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On Extrapolation in Multiple ARMA Processes

Jiří Anděl

We consider a p-dimensional process $\{X_t\}$. If one-step ahead extrapolation is not precise enough in this process, we can try to improve it using a related q-dimensional process $\{Y_t\}$. It is investigated, when $\{Y_t\}$ really improves the extrapolation in $\{X_t\}$ under the assumption that $\{(X_t', Y_t')'\}$ is an ARMA process.

1. INTRODUCTION

We shall investigate multiple stationary discrete processes with zero expectation. If we have such a p-dimensional process $\{X_t\}$ and if its values are known only for $t \leqq s - 1$ (where s is a given point of time), then one of the most important problems is to calculate the best linear extrapolation $\hat{X}_s$ of the vector X_s. The extrapolation $\hat{X}_s$ can be calculated using methods described in the Rozanov's book [1] or by a well-known recurrent procedure based on the Kalman filter. The quality of $\hat{X}_s$ is measured by the residual variance matrix

$$\Delta_X = \mathrm{E}(X_s - \hat{X}_s)(X_s - \hat{X}_s)' .$$

If the diagonal elements of Δ_X are too large, the extrapolation is not good enough and it is necessary to look how to improve it. There is a possibility to try to calculate the best (generally non-linear) extrapolation. Nevertheless, even if we do not take into account the theoretical and practical problems connected with its evaluation, some numerical results show that the improvement can be hardly substantial (see [2]). It remains the only promising possibility to find another (say q-dimensional) process $\{Y_t\}$ which is correlated with our process $\{X_t\}$. Denote $W_t = (X_t', Y_t')'$. It is clear that the best linear extrapolation $\bar{X}_s$ of X_s based on $W_{s-1}, W_{s-2}, \ldots$ cannot be worse than $\hat{X}_s$. More precisely, if we denote

$$\Delta_I = \mathrm{E}(X_s - \bar{X}_s)(X_s - \bar{X}_s)' ,$$

then it can be proved that $\Delta_X - \Delta_I$ is a positive semidefinite matrix.

In the case that $\{X_t\}$ and $\{Y_t\}$ are uncorrelated, no improvement of linear extrapolation is possible and we have $\bar{X}_s = \hat{X}_s$. On the other side, when $\{X_t\}$ and $\{Y_t\}$ are extremely correlated it can happen that also the variables Y_t $(t \leqq s - 1)$ carry no additional information concerning the extrapolation of X_s. Such a situation occurs, for example, when $Y_t = X_t$ for all t, or when $Y_t = X_{t-k}$ for $k \geqq 1$. At first sight it seems that if $\{W_t\}$ is described by a reasonable model (such as an invertible ARMA model) then $\{Y_t\}$ should always improve the original extrapolation $\hat{X}_s$. Surprisingly, this is not true. The conditions for the equality $\bar{X}_s = \hat{X}_s$ were derived in [3] for the case that $\{X_t\}$ and $\{Y_t\}$ are univariate and $\{W_t\}$ is a two-dimensional invertible ARMA (n, m) process. In this paper we generalize these conditions to multiple processes $\{X_t\}$ and $\{Y_t\}$. Some other methods for solving problems of this kind are published in [4] and [5].

2. AUXILIARY ASSERTIONS

The methods used for obtaining the main results contained in Section 3 are based on the matrix theory and on some properties of the matrix of spectral densities. It seems to be convenient to prepare some auxiliary assertions in advance.

Theorem 1. Let $\begin{Vmatrix} K, & L \\ M, & N \end{Vmatrix}$ be a square regular matrix with square blocks K and N. If N is regular, then $K - LN^{-1}M$ is aiso regular and

$$\begin{Vmatrix} K, & L \\ M, & N \end{Vmatrix}^{-1} = \begin{Vmatrix} (K - LN^{-1}M)^{-1} & -(K - LN^{-1}M)^{-1} LN^{-1} \\ -N^{-1}M(K - LN^{-1}M)^{-1}, & N^{-1} + N^{-1}M(K - LN^{-1}M)^{-1} LN^{-1} \end{Vmatrix}.$$

Proof is omitted, because the assertion is well-known.

