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ON THE CONTINUITY OF THE MINIMAL α-ENTROPY

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Sufficient conditions under which the minimal α-entropy is a continuous function of parameters of the family of the probability measures are given. For the special case of the location and scale parameter these conditions are specified. Moreover under the assumption $f(x)$ is a symmetric density, the continuity of the minimal α-entropy at the origine for the location parameter case is proved to be equivalent to the continuity in the quadratic mean of the square root of the density $f(x)$.

1. INTRODUCTION

For i.i.d. random variables H. Chernoff [1] showed that logarithm of the minimal α-entropy is equal to the asymptotic rate of the convergence of the error probabilities when a hypothesis against an alternative is tested. In 1972 A. Perez has generalized Chernoff's results for the case of stationary probability measures.

Now testing a hypothesis against an alternative and utilizing a likelihood ratio as a test statistic one can be iterested in changes of the asymptotic rate of convergence of the error probabilities which will occur when deviations of the hypothesis or of the alternative happen. In the present paper sufficient conditions under which these changes are continuous with respect to the parameters of the family of the probability measures are found. This is made by proving the continuity of the minimal α-entropy. The continuity of the minimal α-entropy guarantees not only continuity of the above described changes when we use the likelihood ratio test but it enables us (it will be done in a next paper) to prove the continuity of these changes even if an approximately best test is used.

It is intuitively clear that to prove the continuity of the minimal α-entropy one will need to assume a continuity of the family of densities of the probability measures with respect to its parameters. But as can be heuristically justified, the continuity of the minimal α-entropy should not be dependent on the choice of the densities.

It leads to an idea that the conditions should be stated in an integral form. From the point of view of an easy utilization of the results a form of the conditions including assumptions on local properties of densities seems to be reasonable, too. Both demands are satisfied in the present paper.

2. NOTATIONS

Let $\mathscr{X}$ and Θ be metric spaces with metrics $\varrho_{\mathscr{X}}$ and ϱ_{Θ}, respectively. Let $\mathscr{B}$ be a Borel σ-algebra over $\mathscr{X}$ and let us have, for every $\theta \in \Theta$, defined on $(\mathscr{X}, \mathscr{B})$ a probability measure P_θ. Let for every $\theta \in \Theta$ P_θ be absolutely continuous with respect to a σ-finite measure μ. Let $f(x, \theta)$ be the Radon-Nikodym derivative of the measure P_θ with respect to the measure μ. Let us write $\|x\|$ instead of $\varrho_{\mathscr{X}}(x, 0)$. Let $\mathscr{N}$ denote the set of all positive integers.

Let $\theta_1, \theta_2 \in \Theta$ and $\alpha \in [0, 1]$. Let $H_\alpha(\theta_1, \theta_2)$ (resp. $H(\theta_1, \theta_2)$) denote the α-entropy (resp. minimal α-entropy) of the measure P_{θ_1} with respect to the measure P_{θ_2}, i.e.

$$H_\alpha(\theta_1, \theta_2) = \int f^\alpha(x, \theta_1) f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu\,, \quad 0 < \alpha < 1\,,$$

$$H_0(\theta_1, \theta_2) = \lim_{\alpha \to 0+} H_\alpha(\theta_1, \theta_2)\,,$$

$$H_1(\theta_1, \theta_2) = \lim_{\alpha \to 1-} H_\alpha(\theta_1, \theta_2)$$

and

$$H(\theta_1, \theta_2) = \min_{0 \leqq \alpha \leqq 1} H_\alpha(\theta_1, \theta_2)\,.$$

(These limits exist because $\int f^\alpha(x, \theta_1) f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu$ is a convex and bounded function of $\alpha \in (0, 1)$.) Finally for $\theta^* \in \Theta$ and $\delta > 0$ let $\mathscr{O}(\theta^*, \delta) = \{\theta \in \Theta;\ \varrho_\Theta(\theta, \theta^*) \leqq \delta\}$.

3. ASSUMPTIONS

Let hereafter $\theta_1, \theta_2 \in \Theta$ be fixed points.

A1. Let for any real $K > 0$ the density $f(x, \theta)$ uniformly in a set $\{x \in \mathscr{X};\ \|x\| \leqq K\} \setminus \mathscr{W}$ be a continuous function of θ at the point θ_1, where $\mu(\mathscr{W}) = 0$. (As in the following in all proofs of assertions only integration (with respect to μ) will be used we can assume to have $f(x, \theta)$ continuous at θ_1 uniformly for all $x \in \in \{x \in \mathscr{X};\ \|x\| \leqq K\}$, i.e.

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall \theta \in \Theta,\ \varrho_\Theta(\theta, \theta_1) < \delta) \quad \text{and}$$

$$(\forall x \in \mathscr{X},\ \|x\| \leqq K) : |f(x, \theta) - f(x, \theta_1)| \leqq \varepsilon\,.)$$

A2. Let for any $\varepsilon > 0$ exist $\delta > 0$ and $K > 0$ so that for every $\theta \in \mathcal{O}(\theta_1, \delta)$

$$\int_{\{x \in \mathscr{X};\, \|x\| > K\}} f(x, \theta)\, \mathrm{d}\mu < \varepsilon\,.$$

A3. Let the density $f(x, \theta)$ (as a function of θ) be continuous in the mean (with respect to μ) at the point θ_1.

4. GENERAL CASE

Lemma 1. Let A1 and A2 hold. Then the α-entropy $H_\alpha(\theta, \theta_2)$ is a continuous function of θ at the point θ_1. Moreover for any $\alpha_1 \in (0, 1)$ the continuity of α-entropy $H_\alpha(\theta, \theta_2)$ is uniform with respect to $\alpha \in (\alpha_1, 1]$.

