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OPTIMAL DESIGNS FOR THE ESTIMATION OF POLYNOMIAL FUNCTIONALS

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Estimates of *homogeneous* polynomial functionals in regression and generalized regression (= infinitedimensional) experiments are investigated. Inequalities are derived showing that designs are to be compared according to the minimal variances of unbiassed estimates of the functionals under the hypothesis that the response function is zero. This is an extension of previous results by the author (cf. [6]). The corresponding new optimality criteria, based on the tensor power of the information matrix, are derived. A method of computation of the optimal designs is presented.

1. INTRODUCTION AND RESULTS

We recall briefly the structure of a standard regression experiment. We are given m linearly independent continuous functions $f_1, \ldots, f_m$ on a compact metric space X. In any point $x \in X$ an elementary experiment can be performed whose outcome is assumed to be a gaussian random variable $y(x)$ having the mean $\mathrm{E}[y(x)] = \sum_{i=1}^{m} a_i f_i(x)$ and the variance $\mathrm{Var}[y(x)] = 1$, respectively. Uncorrelated observations are performed in different points of X. Any probability measure ξ on X supported by a finite set is said to be a design of the experiment ($\xi(x)$ is proportional to the number of repeated observations in the point $x \in X$).

Usually we attempt to construct designs that are optimal for estimation of the parameters $\alpha_1, \ldots, \alpha_m$ or, of some linear functions in $\alpha_1, \ldots, \alpha_m$. In the present paper, we consider designs for the estimation of one or of several homogeneous polynomials in $\alpha_1, \ldots, \alpha_m$. In this case, the variances of the corresponding unbiased estimates depend on the true, but unknown, values of $\alpha_1, \ldots, \alpha_m$. Therefore difficulties arise when looking for optimal a priori designs. However, it turnes out that a good optimality criterion is the variance of unbiased estimate of the polynomial which is minimal in case $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$. We present arguments supporting this approach.

A nonparametric description of the standard regression experiment is also possible. We are given a finite dimensional linear space $\Theta \subset C(X)$ (in the parametric case $\Theta = \{\sum_{i=1}^{m} \alpha_i f_i(\cdot) : \alpha \in R^m\}$). Every $\vartheta \in \Theta$ is referred to as a possible state of the observed object, and $\vartheta \in \Theta$ is the only a priori information about the state available. The role of the functions of the parameters $\alpha_1, \ldots, \alpha_m$ is now played by functionals on Θ. Moreover, we can drop out the assumption that Θ is of finite dimension. In this case we speak of a generalized regression experiment with uncorrelated observations. This general model makes the Hilbert space techniques (namely the properties of Wiener chaos; see [4]) effective and advantageous.

Let q_k denote a k-linear functional defined on Θ^k. The polynomial p_k on Θ defined by the properties

$$p_k(\vartheta) = q_k(\vartheta, \ldots, \vartheta)\,; \quad (\vartheta \in \Theta) \tag{1}$$

is said to be homogeneous of degree k. Any polynomial p on Θ can be given as

$$p(\vartheta) = \sum_{k=0}^{n} p_k(\vartheta)\,; \quad (\vartheta \in \Theta)\,, \tag{2}$$

where the components $p_0, \ldots, p_n$ are homogeneous polynomials of respective degrees $0, \ldots, n$.

We identify the set of all (Borel) probability measures on X with the set Ξ of all designs. In particular, if the dimension of Θ is finite then every $\xi \in \Xi$ is equivalent to some design supported by finitely many points (cf. [2]). Within the generalized regression experiment this is not true.

In Section 2 we prove the following assertion:

Proposition 3a. A homogeneous polynomial p_k of degree k is estimable without bias in an experiment performed according to a design ξ if and only if there is a function $l : X^k \mapsto R$ such that

$$\int_{X^k} l^2 \,\mathrm{d}\xi^k < \infty$$

and

$$p_k(\vartheta) = \int_{X^k} l(x_1, \ldots, x_k)\, \vartheta(x_1) \ldots \vartheta(x_k)\, \mathrm{d}\xi(x_1) \ldots \mathrm{d}\xi(x_k)\,; \quad (\vartheta \in \Theta) \tag{3}$$

The minimal variance, under the hypothesis $\vartheta = 0$, attainable by unbiased estimates of p_k equals

$$k! \int_{X^k} l^2 \,\mathrm{d}\xi^k$$

if and only if the function l belongs to the $L^2(X^k, \xi^k)$ – closure of the linear space spanned by the set

$$\Theta^{\odot k} = \{\vartheta(\cdot) \ldots \vartheta(\cdot);\ \vartheta \in \Theta\}\,.$$

Consequently, a polynomial of the form $p = \sum_{k=0}^{n} p_k$ is estimable without bias if and only if (3) is valid simultaneously for all components $p_1, \ldots, p_n$.

The variance, $\mathrm{Var}_\vartheta U$, of an unbiased estimate U of a polynomial depends, in general, on $\vartheta \in \Theta$. Nevertheless, if the polynomial is *homogeneous* of degree k then the following inequalities take place (see Section 2):

$$\mathrm{Var}_0(U) \leqq \mathrm{Var}_\vartheta(U) \leqq \mathrm{Var}_0(U) \sum_{r=0}^{k-1} \binom{k}{r} \frac{\left[\int_X \vartheta^2 \,\mathrm{d}\xi\right]^r}{r!}\,; \quad (\vartheta \in \Theta) \tag{4}$$

and, for any $\vartheta^* \in \Theta$,

$$\limsup_{\vartheta \to \vartheta^*} \frac{[(\mathrm{Var}_\vartheta(U))^{1/2} - (\mathrm{Var}_{\vartheta^*}(U))^{1/2}]^2}{\int_X (\vartheta - \vartheta^*)^2 \,\mathrm{d}\xi} \leqq \tag{5}$$

$$\leqq \mathrm{Var}_0(U) \sum_{r=1}^{k-1} \binom{k}{r} \frac{\left[\int_X (\vartheta^*)^2 \,\mathrm{d}\xi\right]^{r-1}}{(r-1)!}$$

if $\int_X (\vartheta - \vartheta^*)^2 \,\mathrm{d}\xi$ tends to zero.

Thus, we can specify the bounds for $\mathrm{Var}_\vartheta(U)$ for an arbitrary $\vartheta \in \Theta$ in terms of $\mathrm{Var}_0(U)$, according to (4) and (5).

