Hence the optimal regulator K is given by

$$K = [0.5 \;\; 0]$$

and produces a closed loop system with the characteristic polynomial $2 - d$.

To illustrate the significance of $HB + JA$ not having rank m, consider system (1) described by

$$F = [1], \qquad G = [1 \;\; 1]$$

$$H = [1], \qquad J = [1 \;\; 1].$$

This gives

$$B = [d \;\; 0], \qquad A = \begin{bmatrix} 1 - d & -1 \\ 0 & 1 \end{bmatrix}$$

and

$$HB + JA = [1 \;\; 0].$$

There is no Hurwitz spectral factor C satisfying (8); however, a detailed analysis shows that the regulators

$$K = \begin{bmatrix} x \\ 1 - x \end{bmatrix}$$

whete x is any real number yield the closed-loop system matrix $F - GK = 0$ and hence all must be optimal with respect to (2). Thus the optimal regulator is not unique for non-left invertible systems.

DISCUSSION

The transfer matrix solution to the discrete-time linear regulator problem advanced in this paper has several interesting features. The design procedure consists of three steps: the calculation of an appropriate matrix fraction, the spectral factorization, and the solution of a linear equation in polynomial matrices. Computationally, the central problem is to perform the sectral factorization. There are efficient iterative algorithms reported by Kazanjian [2] and Kučera [3]. However, a matrix version of the polynomial algorithm developed by Vostrý [8] is to be preferred, for it combines efficiency with quadratic convergence. The calculation of right coprime matrix fractions is described e.g. in Kučera [3]. An efficient way of finding the constant solution X, Y of equation (10) is to equate the coefficients at like powers of d. Denoting A_i, B_i, and C_i the coefficients of A, B, and C at d^i, we have to solve the system of linear equations

$$XA_0 = C_0$$

$$YB_i = C_i - XA_i \,, \quad i = 1, 2, \ldots, \deg B \,.$$

From the theoretical point of view, it is interesting that the solution of the algebraic Riccati equation, a complicated quadratic problem, separates here into a simpler quadratic problem of spectral factorization and a linear problem of solving matrix polynomial equation. In fact, the process of solving equation (10) can be interpreted as the assignment of desired invariant polynomials (i.e., dynamics and structure) to the closed loop system, see Kučera [3]; the spectral factorization then simply tells us which polynomials are to be assigned. The relationship between spectral factors and the optimal system dynamics has been around for some time but the direct way of obtaining the regulator via the solution of matrix polynomial equation is believed to be original. Of course, the overall characteristics of the optimal system, such as the return-difference matrix, can be obtained without actually computing the regulator, see Shaked [6].

In addition to providing further insight and relating the state-space and transfer-matrix techniques, this approach seems to be computationally attractive. The computational savings are associated with the reduction of dimensionality at the expense of introducing polynomials: the complexity of calculations with real matrices is proportional to the third power of their dimensions while the complexity of polynomial manipulations grows only with the second power of their degrees.

All the results which have been given for the regulator problem translate at once into dual theorems concerning the discrete-time stationary estimation (filtering). This problem can be posed in terms of the dual of system (1)

$$w_{t+1} = Fw_t + H^T v_t$$
$$z_t \quad = Gw_t + J^T v_t$$

where $v_t \in R^l$, $z_t \in R^m$, and $w_t \in R^n$. The $v = (v_t)$ is a zero-mean, Gaussian, white noise sequence with unit covariance matrix. Note that this model allows for arbitrary cross-correlation between input and output noise processes and noise-free measurements. The estimate $\hat{w}_t$ of w_t which minimizes the variance of $f^T(w_t - \hat{w}_t)$ for arbitrary vector f, given the observations z from the infinite past up to but not including time t, is generated by the system

$$\hat{w}_{t+1} = (F^T - K^T G^T)\, \hat{w}_t + K^T z_t$$

where the gain matrix K is given by (4) and (5). Thus our Theorem can directly be applied to solve the linear estimator problem as well.

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