Proof. Let $\alpha_1 \in (0, 1)$ and $\varepsilon > 0$. Applying A2 let us take $\delta_1 > 0$ and $K > 0$ so that for any $\theta \in \mathcal{O}(\theta_1, \delta_1)$

$$\int_{\{x \in \mathscr{X};\, \|x\| > K\}} f(x, \theta)\, \mathrm{d}\mu < \frac{\varepsilon^{1/\alpha_1}}{4}\,.$$

Let us denote $B_K = \{x \in \mathscr{X};\ \|x\| \leqq K\}$. As the (minimal) α-entropy does not depend on the choice of the densities we may assume $\mu(B_K) < \infty$ (see [4], Lemma 2.3.5.). Considering A1, $\delta_2 > 0$ can be found so that for any $\theta \in \mathcal{O}(\theta_1, \delta_2)$ and any $x \in \{x \in \mathscr{X};\ \|x\| \leqq K\}$

$$|f(x, \theta) - f(x, \theta_1)| < \tfrac{1}{2}\,.\, \varepsilon^{1/\alpha_1}\,.\, \mu^{-1}(B_K)\,.$$

Let $\delta = \min(\delta_1, \delta_2)$. For any $\theta \in \mathcal{O}(\theta_1, \delta)$ we have

$$|H_\alpha(\theta, \theta_2) - H_\alpha(\theta_1, \theta_2)| = \left|\int f^\alpha(x, \theta) f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu - \int f^\alpha(x, \theta_1) f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu\right| \leqq$$

$$\leqq \left\{\int |f^\alpha(x, \theta) - f^\alpha(x, \theta_1)|^{1/\alpha}\, \mathrm{d}\mu\right\}^\alpha \left\{\int f(x, \theta_2)\, \mathrm{d}\mu\right\}^{1-\alpha} \leqq$$

(see APPENDIX, Lemma 1A)

$$\leqq \left\{\int | f(x, \theta) - f(x, \theta_1) |\, \mathrm{d}\mu\right\}^\alpha =$$

$$= \left\{\int_{B_K} |f(x, \theta) - f(x, \theta_1)|\, \mathrm{d}\mu + \int_{B^c{}_K} |f(x, \theta) - f(x, \theta_1)|\, \mathrm{d}\mu\right\}^\alpha \leqq$$

$$\leqq \left\{\int_{B_K} |f(x, \theta) - f(x, \theta_1)|\, \mathrm{d}\mu + \int_{B^c{}_K} f(x, \theta)\, \mathrm{d}\mu + \int_{B^c{}_K} f(x, \theta_1)\, \mathrm{d}\mu\right\}^\alpha \leqq$$

$$\leqq \left\{\frac{\varepsilon^{1/\alpha_1}}{4} + \frac{\varepsilon^{1/\alpha_1}}{4} + \mu(B_K)\,.\, \frac{\varepsilon^{1/\alpha_1}}{2\mu(B_K)}\right\}^\alpha \leqq \varepsilon\,. \qquad \square$$

Lemma 2. Let A1 hold. Then the α-entropy $H_\alpha(\theta_1, \theta_2)$ is a lower semicontinuous function of θ at the point θ_1. Moreover for any $\alpha_1 \in (0, 1)$ the lower semicontinuity of α-entropy $H_\alpha(\theta_1, \theta_2)$ is uniform with respect to $\alpha \in (0, \alpha_1)$.

Proof. Let a real $K > 1$ be found so that

$$\int_{B^c_K} f(x, \theta_2)\, \mathrm{d}\mu < \left(\frac{\varepsilon}{6}\right)^2 .$$

Let $\alpha \in (0, \frac{1}{2})$. Then

$$(1) \qquad \int_{B^c_K} |f^\alpha(x, \theta) - f^\alpha(x, \theta_1)| f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu \leqq$$

$$\leqq \int_{B^c_K} [f^\alpha(x, \theta) + f^\alpha(x, \theta_1)] f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu \leqq$$

$$\leqq \left\{\int f(x, \theta_1)\, \mathrm{d}\mu\right\}^\alpha \left\{\int_{B^c_K} f(x, \theta_2)\, \mathrm{d}\mu\right\}^{1-\alpha} + \left\{\int f(x, \theta)\, \mathrm{d}\mu\right\}^\alpha \left\{\int_{B^c_K} f(x, \theta_2)\, \mathrm{d}\mu\right\}^{1-\alpha} \leqq$$

$$\leqq 2 \cdot \left(\frac{\varepsilon}{6}\right)^{2(1-\alpha)} \leqq \frac{\varepsilon}{3} .$$

Now let $\{t_n\}_{n=1}^\infty$ be a decreasing sequence of real numbers such that $\lim_{n\to\infty} t_n = 0$. Then

$$\{x \in \mathscr{X};\ 0 < f(x, \theta_1) < t_{n+1}\} \subset \{x \in \mathscr{X};\ 0 < f(x, \theta_1) < t_n\}$$

so that

$$\lim_{n\to\infty} \int_{\{x\in\mathscr{X};0<f(x,\theta_1)<t_n\}} f(x, \theta_2)\, \mathrm{d}\mu = 0 .$$

Let $n_0 \in \mathscr{N}$ be such that

$$\int_{\{x\in\mathscr{X};0<f(x,\theta_1)<t_{n_0}\}} f(x, \theta_2)\, \mathrm{d}\mu < \left(\frac{\varepsilon}{6}\right)^2 .$$