Similarly, let us suppose that $U_1, \ldots, U_s$ are unbiased estimates for the homogeneous polynomials $p_{k_1}, \ldots, p_{k_s}$ of degrees $k_1, \ldots, k_s$, respectively. Then the generalized variance of $U_1, \ldots, U_s$ under $\vartheta \in \Theta$ is given as

$$\mathrm{D}_\vartheta(U_1, \ldots, U_s) = \det\left[\{\mathrm{Cov}_\vartheta [U_i \cdot U_j]\}_{i,j=1}^{s}\right] \tag{6}$$

and we have bounds similar to (4):

$$\mathrm{D}_0(U_1, \ldots, U_s) \leqq \mathrm{D}_\vartheta(U_1, \ldots, U_s) \leqq \tag{7}$$

$$\leqq \mathrm{D}_0(U_1, \ldots, U_s) \sum_{r_1=0}^{k_1-1} \cdots \sum_{r_s=0}^{k_s-1} \prod_{i=1}^{s} \binom{k_i}{r_i} \frac{\left[\int_X \vartheta^2 \,\mathrm{d}\xi\right]^{r_i}}{r_i!}\,; \quad (\vartheta \in \Theta)$$

(cf. Proposition 4 in Section 2).

Let $U(\vartheta, \xi, \Theta)$ denote the unbiased estimate for p_k (see (1)) having the minimal variance under ϑ. As a consequence of (4) we get the inequalities

$$\mathrm{Var}_0[U(0, \xi, \Theta)] \leqq \mathrm{Var}_\vartheta[U(\vartheta, \xi, \Theta)] \leqq \tag{8}$$

$$\leqq \mathrm{Var}_0[U(0, \xi, \Theta)] \sum_{r=0}^{k-1} \binom{k}{r} \frac{\left[\int_X \vartheta^2 \,\mathrm{d}\xi\right]^r}{r!}\,; \quad (\vartheta \in \Theta)\,.$$

Similar relations for $\mathrm{D}_\vartheta(U_1(\vartheta, \xi, \Theta), \ldots, U_s(\vartheta, \xi, \Theta))$ are obtained by means of (7).

The above inequalities together with the inequality (27) below suggest the following approach. If we have to estimate the polynomial functionals but we cannot evaluate the true state $\vartheta \in \Theta$ before estimating, then we may base the optimality criteria of the design on the variance

$$\mathrm{Var}_0 \left[U(0, \xi, \Theta)\right]$$

or, on the covariance matrix

$$\{\mathrm{Cov}_0 \left[U_i(0, \xi, \Theta),\ U_j(0, \xi, \Theta)\right]\}_{i,j=1}^{s} .$$

If the regression experiment under consideration is standard (i.e., $\Theta = \{\vartheta : \vartheta(x) = f'(x)\,\alpha;\ \alpha \in R^m\}$) then we have good explicit expressions for those optimality criteria. In Section 3 we deal with $\mathrm{Var}_0 \left[U(0, \xi, \Theta)\right]$. The polynomial p_k (see (1)) can be written as

$$(9) \qquad p_k(\vartheta) = \sum_{i_1,\ldots,i_k=1}^{m} \gamma_{i_1,\ldots,i_k} \alpha_{i_1} \ldots \alpha_{i_k} ; \quad (\vartheta(\cdot) = f'(\cdot)\,\alpha \in \Theta) ,$$

where the given coefficients $\gamma_{i_1 \ldots i_k}$ may be supposed to be symmetric with respect to all permutations of the subscripts $i_1, \ldots, i_k$. The polynomial p_k is estimable without bias if and only if there exist the numbers $\delta_{i_1,\ldots,i_k}$; $(i_j = 1, \ldots, m;\ j = 1, \ldots, k)$ such that

$$(10) \qquad \gamma_{i_1,\ldots,i_k} = \prod_{r=1}^{k} \{\mathbf{M}(\xi)\}_{i_r j_r} \delta_{j_1,\ldots,j_k} .$$

Here, $\mathbf{M}(\xi)$ stands for the information matrix

$$(11) \qquad \mathbf{M}(\xi) = \int_X f(x) f'(x)\, \mathrm{d}\xi(x) .$$

Let $\mathbf{M}^-(\xi)$ denote a g-inverse of $\mathbf{M}(\xi)$. Then

$$(12) \qquad \mathrm{Var}_0 \left[U(0, \xi, \Theta)\right] \approx \sum_{i_1,\ldots,i_k=1}^{m} \sum_{j_1,\ldots,j_k=1}^{m} \gamma_{i_1,\ldots,i_k} \times \\ \times \prod_{r=1}^{k} \{\mathbf{M}^-(\xi)\}_{i_r j_r} \gamma_{j_1,\ldots,j_k} .$$

A design minimizing the right-hand side of (12) can be computed by means of an iterative procedure. We start with a nonsingular design ξ_{n_0}. Then we add a new point according to

$$(13) \quad \xi_{n+1} = \left(1 - \frac{1}{n+1}\right) \xi_n + \frac{1}{n+1} \xi_{x_{n+1}} ; \quad (n = n_0 + 1,\ n_0 + 2, \ldots) ,$$

where $\xi_{x_{n+1}}$ is the measure concentrated at x_{n+1} and x_{n+1} is chosen so that it gives the maximal descent of $\mathrm{Var}_0 \left[U(0, \xi_n, \Theta)\right]$. The convergence proof for the proposed procedure is given in Section 3, Proposition 7. Even the function $\mathbf{M}(\xi) \mapsto$ $\mapsto \mathrm{Var}_0 \left[U(0, \xi, \Theta)\right]$ is convex and differentiable, the convergence proof does not follow from the ideas used to study general algorithms that have appeared recently (see e.g. [1]). The point is that our optimality criterion is not regular (cf. also [8]) so that a special consideration is necessary. Especially, the following auxiliary proposition is used, which is an extension of the convergence theorem in [5] and which is important in the proof of Proposition 7.

Proposition 8. Let $\{x_i\}_{i=1}^{\infty} \subset X$ be an arbitrary sequence and let $\mathbf{M}_n$ denote the matrix

$$\mathbf{M}_n = \sum_{i=1}^{n} \mathbf{f}(x_i)\,\mathbf{f}'(x_i)\,; \quad (n = 1, 2, \ldots)\,. \tag{14}$$

Then there exists n_0 such that

$$\sum_{n=n_0}^{\infty} [\mathbf{f}'(x_{n+1})\,\mathbf{M}_n^{-1}\,\mathbf{f}(x_{n+1})]^2 < \infty \tag{15}$$

if al least one of the following conditions is satisfied:

a) X is finite and $\det \mathbf{M}_{n_0} > 0$;

b) $\liminf_{n\to\infty} \det(\mathbf{M}_n/n) > 0$;

c) $\limsup_{n\to\infty} n^{\beta}\,\mathbf{f}'(x_n)\,\mathbf{M}_n^{-1}\,\mathbf{f}(x_n) < \infty$ for some $\beta \in (0, 1)$.

2. ESTIMATES

Let $\mathscr{F}$ denote the Borel σ-algebra on X. Let ξ be a probability measure on $\mathscr{F}$ (a design). Consider a set $\{Y^{\xi}(F) : F \in \mathscr{F}\}$ of gaussian random variables such that

$$E_{\vartheta}[Y^{\xi}(F)] = \int_F \vartheta \,\mathrm{d}\xi\,; \quad (F \in \mathscr{F})\,, \tag{16}$$

$$\mathrm{Cov}_{\vartheta}\,[Y^{\xi}(F), Y^{\xi}(F')] = \xi(F \cap F')\,; \quad (F, F' \in \mathscr{F})\,,$$

$$Y^{\xi}[\bigcup_{i=1}^{k} F_i] = \sum_{i=1}^{k} Y^{\xi}(F_i) \quad \text{a.e.}\,; \quad (F_i \cap F_j = \emptyset \text{ if } i \neq j)$$

for all $\vartheta \in \Theta$. As known, to any probability space $(X, \mathscr{F}, \xi)$ we can associate a measurable space $(\Omega, \mathscr{S})$, a family $\{P_{\vartheta} : \vartheta \in \Theta\}$ of probability measure on $(\Omega, \mathscr{S})$ and a set $\{Y^{\xi}(F) : F \in \mathscr{F}\}$ of gaussian random variables having the properties (16) (cf. [4] chapt. IV).

We identify $\{Y^{\xi}(F) : F \in \mathscr{F}\}$ with the set of observables in a generalized regression experiment when using the design ξ. In case of a standard experiment ($\dim \Theta < \infty$) we can specify the design by $\xi(x_i) = N_i/N$, $(i = 1, \ldots, k)$, N_i being the number of replicae of observations in x_i, $N = \sum_{i=1}^{k} N_i$ being the total number of observations, respectively. Let $\bar{y}(x_i)$ denote the arithmetic mean of the values observed in x_i. Then

$$Y_N^\xi(F) = \sum_{x_i \in F} N_i\, \bar{y}(x_i)\,; \quad (F \in \mathscr{F})$$

define a set of random variables satisfying (16) up to a multiplicative constant.

Our investigation of the estimation theory within the generalized experiment will be based on the following results on gaussian processes (they are useful even in the standard situation because in problems of nonlinear estimation they allow to omit cumbersome combinatorial computations).

A) There is a linear bijection τ of $L^2(X, \xi)$ onto the Hilbert space $\mathscr{H} \subset L^2(\Omega, \mathscr{S}, P_\vartheta)$ spanned by the set $\{Y^\xi(F) : F \in \mathscr{F}\}$ such that

$$E_\vartheta[\tau(l)] = \int l\vartheta\, d\xi\,; \quad (l \in L^2(\xi))\,,$$

$$\mathrm{Cov}_\vartheta\,[\tau(l_1), \tau(l_2)] = \int l_1 l_2\, d\xi\,; \quad (l_1, l_2 \in L^2(\xi))\,.$$

(Here and in the sequel we omit the integration domain in case we integrate over the whole space X.) $\tau(l)$ can be formally interpreted as $\int l(x)\, Y^\xi(dx)$ and the set $\mathscr{H}$ that does not depend on $\vartheta \in \Theta$ is identified with the set of linear estimates in the experiment (cf. [4, 7]).

B) The σ-algebra $\mathscr{S}$ can be chosen as the minimal σ-algebra with respect to which all the functions $Y^\xi(F)$; $(F \in \mathscr{F})$ are measurable. Then $P_\vartheta \sim P_0$ and

$$\frac{dP_\vartheta}{dP_0} = \exp\{\tau(\vartheta) - \tfrac{1}{2}\, E_0[\tau^2(\vartheta)]\} \tag{17}$$

(cf. [4], chpt. VIII).

C) Let $[L^2(\xi)]^{\odot i}$ denote the i-th symmetric tensor power of the Hilbert space $L^2(\xi)$ (especially, $L^2(\xi)^{\odot 0} = R$). Under the hypothesis $\vartheta = 0$ the Hilbert space $\mathscr{P}_\xi \subset \subset L^2(\Omega, \mathscr{S}, P_0)$ spanned by the set of all polynomials in variables $Y^\xi(F)$; $(F \in \mathscr{F})$ can be mapped by an isomorphism $\varkappa$ onto the direct sum

$$\exp \odot L^2(\xi) := \bigoplus_{i=0}^{\infty} [L^2(\xi)]^{\odot i}\,.$$

It is worth to give a precise definition of the isomorphism $\varkappa$. If $l \in L^2(\xi)$ we define the i-th symmetric tensor power of l by

$$l^{\odot i} := \sqrt{(i!)}\; l \otimes \ldots \otimes l\,. \tag{18}$$

Let

$$\exp \odot l := \bigoplus_{i=0}^{\infty} \frac{l^{\odot i}}{i!}\,. \tag{19}$$

Then

(20) $$\varkappa[\exp\{\tau(l) - \tfrac{1}{2}\,\mathrm{E}_0[\tau^2(l)]\}] = \exp \odot l\,; \quad (l \in L^2(\xi))$$

(cf. [4], chpt. VII).

The set $\mathscr{P}_\xi$ is interpreted as the set of nonlinear estimates having finite variances under the hypothesis $\vartheta = 0$. If $\vartheta \neq 0$, then such estimates can possess infinite variances. However, the polynomials belonging to $\mathscr{P}_\xi$ are in $L^2(\Omega, \mathscr{S}, P_\vartheta)$ for every $\vartheta \in \Theta$. Therefore we use $\mathscr{P}_\xi$ as the set containing unbiased estimates for estimating the polynomials p_k.

Let $\langle\,,\rangle_\xi$ denote the inner product in $\exp \odot L^2(\xi)$. The inner product of any two elements $Y_1, Y_2 \in \mathscr{P}_\xi$ equals $\mathrm{E}_0(Y_1 Y_2)$. We denote by $\{h_i\}_{i=1}^\infty$ an orthonormal base of $L^2(\xi)$. Let $l \in [L^2(\xi)]^{\odot k}$. The Fourier coefficients of l are defined by the properties

$$b_{i_1,\ldots,i_k} := \langle l, h_{i_1} \otimes \ldots \otimes h_{i_k}\rangle_\xi\,.$$

Thus,

(21) $$l = \sum_{i_1,\ldots,i_k=1}^{\infty} b_{i_1,\ldots,i_k} h_{i_1} \otimes \ldots \otimes h_{i_k}$$

and

(22) $$\|l\|_\xi^2 = \sum_{i_1,\ldots,i_k=1}^{\infty} b_{i_1,\ldots,i_k}^2\,.$$

Next introduce the random variable

(23) $$Z_\vartheta(l) := \sum_{r=0}^{k} (r!)^{-1/2} \binom{k}{r}^{1/2} \varkappa\Big\{ \sum_{i_{r+1},\ldots,i_k=1}^{\infty} \Big[\sum_{i_1,\ldots,i_r=1}^{\infty} b_{i_1,\ldots,i_k} \times \times \prod_{s=1}^{r} \int h_{i_s}\vartheta\,\mathrm{d}\xi\Big]\, h_{i_{r+1}} \otimes \ldots \otimes h_{i_k}\Big\}\,.$$