Now it is easy to see that for $t = t_{n_0}$ and for $\alpha \in (0, \frac{1}{2})$

$$(2) \qquad \int_{\{x\in\mathscr{X};0<f(x,\theta_1)<t\}} |f^\alpha(x, \theta_1) - f^\alpha(x, \theta)| f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu \leqq$$

$$\leqq \left\{\int_{\{x\in\mathscr{X};0<f(x,\theta_1)<t\}} f(x, \theta_1)\, \mathrm{d}\mu\right\}^\alpha \left\{\int_{\{x\in\mathscr{X};0<f(x,\theta_1)<t\}} f(x, \theta_2)\, \mathrm{d}\mu\right\}^{1-\alpha} +$$

$$+ \left\{\int_{\{x\in\mathscr{X};0<f(x,\theta_1)<t\}} f(x, \theta)\, \mathrm{d}\mu\right\}^\alpha \left\{\int_{\{x\in\mathscr{X};0<f(x,\theta_1)<t\}} f(x, \theta_2)\, \mathrm{d}\mu\right\}^{1-\alpha} \leqq \frac{\varepsilon}{3} .$$

Now we are going to show that for K as in (1) and for t as in (2) there exist $\delta > 0$ and $\alpha_2 \in (0, 1)$ such that for every $\theta \in \mathcal{O}(\theta_1, \delta)$ and $\alpha \in (0, \alpha_2)$ we have

$$(3) \qquad \int_{\{x\in\mathscr{X};\|x\|\leqq K, f(x,\theta_1)\geqq t\}} |f^\alpha(x, \theta) - f^\alpha(x, \theta_1)| f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu \leqq \frac{\varepsilon}{3} .$$

To this end, let us put $C = \max\{\mu(B_K), 1\}$ (see the proof of the Lemma 1). Utilizing A1 let us find $\delta > 0$ so that for every

$\theta \in \mathcal{O}(\theta_1, \delta)$ and x, $\|x\| \leqq K$ the inequality

$$|f(x, \theta) - f(x, \theta_1)| \leqq \min\left\{\frac{\varepsilon}{3C}, \frac{t}{2}\right\}$$

holds. If $z \in [t/2, 1]$ and $\alpha > 0$ then $(t/2)^\alpha \leqq z^\alpha \leqq 1$ and

$$\lim_{\alpha \to 0+} \left(\frac{t}{2}\right)^\alpha = 1 .$$

From it follows we can find $\alpha_2 \in (0, 1)$ so that for every pair $z_1, z_2 \in [t/2, 1]$ and $\alpha \in (0, \alpha_2)$

$$|z_1^\alpha - z_2^\alpha| \leqq \frac{\varepsilon}{3C} .$$

Now for any $\theta \in \mathcal{O}(\theta_1, \delta)$ and $\alpha \in (0, \alpha_2)$ the following inequality is true.

$$\int_{\{x \in \mathscr{X};\, \|x\| \leqq K, f(x,\theta_1) \geqq t\}} |f^\alpha(x, \theta) - f^\alpha(x, \theta_1)| f^{1-\alpha}(x, \theta_4)\, \mathrm{d}\mu \leqq$$

$$\leqq \int_{\{x \in \mathscr{X};\, \|x\| \leqq K, f(x,\theta_1) \geqq 1\}} |f^\alpha(x, \theta) - f^\alpha(x, \theta_1)| f^{1-\alpha}(x, \theta_4)\, \mathrm{d}\mu +$$

$$+ \int_{\{x \in \mathscr{X};\, \|x\| \leqq K, f(x,\theta) \geqq 1,\, t \leqq f(x,\theta_1) < 1\}} |f^\alpha(x, \theta) - f^\alpha(x, \theta_1)| f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu +$$

$$+ \int_{\{x \in \mathscr{X};\, \|x\| \leqq K, t \leqq f(x,\theta_1) < 1, f(x,\theta) < 1\}} |f^\alpha(x, \theta) - f^\alpha(x, \theta_1)| f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu \leqq$$

$$\leqq \int_{\{x \in \mathscr{X};\, \|x\| \leqq K, f(x,\theta_1) \geqq 1\}} |f(x, \theta) - f(x, \theta_1)| f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu +$$

$$+ \int_{\{x \in \mathscr{X};\, \|x\| \leqq K, f(x,\theta) \geqq 1\}} |f(x, \theta) - f(x, \theta_1)| f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu +$$

$$+ \frac{\varepsilon}{3C} \int_{\{x \in \mathscr{X};\, \|x\| \leqq K, t \leqq f(x,\theta_1) < 1, f(x,\theta) < 1\}} f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu \leqq$$

$$\leqq \frac{\varepsilon}{3C} \int_{\{x \in \mathscr{X};\, \|x\| \leqq K, f(x,\theta_1) \geqq t\}} f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu \leqq$$

$$\leqq \frac{\varepsilon}{3C} \{\mu(B_K)\}^\alpha \left\{\int f(x, \theta_2)\, \mathrm{d}\mu\right\}^{1-\alpha} \leqq \frac{\varepsilon}{3} .$$

Let $\theta \in \mathcal{O}(\theta_1, \delta)$ and $\alpha \in (0, \alpha_2)$. Then

$$H_\alpha(\theta, \theta_2) - H_\alpha(\theta_1, \theta_2) \geqq$$

$$\geqq \int_{\{x \in \mathcal{X}; f(x, \theta_1) > 0\}} (f^\alpha(x, \theta) - f^\alpha(x, \theta_1)) f^{1-\alpha}(x, \theta_2) \, d\mu \geqq -\varepsilon$$

as it follows from (1), (2), (3). Making use of Lemma 1 one may easy finish the proof. □

Theorem 1. Let A1 and A2 hold. Then the minimal α-entropy $H(\theta, \theta_2)$ is a continuous function of θ at the point θ_1.

Proof. Let us suppose the theorem does not hold, i.e. there exists a positive ε_0 such that for every $\delta > 0$ we can find $\theta \in \mathcal{O}(\theta_1, \delta)$ such that

$$|H(\theta, \theta_2) - H(\theta_1, \theta_2)| > \varepsilon_0 . \tag{4}$$

Let us put $\delta_n = 1/n$ for $n > 2$. For every $n > 2$ let us find θ_n so that $|H(\theta_n, \theta_2) - H(\theta_1, \theta_2)| > \varepsilon_0$. Let $\{\alpha_n\}_{n=3}^\infty \subset [0, 1]$ be a sequence such that $H(\theta_n, \theta_2) = H_{\alpha_n}(\theta_n, \theta_2)$ for $n \geqq 3$. The sequence $\{\alpha_n\}_{n=3}^\infty$ is bounded and therefore there exists a subsequence $\{\alpha_{n(k)}\}_{k=1}^\infty$ so that $\lim_{k \to \infty} \alpha_{n(k)} = \alpha_0 \in [0, 1]$. The α-entropy $H_\alpha(\theta_1, \theta_2)$ is a (convex) continuous function of $\alpha \in [0, 1]$, hence there exists $\delta_1 > 0$ so that for every $\alpha \in (0, 1)$, $|\alpha - \alpha_0| < \delta_1$ we have

$$|H_\alpha(\theta_1, \theta_2) - H_{\alpha_0}(\theta_1, \theta_2)| \leqq \frac{\varepsilon_0}{2} .$$

Assuming Lemma 1 and Lemma 2 we can find δ_0 such that for any $\alpha \in (0, 1)$ and $\theta \in \mathcal{O}(\theta_1, \delta_0)$ an inequality

$$H_\alpha(\theta, \theta_2) - H_\alpha(\theta_1, \theta_2) \geqq -\frac{\varepsilon_0}{2}$$

is true. Then

$$H_\alpha(\theta, \theta_2) - H_{\alpha_0}(\theta_1, \theta_2) = H_\alpha(\theta, \theta_2) - H_\alpha(\theta_1, \theta_2) +$$

$$+ H_\alpha(\theta_1, \theta_2) - H_{\alpha_0}(\theta_1, \theta_2) \geqq -\varepsilon_0$$

for any $\alpha \in (0, 1)$, $|\alpha - \alpha_0| < \delta_1$ and $\theta \in \mathcal{O}(\theta_1, \delta_0)$. Let us find $n_0 \in \mathcal{N}$ such that $1/n_0 < \delta_0$ and for every $k > n_0$ we have $|\alpha_{n(k)} - \alpha_0| < \delta_1$. Then for any $k > n_0$; $\theta_{n(k)} \in \mathcal{O}(\theta_1, \delta_0)$ and $|\alpha_{n(k)} - \alpha_0| < \delta_1$ and so

$$H_{\alpha_{n(k)}}(\theta_{n(k)}, \theta_2) - H_{\alpha_0}(\theta_1, \theta_2) \geqq -\varepsilon_0 .$$

As $H_{\alpha_0}(\theta_1, \theta_2) \geqq H(\theta_1, \theta_2)$, it follows for any $k > n_0$

$$H_{\alpha_{n(k)}}(\theta_{n(k)}, \theta_2) \geqq H(\theta_1, \theta_2) - \varepsilon_0 .$$

Combining it with (4) we obtain for $k > n_0$

$$H_{\alpha_{n(k)}}(\theta_{n(k)}, \theta_2) \geqq H(\theta_1, \theta_2) + \varepsilon_0 . \tag{5}$$

On the other hand $H_\alpha(\theta, \theta_2)$ is a continuous function of θ at the point θ_1 and for any fixed α (see Lemma 1). So for $\alpha = \alpha_1$ such that $H_{\alpha_1}(\theta_1, \theta_2) = H(\theta_1, \theta_2)$

$$\lim_{\theta \to \theta_1} H_{\alpha_1}(\theta, \theta_2) = H(\theta_1, \theta_2) .$$

However $H_{\alpha_n}(\theta_n, \theta_2) = \min_{0 \leqq \alpha \leqq 1} H_\alpha(\theta_n, \theta_2)$ and so $H_{\alpha_n}(\theta_n, \theta_2) \leqq H_{\alpha_1}(\theta_n, \theta_2)$. From it we conclude

$$\limsup_{k \to \infty} H_{\alpha_{n(k)}}(\theta_{n(k)}, \theta_2) \leqq \lim_{k \to \infty} H_{\alpha_1}(\theta_{n(k)}, \theta_2) = H(\theta_1, \theta_2) .$$

The last inequality, however, contradicts with the inequality (5). □

Lemma 3. Let A3 hold. Then the α-entropy $H_\alpha(\theta, \theta_2)$ is a continuous function of θ at the point θ_1. Moreover for any $\alpha_1 \in (0, 1)$ the continuity of α-entropy $H_\alpha(\theta, \theta_2)$ is uniform with respect to $\alpha \in (\alpha_1, 1)$.

Proof. Let $\varepsilon > 0$, $\alpha_1 \in (0, 1)$. Let us find $\delta > 0$ such that for every $\theta \in \mathcal{O}(\theta_1, \delta)$ we have $\int |f(x, \theta) - f(x, \theta_1)| \, d\mu < \varepsilon^{1/\alpha_1}$. Then for $\alpha \in (\alpha_1, 1)$ (see Lemma 1A)

$$\int |f^\alpha(x, \theta) - f^\alpha(x, \theta_1)| \, f^{1-\alpha}(x, \theta_2) \, d\mu \leqq$$

$$\leqq \left\{ \int |f^\alpha(x, \theta) - f^\alpha(x, \theta_1)|^{1/\alpha} \, d\mu \right\}^\alpha \leqq \varepsilon^{1/\alpha_1 . \alpha} \leqq \varepsilon . \qquad \square$$

Lemma 4. Let A3 hold. Then the α-entropy $H_\alpha(\theta, \theta_2)$ is a lower semicontinuous function of θ at the point θ_1. Moreover for any $\alpha_1 \in (0, 1)$ the lower semicontinuity of α-entropy $H_\alpha(\theta, \theta_2)$ is uniform with respect to $\alpha \in (0, \alpha_1)$.