Proposition 1. For every $\vartheta \in \Theta$;

$$\mathrm{E}_\vartheta[\varkappa(l)] = \mathrm{E}_0[Z_\vartheta(l)]\,,$$
$$\mathrm{Var}_\vartheta\,[\varkappa(l)] = \mathrm{Var}_0\,[Z_\vartheta(l)] < \infty\,.$$

Proof. The second assertion is proved in [6]. Let us prove the first one. By the definitions of $\varkappa$ and $\mathscr{P}_\xi$ we have $\mathrm{E}_0[\varkappa(l)] = 0$ for all $l \in \exp \odot L^2(\xi)$. Hence the terms in (23) that correspond to $r < k$ are centered as well. Therefore using (17)–(21) we obtain

$$\mathrm{E}_0[Z_\vartheta(l)] = \left\langle l, \frac{\vartheta^{\odot k}}{k!}\right\rangle_\xi = \langle l, \exp \odot \vartheta\rangle_\xi = \int_\Omega \varkappa(l)\,[\mathrm{d}P_\vartheta/\mathrm{d}P_0]\,\mathrm{d}P_0 = E_\vartheta[\varkappa(l)]\,. \quad \square$$

Now let $l \in [L^2(\xi)]^{\odot k}$ and $U = \varkappa(l)$, respectively.

Proposition 2. The inequalities (4) and (5) are valid.

Proof. The terms in (23) that correspond to different $r \in \{0, \ldots, k-1\}$ and $i_{r+1}, \ldots, i_k$ are orthogonal. Hence

$$\operatorname{Var}_\vartheta[\varkappa(l)] = \mathrm{E}_0[Z_\vartheta(l) - \mathrm{E}_0[Z_\vartheta(l)]]^2 = \tag{24}$$

$$= \sum_{r=0}^{k-1} \frac{1}{r!}\binom{k}{r} \sum_{i_{r+1},\ldots,i_k=1}^{\infty} [\langle \sum_{i_1,\ldots,i_r=1}^{\infty} b_{i_1,\ldots,i_k} h_{i_1} \otimes \ldots \otimes h_{i_r}, \vartheta^{\otimes r}\rangle_\xi]^2 .$$

Apply the Schwarz inequality to the inner products in (24) and use (22). Then we get from

$$\|l\|_\xi^2 = \operatorname{Var}_0[\varkappa(l)] \tag{25}$$

($\varkappa$ is an isomorphism!) the right-hand inequality in (4). Since $\operatorname{Var}_0[\varkappa(l)]$ equals the term in (24) that corresponds to $r = 0$, the other inequality in (4) is established as well. Now let

$$\Delta_\vartheta(l) = Z_\vartheta(l) - \mathrm{E}_0[Z_\vartheta(l)] .$$

Similarly as in (24) we prove that

$$\mathrm{E}_0\{[\Delta_\vartheta(l) - \Delta_{\vartheta^*}(l)]^2\} \leqq$$

$$\leqq \operatorname{Var}_0[\varkappa(l)] \sum_{r=1}^{k-1} \frac{1}{r!}\binom{k}{r} \|\vartheta^{\otimes r} - \vartheta^{*\otimes r}\|_\xi^2 .$$

Evidently

$$\|\vartheta^{\otimes r} - \vartheta^{*\otimes r}\|_\xi^2 = \langle\vartheta, \vartheta\rangle_\xi^r - 2\langle\vartheta, \vartheta^*\rangle_\xi^r + \langle\vartheta^*, \vartheta^*\rangle_\xi^r =$$

$$= \langle\vartheta, \vartheta - \vartheta^*\rangle_\xi \left[\sum_{i=0}^{r-1} \langle\vartheta, \vartheta\rangle_\xi^i \langle\vartheta, \vartheta^*\rangle_\xi^{r-i-1}\right] +$$

$$+ \langle\vartheta^* - \vartheta, \vartheta^*\rangle_\xi \left[\sum_{i=0}^{r-1} \langle\vartheta^*, \vartheta^*\rangle_\xi^i \langle\vartheta, \vartheta^*\rangle_\xi^{r-i-1}\right].$$

Hence

$$\lim_{\vartheta \to \vartheta^*} \frac{\|\vartheta^{\otimes r} - \vartheta^{*\otimes r}\|_\xi^2}{\|\vartheta - \vartheta^*\|_\xi^2} = r\langle\vartheta^*, \vartheta^*\rangle_\xi^{r-1} .$$

It follows that

$$\limsup_{\vartheta \to \vartheta^*} \frac{\mathrm{E}_0\{[\Delta_\vartheta(l) - \Delta_{\vartheta^*}(l)]^2\}}{\int(\vartheta - \vartheta^*)^2 \, \mathrm{d}\xi} \leqq$$

$$\leqq \operatorname{Var}_0[\varkappa(l)] \sum_{r=1}^{k-1} \binom{k}{r} \frac{1}{(r-1)!} \|\vartheta^*\|_\xi^{2(r-1)} .$$

Finally, we may write

$$[(\operatorname{Var}_\vartheta[\varkappa(l)])^{1/2} - (\operatorname{Var}_{\vartheta^*}[\varkappa(l)])^{1/2}]^2 \leqq$$

$$\leqq \mathrm{E}_0[\Delta_\vartheta(l) - \Delta_{\vartheta^*}(l)]^2$$

and this completes the proof of (5). □

Let us consider the polynomial p_k defined in (1). The random variable $\varkappa(l) \in \mathscr{P}_\xi$ is an unbiased estimate of p_k if

$$E_\vartheta[\varkappa(l)] = p_k(\vartheta)\,; \quad (\vartheta \in \Theta)$$

and

$$\mathrm{Var}_\vartheta\,[\varkappa(l)] < \infty\,; \quad (\vartheta \in \Theta)\,.$$

Proposition 3. $\varkappa(l) \in \mathscr{P}_\xi$ is an unbiased estimate of p_k if and only if there exists $l \in [L^2(\xi)]^{\odot k}$ such that

$$p_k(\vartheta) = (k!)^{-1}\,\langle l, \vartheta^{\odot k}\rangle_\xi\,; \quad (\vartheta \in \Theta)\,. \tag{26}$$

Proof. Let $l_i \in [L^2(\xi)]^{\odot i}$ $(i = 0, 1, \ldots)$ denote the components of l in the decomposition $l = \bigoplus_{i=0}^{\infty} l_i$. If $\varkappa(l)$ is unbiased then both (17) and (20) entail the relations

$$p_k(\vartheta) = \int_\Omega \varkappa(l)\,\frac{\mathrm{d}P_\vartheta}{\mathrm{d}P_0}\,\mathrm{d}P_0 = \langle l, \exp \odot\, \vartheta\rangle_\xi = \sum_{i=0}^{\infty} \frac{1}{i!}\,\langle l_i, \vartheta^{\odot i}\rangle_\xi\,; \quad (\vartheta \in \Theta).$$

Since p_k is homogeneous of degree k, we see that $l_i \neq 0$ whenever $i \neq k$.