Proof. Let us again assume the lemma is not true. Then there exist $\varepsilon_0 > 0$ such that for every $\delta_n = 1/n$ $(n = 3, 4, 5, \ldots)$, θ_n and α_n can be found so that $\theta_n \in \mathcal{O}(\theta_1, \delta_n)$ and $\alpha_n \in (0, \delta_n)$ and we have

$$H_{\alpha_n}(\theta_n, \theta_2) < H_{\alpha_n}(\theta_1, \theta_2) - \varepsilon_0 . \tag{6}$$

As $0 \leqq H_\alpha(\theta, \theta') \leqq 1$ we can find a sequence $\{n(k)\}_{k=1}^\infty \subset \{n\}_{n=3}^\infty$ such that $\lim_{k \to \infty} H_{\alpha_{n(k)}}(\theta_{n(k)}, \theta_2)$ exists. Then we have from (6)

$$\lim_{k \to \infty} H_{\alpha_{n(k)}}(\theta_{n(k)}, \theta_2) - H_0(\theta_1, \theta_2) =$$

$$= \lim_{k \to \infty} [H_{\alpha_{n(k)}}(\theta_{n(k)}, \theta_2) - H_{\alpha_{n(k)}}(\theta_1, \theta_2)] + \lim_{k \to \infty} H_{\alpha_{n(k)}}(\theta_1, \theta_2) - H_0(\theta_1, \theta_2) =$$

$$= \lim_{k \to \infty} [H_{\alpha_{n(k)}}(\theta_{n(k)}, \theta_2) - H_{\alpha_{n(k)}}(\theta_1, \theta_2)] < -\varepsilon_0 .$$

As $\lim_{k\to\infty} \theta_{n(k)} = \theta_1$ it follows (see A3) $\{f(x, \theta_{n(k)})\}_{k=1}^{\infty}$ converges in mean $[\mu]$ to $f(x, \theta_1)$ and therefore it converges also in measure μ. Now from lemma 2A we have that $\{f(x, \theta_{n(k)}\}_{k=1}^{\infty}$ converges in probability P_{θ_2} and so we can choose a subsequence $\{n(k_l)\}_{l=1}^{\infty} \subset \{n(k)\}_{k=1}^{\infty}$ such that $\{f(x, \theta_{n(k_l)})\}_{l=1}^{\infty}$ converges to $f(x, \theta_1)$ P_{θ_2}-a.e.. Using Egorov's theorem we find a set $A \in \mathscr{B}$ such that $P_{\theta_2}(A) < \varepsilon_0/16$ and $\{f(x, \theta_{n(k_l)})\}_{l=1}^{\infty}$ converges uniformly for $x \in A^c$ to $f(x, \theta_1)$. Let us finally denote for $i = 1, 2, \ldots, \theta_i^* = \theta_{n(k_l)}$ for $i = l$. Then we have

$$\{\theta_i^*\}_{i=1}^{\infty} \subset \Theta\,, \quad \lim_{i\to\infty} \theta_i^* = \theta_1\,, \quad \lim_{i\to\infty} f(x, \theta_i^*) = f(x, \theta_1)$$

uniformly in $x \in A^c$ and

$$\lim_{i\to\infty} H_{\alpha_i}(\theta_i^*, \theta_2) < H_0(\theta_1, \theta_2) - \varepsilon_0\,.$$

The rest of the proof is analogous to the proof of Lemma 2 so some details will be omitted. Let us find $t \in (0, \frac{1}{2})$ and $\delta_1 > 0$ so that for any $\theta \in \mathcal{O}(\theta_1, \delta_1)$ and any $\alpha \in (0, \frac{1}{2})$ we have

$$\int_{\{x\in\mathscr{X};0<f(x,\theta_1)<t\}} |f^\alpha(x, \theta) - f^\alpha(x, \theta_1)|\, f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu \leqq \frac{\varepsilon_0}{8}\,. \tag{7}$$

Now let $K > 1$ be a real number such that

$$\int_{\{x\in\mathscr{X};\|x\|>K\}} f(x, \theta_2)\, \mathrm{d}\mu < \left(\frac{\varepsilon_0}{16}\right)^2.$$

Then

$$\int_{\{x\in\mathscr{X};\|x\|>K\}} |f^\alpha(x, \theta) - f^\alpha(x, \theta_1)|\, f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu \leqq \tag{8}$$

$$\leqq 2\left\{\int_{\{x\in\mathscr{X};\|x\|>K\}} f(x, \theta_2)\, \mathrm{d}\mu\right\}^{1-\alpha} \leqq 2\left(\frac{\varepsilon_0}{16}\right)^{2(1-\alpha)} \leqq \frac{\varepsilon_0}{8}\,.$$

In the same way we can find (using that $P_{\theta_2}(A) < \varepsilon_0/16$)

$$\int_A |f^\alpha(x, \theta) - f^\alpha(x, \theta_1)|\, f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu \leqq \frac{\varepsilon_0}{8}\,. \tag{9}$$

Now it is easy to verify (see the proof of (3)) that there exist $i_0 \in \mathscr{N}$ and $\delta_2 > 0$ so that for any $i \geqq i_0$, $\alpha \in (0, \delta_2)$ we have

$$\int_{\{x\in\mathscr{X};\|x\|\leqq K, x\in A^c, f(x,\theta_1)\geqq t\}} |f^\alpha(x, \theta_i^*) - f^\alpha(x, \theta_1)|\, f^{1-\alpha}(x, \theta_2)\, \mathrm{d}\mu \leqq \frac{\varepsilon_0}{8}\,. \tag{10}$$

Further let $i_1 \in \mathcal{N}$ be a number chosen so that $i_1 \geqq i_0$ and $i_1 > \delta_1^{-1}$. Let us take $i \geqq i_1$ and $\alpha \in (0, \delta_2)$. Then utilizing (7)–(10) we have

$$(11) \qquad \int_{\{x \in \mathcal{X}; f(x,\theta_1) > 0\}} |f^\alpha(x, \theta_i^*) - f^\alpha(x, \theta_1)| f^{1-\alpha}(x, \theta_2) \, d\mu \leqq$$
$$\leqq \int_{\{x \in \mathcal{X}; \|x\| > K\}} |f^\alpha(x, \theta_i^*) - f^\alpha(x, \theta_1)| f^{1-\alpha}(x, \theta_2) \, d\mu +$$
$$+ \int_A |f^\alpha(x, \theta_i^*) - f^\alpha(x, \theta_1)| f^{1-\alpha}(x, \theta_2) \, d\mu +$$
$$+ \int_{\{x \in \mathcal{X}; 0 < f(x,\theta_1) < t\}} |f^\alpha(x, \theta_i^*) - f^\alpha(x, \theta_1)| f^{1-\alpha}(x, \theta_2) \, d\mu +$$
$$+ \int_{\{x \in \mathcal{X}; \|x\| \leqq K, x \in A^c, f(x,\theta_1) \geqq t\}} |f^\alpha(x, \theta_i^*) - f^\alpha(x, \theta_1)| f^{1-\alpha}(x, \theta_2) \, d\mu \leqq \frac{\varepsilon_0}{2}.$$

As $H_0(\theta_1, \theta_2) = \lim_{\alpha \to 0+} H_\alpha(\theta_1, \theta_2)$ we can find $\tau > 0$ such that for any $\alpha \in (0, \tau)$ we have

$$(12) \qquad |H_\alpha(\theta_1, \theta_2) - H_0(\theta_1, \theta_2)| \leqq \frac{\varepsilon_0}{4}.$$

Finally let us choose $i_2 \in \mathcal{N}$, $i_2 \geqq i_0$ such that for every $i \geqq i_2$

$$(13) \qquad |H_{\alpha_i}(\theta_i^*, \theta_2) - \lim_{j \to \infty} H_{\alpha_j}(\theta_j^*, \theta_2)| \leqq \frac{\varepsilon_0}{4}.$$

Let $\delta = \min\{\delta_1, \delta_2, \tau\}$ and $i_3 = \min\{i \in \mathcal{N};\ i \geqq i_2,\ i > \delta^{-1}\}$. Now for $i \in \mathcal{N}$, $i \geqq i_3$ from (11), (12) and (13) it follows:

$$\lim_{j \to \infty} H_{\alpha_j}(\theta_j^*, \theta_2) - H_0(\theta_1, \theta_2) = \{\lim_{j \to \infty} H_{\alpha_j}(\theta_j^*, \theta_2) - H_{\alpha_i}(\theta_i^*, \theta_2)\} +$$
$$+ \{H_{\alpha_i}(\theta_i^*, \theta_2) - H_{\alpha_i}(\theta_1, \theta_2)\} + \{H_{\alpha_i}(\theta_1, \theta_2) - H_0(\theta_1, \theta_2)\} \geqq$$
$$\geqq \{\lim_{j \to \infty} H_{\alpha_j}(\theta_j^*, \theta_2) - H_{\alpha_i}(\theta_i^*, \theta_2)\} +$$
$$+ \left\{\int_{\{x \in \mathcal{X}; 0 < f(x,\theta_1)\}} [f^{\alpha_i}(x, \theta_i^*) - f^{\alpha_i}(x, \theta_1)] f^{1-\alpha_i}(x, \theta_2) \, d\mu\right\} +$$
$$+ \{H_{\alpha_i}(\theta_1, \theta_2) - H_0(\theta_1, \theta_2)\} \geqq -\frac{\varepsilon_4}{4} -$$
$$- \int_{\{x \in \mathcal{X}; 0 < f(x,\theta_1)\}} |f^{\alpha_i}(x, \theta_i^*) - f^{\alpha_i}(x, \theta_1)| f^{1-\alpha_i}(x, \theta_2) \, d\mu - \frac{\varepsilon_0}{4} \geqq -\varepsilon_0.$$

But it contradict with (6). □

Theorem 2. Let A3 hold. Then the minimal α-entropy $H(\theta, \theta_2)$ is a continuous function of θ at the point θ_1.

Proof of the theorem is analogous to the one of the Theorem 1 and will be omitted.

5. LOCATION AND SCALE PARAMETER

Remark 1. In the next two corollaries we will consider a continuous density $f(x)$. From Lemma 3A (see APPENDIX) one can easy find out that Corollaries 1 and 2 hold under an assumption of a.e.-μ continuity of $f(x)$*), too.

Corollary 1. Let $f(x)$ be a continuous density defined on the real line R and let us construct a set $\{f(x, \theta) = f(x - \theta)\}_{\theta \in R}$. Then for $\{f(x, \theta)\}_{\theta \in R}$ the minimal α-entropy $H(\theta, \theta_2)$ is a continuous function of θ.