Conversely, assume that (26) is valid. By Proposition 1, $\mathrm{Var}_\vartheta\,[\varkappa(l)] < \infty$. Finally, using again (17) and (20) we get

$$\mathrm{E}_\vartheta[\varkappa(l)] = (k!)^{-1}\,\langle l, \vartheta^{\odot k}\rangle_\xi = p_k(\vartheta)\,. \qquad \square$$

Since $\int l^2\,\mathrm{d}\xi^k = \|l\|_\xi^2 = \mathrm{Var}_0\,[\varkappa(l)]$ and $\langle l, \vartheta^{\otimes k}\rangle_\xi = \int l(x_1, \ldots, x_k)\,\vartheta(x_1)\ldots\vartheta(x_k)\,.$ $.\,\mathrm{d}\xi(x_1)\ldots\mathrm{d}\xi(x_k)$, Proposition 3a of Section 1 follows. At the same time, Propositions 2 and 3 imply that (4) and (5) remain valid for every unbiased estimate of p_k. The inequalities (8) follow then from (4).

Now let us consider the simultaneous estimation of s homogeneous polynomials $p^{(1)}, \ldots, p^{(s)}$ of the same degree k. Let $\mathbf{\Sigma}_\vartheta^\xi$ designate the covariance matrix of the unbiased estimate for $p^{(1)}, \ldots, p^{(s)}$ that has the minimal variance under ϑ. As a corollary to Proposition 2 and 3 we get

$$\mathbf{\Sigma}_0^\xi \leqq \mathbf{\Sigma}_\vartheta^\xi \leqq \mathbf{\Sigma}_0^\xi \left[\sum_{r=0}^{k-1} \binom{k}{r} \frac{\left[\int \vartheta^2\,\mathrm{d}\xi\right]^r}{r!}\right]. \tag{27}$$

Consequently,

$$\det\,(\mathbf{\Sigma}_0^\xi) \leqq \det\,(\mathbf{\Sigma}_\vartheta^\xi) \leqq [\det\,(\mathbf{\Sigma}_0^\xi)] \left\{\sum_{r=0}^{k-1} \binom{k}{r} \frac{\left[\int \vartheta^2\,\mathrm{d}\xi\right]^r}{r!}\right\}^s. \tag{28}$$

If we allow the polynomials $p_1, \ldots, p_s$ to have different degrees, say $k_1, \ldots, k_s$, then bounds similar to (28) can be obtained as shown in the following

Proposition 4. If $U_1, \ldots, U_s$ are unbiased estimates of $p_1, \ldots, p_s$, respectively, then the inequalities (7) take place.

Proof. Let $l_t \in [L^2(\xi)]^{\odot k_t}$ and write U_t for $\varkappa(l_t)$. Take $Z_\vartheta(l_t)$ as in (23) and put $\Delta_\vartheta(l_t) = Z_\vartheta(l_t) - \mathrm{E}_0[Z_\vartheta(l_t)]$. We use the symbol $\Delta_\vartheta(l_1) \wedge \ldots \wedge \Delta_\vartheta(l_s)$ to designate the antisymmetric tensor product

$$(s!)^{-1/2} \sum_\sigma \varepsilon_\sigma \, \Delta_\vartheta(l_{\sigma 1}) \otimes \ldots \otimes \Delta_\vartheta(l_{\sigma s}),$$

where the summation is taken over all permutations $\sigma := (\sigma 1, \ldots, \sigma s)$ of the set $(1, \ldots, s)$ and $\varepsilon_\sigma = \pm 1$, respectively. With

$$\delta_\vartheta(l_t) = \varkappa^{-1}[\Delta_\vartheta(l_t)] = \sum_{r=0}^{k_t - 1} (r!)^{-1/2} \binom{k_t}{r}^{1/2} \times \tag{29}$$

$$\times \sum_{i_{n+1}, \ldots, i_{k_t} = 1}^{\infty} \Big\langle \sum_{i_1, \ldots, i_r = 1}^{\infty} b_{i_1, \ldots, i_k} h_{i_1} \otimes \ldots \otimes h_{i_k}, \vartheta^{\otimes r} \Big\rangle_\xi \times h_{i_{r+1}} \otimes \ldots \otimes h_{i_{k_t}}$$

we have

$$\mathrm{D}_\vartheta(U_1, \ldots, U_s) = \mathrm{E}_0[\Delta_\vartheta(l_1) \wedge \ldots \wedge \Delta_\vartheta(l_s)]^2 =$$
$$= \|\delta_\vartheta(l_1) \wedge \ldots \wedge \delta_\vartheta(l_s)\|_\xi^2 .$$

The equation (29) can be rewritten in the form

$$\delta_\vartheta(l_t) = \sum_j c_j^{(t)} \varphi_j ,$$

where φ_j's are orthogonal elements like $h_{i_{r+1}} \otimes \ldots \otimes h_{i_{k_t}}$ and $c_j^{(t)}$'s are the corresponding coefficients in (29). Thus

$$\|\delta_\vartheta(l_1) \ldots \delta_\vartheta(l_s)\|_\xi^2 =$$
$$= \sum_{j_1, \ldots, j_s} (c_{j_1}^{(1)} \ldots c_{j_s}^{(s)})^2 \| \varphi_{j_1} \wedge \ldots \wedge \varphi_{j_s}\|_\xi^2 \leqq \sum_{j_1, \ldots, j_s} (c_{j_1}^{(1)} \ldots c_{j_s}^{(s)})^2 \leqq$$
$$\leqq \prod_{j=1}^{s} \left[\sum_{r=0}^{k-1} (r!)^{-1} \binom{k_j}{r} \|\vartheta\|_\xi^{2r} \right].$$

The last inequality has been obtained by application of Schwarz inequality to the inner products in (29). This proves the right-hand inequality in (7). If $\vartheta = 0$, only the terms with $r = 0$ occur in (29) so that we get easily also the left-hand inequality in (7). □

3. DESIGNS MINIMIZING $\mathrm{Var}_0\,[U(0, \xi, \Theta)]$

Throughout this section we consider the standard regression experiment with $\Theta = \{\vartheta : \vartheta(x) = \boldsymbol{\alpha}' \boldsymbol{f}(x), \boldsymbol{\alpha} \in R^m\}$.

Proposition 5. A polynomial p_k of the form (9) is estimable if and only if (10) is valid. In this case $\mathrm{Var}_0\,[U(0, \xi, \Theta)]$ can be expressed by the formula (12).