Proof. It is easy to see that A1 holds. Now let us find $K > 2|\theta_1|$ (now $\mathscr{X} = R$, hence θ_1 is a real number) such that $\int_{\{x \in R; |x| > K/4\}} f(x) \,\mathrm{d}x < \varepsilon$ and let $\theta \in R$, $|\theta| < K/4$. Then

$$\int_{\{x \in R; |x| > K\}} f(x - \theta_1 - \theta) \,\mathrm{d}x \leqq \int_{\{x \in R; |x| > K/4\}} f(x) \,\mathrm{d}x < \varepsilon .$$

So the proof of A2 is a straightforward one, too. □

Corollary 2. Let $f(x)$ be a continuous density defined on the real line R and let us consider a set $\{f(x, \theta) = \theta \,.\, f(x \,.\, \theta)\}_{\theta > 0}$ (i.e. $\theta \in R$, $\theta > 0$). Then for $\{f(x, \theta)\}_{\theta > 0}$ the minimal α-entropy is a continuous function of θ.

Proof. Let ε be a positive number. As $f(x \,.\, \theta_1)$ is a continuous function (on $[-K, K]$, where K is any positive number) let us put $M_1 = \max_{|x| \leqq K} f(x \,.\, \theta_1)$. Let us find $\delta_1 > 0$ so that for any pair $x, y \in [-\theta_1 K, \theta_1 K]$, $|x - y| < \delta_1$ we have $|f(x) - f(y)| < \varepsilon/3\theta_1$. Let $\delta = \min\{\delta_1/K, \varepsilon/2M_1, \theta_1/2\}$. For any $\theta \in \mathcal{O}(\theta_1, \delta)$ and $x \in [-K, K]$ the following inequalities hold:

$$\begin{aligned} |f(x, \theta) - f(x, \theta_1)| &= |\theta \,.\, f(x \,.\, \theta) - \theta_1 \,.\, f(x \,.\, \theta_1)| \leqq \\ &\leqq |\theta[f(x \,.\, \theta) - f(x \,.\, \theta_1)]| + |(\theta - \theta_1) \,.\, f(x \,.\, \theta_1)| \leqq \\ &\leqq \frac{3}{2}\theta_1 \,.\, \frac{\varepsilon}{3\theta_1} + \frac{\varepsilon}{2M_1} \,.\, M_1 \,. \end{aligned}$$

It proves that A1 holds. The proof that A2 is fulfilled is a simple one and that is why it will be omitted. □

Lemma 5. Let $f(x)$ be a density defined on the real line R and let us suppose that for any real x, $f(x) = f(-x)$ and that $f^{1/2}(x)$ is continuous in quadratic mean with

*) We say $f(x)$ is a.e.-μ continuous if there exists a set W such that $\mu(W) = 0$ and $f(x)$ is continuous on $R \setminus W$.

respect to Lebesgue measure. Let us again construct $\{f(x, \theta) = f(x - \theta)\}_{\theta \in R}$. Then the minimal α-entropy $H(\theta, \theta_2)$ is a continuous function of θ for all points $\theta \in \Theta$.

Proof. It is easy to see that

$$H(\theta', \theta'') = H_{1/2}(\theta', \theta'') \tag{14}$$

(for any pair $\theta', \theta'' \in R$). A proof can be given as follows. Let $\alpha \in (0, 1)$. Then

$$H_\alpha(\theta', \theta'') = \int f^\alpha(x - \theta') f^{1-\alpha}(x - \theta'') \, dx = \int f^\alpha(-z) f^{1-\alpha}(-z + \theta' - \theta'') \, dz =$$

$$= \int f^\alpha(z) f^{1-\alpha}(z - \theta' + \theta'') \, dz = \int f^\alpha(z - \theta'') f^{1-\alpha}(z - \theta') \, dz = H_{1-\alpha}(\theta', \theta'').$$

And finally

$$H_{1/2}(\theta', \theta'') \leqq H_\alpha^{1/2}(\theta', \theta'') \, . \, H_{1-\alpha}^{1/2}(\theta', \theta'') = H_\alpha(\theta', \theta'') \, .$$

That proves (14). Now we can finish the proof of the lemma. Let $\theta, \theta' \in R$.

$$|H_{1/2}(\theta, \theta_2) - H_{1/2}(\theta', \theta_2)| \leqq \left\{\int f(x - \theta_2) \, dx\right\}^{1/2} \times$$

$$\times \left\{\int [f^{1/2}(x - \theta) - f^{1/2}(x - \theta')]^2 \, dx\right\}^{1/2} = \left\{\int [f^{1/2}(x - \theta) - f^{1/2}(x - \theta')]^2 \, dx\right\}^{1/2}.$$

□

Lemma 6. Let us consider a set $\{f(x - \theta)\}_{\theta \in R}$, where $f(x)$ is a symmetric density on the real line R. Let moreover the minimal α-entropy $H(\theta, 0)$ be a continuous function of θ at the origine. Then the function $f^{1/2}(x)$ is continuous in quadratic mean with respect to the Lebesgue measure.

Proof. To prove the lemma it is sufficient to verify the equality:

$$[H(\Delta, 0) - H(0, 0)] + [H(-\Delta, 0) - H(0, 0)] =$$

$$= \int [f^{1/2}(x - \Delta) - f^{1/2}(x)] f^{1/2}(x) \, dx + \int [f^{1/2}(x + \Delta) - f^{1/2}(x)] f^{1/2}(x) \, dx =$$

$$= \int [f^{1/2}(x - \Delta) - f^{1/2}(x)] \, [f^{1/2}(x) - f^{1/2}(x - \Delta)] \, dx \, .$$

□

Theorem 3. Let $f(x)$ be a symmetric density defined on the real line R. Let us define a family $\{f(x, \theta) = f(x - \theta)\}_{\theta \in R}$. Then the minimal α-entropy $H(\theta, \theta_2)$ is a continuous function of θ at the point θ_2 iff the function $f^{1/2}(x)$ is continuous in quadratic mean.

Proof follows from the Lemmas 5 and 6.