Proof. Consider an experiment with the set X^k of controlled variables, the set $\Theta^{\odot k}$ of states, and the design ξ^k, respectively. Let a linear functional g on $\Theta^{\odot k}$ be defined by the properties that

$$g(\vartheta^{\odot k}) = p(\vartheta) ; \quad (\vartheta \in \Theta) . \tag{30}$$

Apply Proposition 3 to the experiment specified above and to the homogeneous polynomial g. We see that (26) is a necessary and sufficient condition for the estimability of both p_k (under ξ) and g (under ξ^k). Similarly, using Proposition 3a of Section 1 we deduce that $\mathrm{Var}_0 [U(0, \xi, \Theta)]$ is but the variance of the BLUE for $(k!)^{-1} g$ in the new experiment. From (9) it follows that

$$g(\varphi) = \sum_{i_1,\ldots,i_k=1}^{m} \gamma_{i_1,\ldots,i_k} \varphi(x_{i_1}, \ldots, x_{i_k}) ; \quad (\varphi \in \Theta^{\odot k})$$

so that the variance of the BLUE for g is proportional to

$$\sum_{i_1,\ldots,i_k=1}^{m} \sum_{j_1,\ldots,j_k=1}^{m} \gamma_{i_1,\ldots,i_k} \{[\mathbf{M}(\xi^k)]^-\}_{\substack{i_1,\ldots,i_k\\ j_1,\ldots,j_k}} \gamma_{j_1,\ldots,j_k} ,$$

where $\mathbf{M}(\xi^k)$ stands for the information matrix of the new experiment. The expression (12) is now obtained using the symmetry of $\gamma_{i_1,\ldots,i_k}$ with respect to $i_1, \ldots, i_k$. □

Let us consider the mapping

$$(i_1, \ldots, i_k) \mapsto \gamma_{i_1,\ldots,i_k} : \{1, 2, \ldots, m\}^k \mapsto R$$

as an element γ of the k-th tensor power $[R^m]^{\otimes k}$. If we denote by $\gamma'\boldsymbol{\delta}$ the inner product in this space, the relations (9)–(12) take on the following simple forms:

$$p_k(\vartheta) = \gamma' \boldsymbol{\alpha}^{\otimes k} ; \quad (\vartheta(\cdot) = \mathbf{f}'(\cdot)\, \boldsymbol{\alpha} \in \Theta) ;$$

$$\gamma = [\mathbf{M}(\xi)]^{\otimes k} \boldsymbol{\delta} ,$$

$$\mathrm{Var}_0 [U(0, \xi, \Theta)] \approx \gamma' [\mathbf{M}^-(\xi)]^{\otimes k} \gamma .$$

On the set $\mathscr{M}$ of all information matrices we define a function Φ by

$$\Phi(\mathbf{M}) = \log \gamma' [\mathbf{M}^-]^{\otimes k} \gamma , \quad \text{if there exists} \quad \boldsymbol{\delta} \in [R^m]^{\otimes k}$$
$$\text{such that} \quad \gamma = \mathbf{M}^{\otimes k} \boldsymbol{\delta} ;$$
$$= \infty \quad \text{otherwise} .$$

The formula $\Phi(\mathbf{A}) = \log \gamma' (\mathbf{A}^{-1})^{\otimes k} \gamma$ extends Φ to the set of all nonsingular $m \times m$ matrices. The gradient $\nabla \Phi$,

$$\{\nabla \Phi(\mathbf{M})\}_{ij} := \partial \Phi(\mathbf{A}) / \partial \{\mathbf{A}\}_{ij} \big|_{\mathbf{A}=\mathbf{M}}$$

is well-defined for all nonsingular $\mathbf{M} \in \mathscr{M}$.

Proposition 6. The function Φ is convex on $\mathscr{M}$ and, moreover, it is strictly convex and differentiable on $\mathcal{M}_+ := \{\mathbf{M} : \mathbf{M} \in \mathcal{M}, \det(\mathbf{M}) > 0\}$. Its gradient can be expressed as

$$\{\nabla\Phi(\mathbf{M})\}_{ij} = -k\begin{pmatrix}\gamma\\ \mathbf{e}_i\end{pmatrix}' [\mathbf{M}^{-1}]^{\otimes(k+1)} \begin{pmatrix}\mathbf{e}_j\\ \gamma\end{pmatrix} \Big/ \gamma'[\mathbf{M}^{-1}]^{\otimes k}\,\gamma \tag{31}$$

$(i, j = 1, \ldots, m)$, where $\mathbf{e}_i$ is the i-th unit vector in R^m, $(\{\mathbf{e}_i\}_j = \delta_{ij})$.

Proof. The expression (31) is obtained by a direct computation using that

$$\nabla\{\mathbf{M}^{-1}\}_{ij} = -\mathbf{M}^{-1}\mathbf{I}^{(ij)}\mathbf{M}^{-1},$$

where $\{\mathbf{I}^{(ij)}\}_{r,s} = 0$ if $r \neq i$ or $s \neq j$ and $\{\mathbf{I}^{(ij)}\}_{r,s} = 1$ if $r = i$, $s = j$.

To prove the strict convexity of Φ on $\mathcal{M}_+$ take $\mathbf{M}_1, \mathbf{M}_0 \in \mathcal{M}_+$, $\beta \in (0, 1)$. Denote $\mathbf{M}_\beta := (1 - \beta)\,\mathbf{M}_0 + \beta\mathbf{M}_1$ and $\Psi(\mathbf{M}) := \gamma'[\mathbf{M}^{-1}]^{\otimes k}\gamma$. We have

$$\frac{\partial^2\Phi(\mathbf{M}_\beta)}{\partial\beta^2} = \frac{1}{\Psi^2(\mathbf{M}_\beta)}\left\{\Psi(\mathbf{M}_\beta)\,\frac{\partial^2\Psi(\mathbf{M}_\beta)}{\partial\beta^2} - \left(\frac{\partial\Psi(\mathbf{M}_\beta)}{\partial\beta}\right)^2\right\}. \tag{32}$$

Using again that

$$\frac{\partial}{\partial\beta}\,\mathbf{M}_\beta^{-1} = -\mathbf{M}_\beta^{-1}(\mathbf{M}_1 - \mathbf{M}_0)\,\mathbf{M}_\beta^{-1}$$

we obtain

$$\frac{\partial\Psi(\mathbf{M}_\beta)}{\partial\beta} = -k\gamma'[\mathbf{M}_\beta^{-1}(\mathbf{M}_1 - \mathbf{M}_0)\,\mathbf{M}_\beta^{-1}] \otimes [\mathbf{M}_\beta^{-1}]^{\otimes(k-1)}\,\gamma\,,$$

$$\begin{aligned}\frac{\partial^2\Psi(\mathbf{M}_\beta)}{\partial\beta^2} &= 2k\gamma'[\mathbf{M}_\beta^{-1}(\mathbf{M}_1 - \mathbf{M}_0)\,\mathbf{M}_\beta^{-1}(\mathbf{M}_1 - \mathbf{M}_0)\,\mathbf{M}_\beta^{-1}] \otimes [\mathbf{M}_\beta^{-1}]^{\otimes(k-1)}\,\gamma\, + \\ &+ (k-1)\,k\,\gamma'[\mathbf{M}_\beta^{-1}(\mathbf{M}_1 - \mathbf{M}_0)\,\mathbf{M}_\beta^{-1}] \otimes \\ &\otimes [\mathbf{M}_\beta^{-1}(\mathbf{M}_1 - \mathbf{M}_0)\,\mathbf{M}_\beta^{-1}] \otimes [\mathbf{M}_\beta^{-1}]^{\otimes(k-2)}\,\gamma\,.\end{aligned}$$

This allows to prove that

$$\frac{\partial^2\Psi(\mathbf{M}_\beta)}{\partial\beta^2}\,\Psi(\mathbf{M}_\beta) > \left[\frac{\partial\Psi(\mathbf{M}_\beta)}{\partial\beta}\right]^2$$

(using the Schwarz inequality), thus $\partial^2\Phi(\mathbf{M}_\beta)/\partial\beta^2 > 0$. The function Φ is strictly convex on $\{\mathbf{M}_\beta : \beta \in (0, 1)\}$ and therefore also on $\mathcal{M}_+$.

If $\mathbf{M}_1, \mathbf{M}_0 \in \mathcal{M}$, $\Phi(\mathbf{M}_0) < \infty$, $\Phi(\mathbf{M}_1) < \infty$, take $\mathbf{M}_i^- = \lim_{\varepsilon\to 0}(\mathbf{M}_i + \varepsilon\mathbf{I})^{-1}$; $(i = 0, 1)$. Hence the inequality $\Phi(\mathbf{M}_\beta) \leqq (1 - \beta)\,\Phi(\mathbf{M}_0) + \beta\,\Phi(\mathbf{M}_1)$ follows from the convexity of Φ on $\mathcal{M}_+$. If $\Phi(\mathbf{M}_1) = \infty$ of $\Phi(\mathbf{M}_0) = \infty$ this inequality is evident. □

Next we shall consider the algorithm proposed in Section 1. Let $\mathbf{M}_n$ be as in (14), $\mathbf{M}(\xi_n) = \mathbf{M}_n/n$, and write $\mathbf{f}_n$ for $\mathbf{f}(x_n)$. Let

$$\mathrm{d}(x, \mathbf{M}(\xi)) := \mathbf{f}'(x)\,\mathbf{M}^{-1}(\xi)\,\mathbf{f}(x)\,.$$

Then, according to the relation $\mathbf{M}_{n+1}\mathbf{M}_{n+1}^{-1} = \mathbf{I}$, we can prove that

$$\text{(33)} \qquad \mathbf{M}_{n+1}^{-1} = \mathbf{M}_n^{-1} - [1 - \mathrm{d}(x_{n+1}, \mathbf{M}(\xi_n))/n]^{-1}\, \mathbf{M}_n^{-1} \boldsymbol{f}_{n+1} \boldsymbol{f}'_{n+1} \mathbf{M}_n^{-1}\,.$$

Now use the symmetry of $\gamma_{i_1,\dots,i_k}$ and (33); then we get

$$\text{(34)} \qquad \gamma'[\mathbf{M}^{-1}(\xi_{n+1})]^{\otimes k}\,\gamma =$$
$$= \left(\frac{n+1}{n}\right)^k \sum_{i=0}^{k} \binom{k}{i} \left[\frac{-1}{n + d(x_{n+1}, \mathbf{M}(\xi_n))}\right]^i \varepsilon^{(i)\prime}[\mathbf{M}^{-1}(\xi_n)]^{\otimes(k+i)}\, \boldsymbol{\eta}^{(i)}$$

where

$$\varepsilon^{(i)} = \begin{pmatrix} \gamma \\ \boldsymbol{f}_{n+1} \\ \vdots \\ \boldsymbol{f}_{n+1} \end{pmatrix}, \qquad \boldsymbol{\eta}^{(i)} = \begin{pmatrix} \boldsymbol{f}_{n+1} \\ \vdots \\ \boldsymbol{f}_{n+1} \\ \gamma \end{pmatrix},$$

the component $\boldsymbol{f}_{n+1}$ being repeated i times. Using the Schwarz inequality

$$[\sum_{i,j=1}^{m} (\boldsymbol{f}_{n+1})_i \{\mathbf{M}^{-1}(\xi_n)\}_{ij}\, \gamma_{j,i_2,\dots,i_k}]^2 \leqq$$
$$\leqq \boldsymbol{f}'_{n+1}\mathbf{M}^{-1}(\xi_n)\, \boldsymbol{f}_{n+1} \sum_{i,j=1}^{m} \gamma_{i,i_2,\dots,i_k} \{\mathbf{M}^{-1}(\xi_n)\}_{ij}\, \gamma_{j,j_2,\dots,j_k}$$

we obtain

$$\text{(35)} \qquad \varepsilon^{(i)\prime}[\mathbf{M}^{-1}(\xi_n)]^{\otimes(k+i)}\, \boldsymbol{\eta}^{(i)} \leqq$$
$$\leqq (\boldsymbol{f}'_{n+1}\mathbf{M}^{-1}(\xi_n)\, \boldsymbol{f}_{n+1})^i\, \gamma'[\mathbf{M}^{-1}(\xi_n)]^{\otimes k}\, \gamma\,.$$

Denote

$$\partial\Phi(\mathbf{M}, \boldsymbol{f}\boldsymbol{f}') := \frac{\partial}{\partial\beta}\, \Phi[(1-\beta)\,\mathbf{M} + \beta \boldsymbol{f}\boldsymbol{f}']\,|_{\beta=0^+}$$

and use the fact that $\lim_{n\to\infty} \mathrm{d}(x_{n+1}, \mathbf{M}(\xi_n))/n = 0$ (cf. [6]). Using (34) and taking (35) into account we get

$$\text{(36)} \qquad \log \Phi[\mathbf{M}(\xi_{n+1})] - \log \Phi[\mathbf{M}(\xi_n)] \leqq$$
$$\leqq \log 2 \left\{1 + \partial\Phi[\mathbf{M}(\xi_n), \boldsymbol{f}_{n+1}\boldsymbol{f}'_{n+1}] + \sum_{i=2}^{k} \binom{k}{i} \frac{\mathrm{d}^2(x_{n+1}, \mathbf{M}(\xi_n))}{n^i}\right\}.$$