Remark 2. In the Theorem 3 the assumption of symmetry of the density $f(x)$ is shown to guarantee that the continuity of the minimal α-entropy is equivalent to the continuity in the quadratic mean of the square root of the density. From the continuity in the mean of the density we can find the continuity in the quadratic mean of the square root of the density, but the opposite must not be generally true. So the assumption of the continuity in quadratic mean of the square root of the density may be weaker than the continuity in the mean of the density and therefore the theorem 3 may occur useful.

6. APPENDIX

Lemma 1A. Let $a, b \geqq 0$, $0 < \alpha < 1$. Then

$$|a^\alpha - b^\alpha| \leqq |a - b|^\alpha .$$

Proof. If $a = b$, then lemma holds. Let $a > b$. Then $a \neq 0$. So we would like to prove

$$1 - x^\alpha \leqq (1 - x)^\alpha$$

where $x = b/a$, i.e. $0 \leqq x < 1$, $0 < \alpha < 1$. From $0 \leqq x < 1$ follows $x^\alpha \geqq x$ and also $0 \leqq 1 - x \leqq 1$ and finally $(1 - x)^\alpha \geqq 1 - x$. Taking sum of the left and right sides of the second and the fourth inequality we obtain $x^\alpha + (1 - x)^\alpha \geqq \geqq 1 - x + x$ i.e. $(1 - x)^\alpha \geqq 1 - x^\alpha$. It was to be proved. The case $b < a$ is a symmetric one with respect to the above case. □

Lemma 2A. Let P be a probability measure absolutely continuous with respect to a measure μ. Then convergence with respect to the measure μ implies convergence with respect to the probability P.

Proof. At first let us show that

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall A \in \mathscr{B},\ \mu(A) < \delta)\ ; \quad P(A) < \varepsilon .$$

Let us assume it is not true. Then

$$\exists(\varepsilon_0 > 0)\ \forall(n \in \mathscr{N})\ \exists\left(A_n \in \mathscr{B}),\ \mu(A_n) < \frac{1}{n^2}\right) \quad \text{but} \quad P(A_n) > \varepsilon_0 .$$

Let $B_n = \bigcup_{k=n}^{\infty} A_k$ and $B = \bigcap_{n=1}^{\infty} B_n$. Then

$$\mu(B) \leqq \lim_{n\to\infty} \mu(B_n) \leqq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(A_k) = 0$$

and $P(B) = \lim_{n\to\infty} P(B_n) \geqq \varepsilon_0$. But it contradicts with $P \ll \mu$. Now let a sequence $\{f_n\}_{n=1}^{\infty}$ converge to f with respect to the measure μ, i.e.

$$\forall(\varepsilon > 0 \text{ and } \delta > 0)\ \ \exists(n_0 \in \mathscr{N})\ \ \forall(n \in \mathscr{N},\ n \geqq n_0)$$
$$\mu(\{x \in \mathscr{X};\ |f_n(x) - f(x)| > \delta\}) < \varepsilon .$$

Let ω and ν be positive numbers. Let us find γ so that for any $A \in \mathscr{B}$, $\mu(A) < \gamma$ we have $P(A) < \omega$. Let us finally choose $n_0 \in \mathscr{N}$, $n_0 = n_0(\omega, \nu)$ so that for any $n \in \mathscr{N}$, $n \geqq n_0$ we have

$$\mu(\{x \in \mathscr{X};\ |f_n(x) - f(x)| > \nu\}) < \gamma .$$

Then

$$P(\{x \in \mathscr{X};\ |f_n(x) - f(x)| > v\}) < \omega . \qquad \square$$

Lemma 3A. Let $f(x)$ be an a.e.-λ (Lebesgue measure) continuous and bounded density defined on the real line R and let us construct a family of densities $\{f(x, \theta) = f(x - \theta)\}_{\theta \in R}$. Then for $\varepsilon > 0$ there exists $g(x)$ continuous on R and $\delta > 0$ so that for any $\theta \in \mathcal{O}(\theta_1, \delta)$ and $\alpha \in [0, 1]$

$$(15) \qquad \left|H_\alpha(\theta, \theta_2) - \int g^\alpha(x - \theta)\, g^{1-\alpha}(x - \theta_2)\, \mathrm{d}x\right| < \varepsilon .$$

Proof. Let M be a real number such that $f(x) < M$ for any real x. Let us take $K \in R$ and $\delta > 0$ so that for any $\theta \in \mathcal{O}(\theta_1, \delta)$

$$(16) \qquad \int_{\{x \in R;\, |x| > K\}} f(x, \theta)\, \mathrm{d}x < \frac{\varepsilon}{3}$$

and

$$(17) \qquad \int_{\{x \in R;\, |x| > K\}} f(x, \theta_2)\, \mathrm{d}x < \frac{\varepsilon}{3} .$$

Now let D_1 denote a set of measure zero and the function f is continuous on $R \setminus D_1$. Let $D = D_1 \cap [-K, K]$. Let us find an open set G such that $D \subset G$ and $\lambda(G) < \varepsilon/6M$. As $G^c \cap [-K, K]$ is a compact set we can find a continuous function $g(x)$ such that for $x \in G^c \cap [-K, K]$ we have $g(x) = f(x)$, for $x \in R$ $|g(x)| < M$ and

$$(18) \qquad \int_{\{x \in R;\, |x| > K\}} g(x)\, \mathrm{d}x < \frac{\varepsilon}{3} .$$

Now using the fact that

$$R \subset (-\infty, -K) \cup G \cup \{G^c \cap [-K, K]\} \cup (K, +\infty)$$

and (16), (17) and (18) one can easy conclude the proof of the lemma. $\square$

Remark 3. To prove (15) uniformly with respect to $\alpha \in (0, 1)$ both (16) and (17) must be used.

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