The algorithm in Section 1 leads to an $\boldsymbol{f}_{n+1}$ that minimizes $\partial\Phi[\mathbf{M}(\xi_n), \boldsymbol{f}\boldsymbol{f}']$. Since Φ is convex, $\partial\Phi[\mathbf{M}(\xi_n), \boldsymbol{f}_{n+1}\boldsymbol{f}'_{n+1}] < 0$ whenever ξ_n is not Φ-optimal. If at least one of the assumptions of Proposition 8 is valid then the inequality

$$\lim_{n\to\infty} \log \Phi[\mathbf{M}(\xi_{n+1})] - \log \Phi[\mathbf{M}(\xi_{n_0})] \leqq$$
$$\leqq \log 2 \prod_{n=n_0}^{\infty} \left\{1 + \partial\Phi[\mathbf{M}(\xi_n), \boldsymbol{f}_{n+1}\boldsymbol{f}'_{n+1}] + \sum_{i=2}^{k} \binom{k}{i} \frac{\mathrm{d}^i(x_{n+1}, \xi_n)}{n^i}\right\}$$

necessarily entails

$$0 \geqq \sum_{n=n_0}^{\infty} \partial \Phi[\mathbf{M}(\xi_n), \mathbf{f}_{n+1}\mathbf{f}'_{n+1}] > -\infty .$$

Hence

$$\lim_{n\to\infty} \partial \Phi[\mathbf{M}(\xi_n), \mathbf{f}_{n+1}\mathbf{f}'_{n+1}] = 0 .$$

On the other hand, again from the convexity of Φ, it follows that $\partial\Phi[\mathbf{M}(\xi_n), \mathbf{f}_{n+1}\mathbf{f}'_{n+1}] > -\varepsilon$ implies $\Phi[\mathbf{M}(\xi_n)] \leqq \inf_{\xi} \Phi[\mathbf{M}(\xi)] + \varepsilon$. We have just proved the following

Proposition 7. The iterative procedure (13) converges to a Φ-optimal design, i.e.

$$\lim_{n\to\infty} \Phi[\mathbf{M}(\xi_n)] = \inf_{\xi} \Phi[\mathbf{M}(\xi)] .$$

Finally, let us prove Proposition 8.

a) Suppose that the set X is finite. Take $x \in X$ and consider the constant (finite or unfinite) subsequence of $\{\mathbf{f}_n\}_{n=1}^{\infty}$:

$$\mathbf{f}_{n_k} = \mathbf{f}(x) ; \quad (k = 1, 2, \ldots, L(x)) .$$

Then

$$\sum_{k=1}^{L(x)} [\mathbf{f}'_{n_k}(\sum_{i=1}^{n_k-1} \mathbf{f}_i\mathbf{f}'_i)^{-1} \mathbf{f}_{n_k}]^2 \leqq$$

$$\leqq \sum_{k=1}^{L(x)} [\mathbf{f}'(x)(\mathbf{M}_{n_0} + (k-1)\mathbf{f}(x)\mathbf{f}'(x))^{-1}\mathbf{f}(x)]^2 \leqq \sum_{k=1}^{L(x)} \frac{1}{(k-1)^2} < \infty .$$

Thus

$$\sum_{n=n_0}^{\infty} [\mathbf{f}'_{n+1}(\sum_{i=1}^{n} \mathbf{f}_i\mathbf{f}'_i)^{-1} \mathbf{f}_{n+1}]^2 \leqq \sum_{x\in X} \sum_{k=1}^{L(x)} \frac{1}{(k-1)^2} < \infty .$$

b) Suppose that the assumptions b) or c) in Proposition 8 are satisfied. Using (33) we may prove that

$$\mathbf{f}'_{n+1}\mathbf{M}_n^{-1}\mathbf{f}_{n+1} = \frac{\mathbf{f}'_{n+1}\mathbf{M}_{n+1}^{-1}\mathbf{f}_{n+1}}{1-\mathbf{f}'_{n+1}\mathbf{M}_{n+1}^{-1}\mathbf{f}_{n+1}} .$$

It follows that the series (15) converges if and only if

(37) $$\sum_{n=n_0+1}^{\infty} [\mathbf{f}'_n\mathbf{M}_n^{-1}\mathbf{f}_n]^2 < \infty .$$

Let us introduce a mapping $\varphi : (0, \infty) \mapsto R^m$:

$$\varphi(t) = \mathbf{f}(x_n) ; \quad (t \in (n-1, n\rangle , \quad n = 1, 2, \ldots)$$

and denote

(38) $$\mathbf{M}_t = \int_0^t \varphi(u)\,\varphi'(u)\,\mathrm{d}u .$$

Evidently

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \det \left(\int_0^t \varphi(u)\, \varphi'(u)\, \mathrm{d}u \right) = -\varphi'(t)\, \mathbf{M}_t^{-1}\, \varphi(t)\,.$$

Hence in order to prove (37) we have merely to prove that the integral

$$J := \int_{n_0+1}^{\infty} \left[\frac{\mathrm{d}}{\mathrm{d}t} \log \det \mathbf{M}_t \right]^2 \mathrm{d}t$$

is finite. Denote

(39) $$u(t) := \det \mathbf{M}_t\,.$$

Then

(40) $$J = \int_{u(n_0+1)}^{\infty} \varphi'(t)\, \mathbf{M}_t^{-1}\, \varphi(t)\big|_{t^{-1}(u)} \frac{\mathrm{d}u}{u}\,.$$

The set X is compact, hence $\|\varphi(t)\| \leqq C$ for some $C < \infty$. We may write

$$\varphi'(t)\, \mathbf{M}_t^{-1} \varphi(t) \leqq \frac{C^2}{\det \mathbf{M}(t)} \sum_{i,j=1}^{m} \left| \mathbf{M}_t^{(i,j)} \right|,$$

where $\mathbf{M}_t^{(i,j)} := \det \mathbf{M}_t \{\mathbf{M}_t^{-1}\}_{ij}$ may be written as

$$\mathbf{M}_t^{(i,j)} = \sum_{\sigma} \varepsilon_\sigma \prod_{\substack{l \neq i \\ \sigma(l) \neq j}} \int_0^t \varphi_l(u)\, \varphi_{\sigma l}(u)\, \mathrm{d}u\,.$$

Hence

$$\left| \mathbf{M}_t^{(i,j)} \right| \leqq (m-1)!\, C^{2(m-1)} t^{(m-1)}\,.$$

It follows that

$$\varphi'(t)\, \mathbf{M}_t^{-1} \varphi(t) \leqq \frac{C^{2m}(m-1)!}{t \det (\mathbf{M}_t/t)}\,.$$

Setting

$$u(t) = t^m \det (\mathbf{M}_t/t)$$

in this inequality, we obtain

$$\varphi'(t)\, \mathbf{M}_t^{-1} \varphi(t) \leqq \frac{C^{2m}(m-1)!}{[\liminf_{n \to \infty} \det (\mathbf{M}_n/n)]^{(1-1/m)}} \cdot \frac{1}{u^{1/m}(t)}\,.$$

It follows that

$$J \leqq \mathrm{const} \int u^{-(1+1/m)}\, \mathrm{d}u < \infty\,.$$

Analogically, from assumption c) it follows that

$$\varphi'(t)\, \mathbf{M}_t^{-1} \varphi(t) \leqq \mathrm{const.} \frac{[\det \mathbf{M}(\xi^*)]^{\beta/m}}{[u(t)]^{\beta/m}}\,,$$

where ξ^* is the D-optimum design. The integral J is finite again. □

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