Theorem 2. Let $A_0, \ldots, A_n$ be $p \times p$ matrices such that

$$\operatorname{Det}\left(\sum_{k=0}^{n} A_k z^k\right) \neq 0 \quad \text{for} \quad |z| \leqq 1 .$$

Let $B_0, \ldots, B_m$ be $p \times q$ matrices, where $B_0 \neq 0$. Denote $\{Z_t\}$ a q-dimensional white noise, i.e. a process with

$$\mathrm{E}Z_t = 0 , \quad \operatorname{Var} Z_t = I , \quad \operatorname{Cov}(Z_s, Z_t) = 0 \quad \text{for} \quad s \neq t ,$$

where I is the unit matrix. Then there exists a stationary process $\{X_t\}$ given by

$$\sum_{k=0}^{n} A_k X_{t-k} = \sum_{j=0}^{m} B_j Z_{t-j} \tag{1}$$

such that each component of X_t belongs to the Hilbert space H_t generated by all components of vectors Z_s for $s \leqq t$. The process $\{X_t\}$ is determined uniquely. Put

$$A = \sum_{k=0}^{n} A_k \, e^{-ik\lambda}, \quad B = \sum_{j=0}^{m} B_j \, e^{-ij\lambda}.$$

Then the matrix $f(\lambda)$ of the spectral densities of the process $\{X_t\}$ is given by the formula

$$f(\lambda) = (2\pi)^{-1} A^{-1} B B^* A^{*-1}, \tag{2}$$

where the symbol * denotes the transposition and complex conjugation.

Proof. The assertion is well-known in the case when the matrices B_j are of type $p \times p$. Our proof will be similar to that in the mentioned special case. Denote

$$A(z) = \sum_{k=0}^{n} A_k z^k, \quad B(z) = \sum_{j=0}^{m} B_j z^j.$$

It follows from our assumptions that the function $\{\text{Det}\,[A(z)]\}^{-1}$ is analytic on the set $\{z : |z| \leqq 1\}$ and thus it can be expanded into a power series, which converges absolutely for $|z| = 1$. The elements of the both matrices Adj $[A(z)]$ and $B(z)$ are polynomials in z. From

$$[A(z)]^{-1} B(z) = \{\text{Det}\,[A(z)]\}^{-1} \,\text{Adj}\,[A(z)] \,.\, B(z)$$

we can see that

$$[A(z)]^{-1} B(z) = \sum_{s=0}^{\infty} D_s z^s, \tag{3}$$

where the matrices D_s are of type $p \times q$. If the elements of D_s are d^s_{uv}, then

$$\sum_{s=0}^{\infty} |d^s_{uv}| < \infty \tag{4}$$

obviously holds for every pair (u, v).

Put $B_j = 0$ for $j > m$. Then (3) implies

$$\sum_{k=0}^{\min(h,n)} A_k D_{h-k} = B_h, \quad h = 0, 1, 2, \ldots \tag{5}$$

We can define X_t by

$$X_t = \sum_{s=0}^{\infty} D_s Z_{t-s}, \tag{6}$$

because every component in (6) converges in the quadratic mean with respect to (4). Using (5) it can be proved that X_t defined in (6) satisfies relation (1). The condition concerning the space H_t is fulfilled automatically. It is not difficult to see that (5) is necessary for X_t of type (6) to be a solution of (1).

Denote Z the vector-valued random measure corresponding to the process $\{Z_t\}$. From (6) and (3) we have

$$X_t = \int_{-\pi}^{\pi} e^{it\lambda} A^{-1} B \, dZ(\lambda) .$$

Since the process $\{Z_t\}$ possesses the matrix of spectral densities $(2\pi)^{-1} I$, we obtain

$$E X_{s+t} X_s' = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{it\lambda} A^{-1} B B^* A^{*-1} \, d\lambda .$$

From here we see that the matrix $f(\lambda)$ of spectral densities of the process $\{X_t\}$ exists and equals to (2). □

Theorem 3. Let $\{X_t\}$ be the process defined in Theorem 2. Denote

$$\Delta_X = E(X_s - \hat{X}_s)(X_s - \hat{X}_s)' , \quad \Delta_0 = A_0^{-1} B_0 B_0' A_0'^{-1} .$$

Then the matrix $\Delta_X - \Delta_0$ is positive semidefinite. If the equality $\Delta_X = \Delta_0$ holds, then there exist $p \times p$ matrices $C_0, \ldots, C_m$ such that

$$B_j = C_j B_0 , \quad j = 0, 1, \ldots, m . \tag{7}$$

If there exist matrices $C_0, \ldots, C_m$ such that (7) holds and if the condition

$$\operatorname{Det} \left(\sum_{j=0}^{m} C_j z^j \right) \neq 0 \quad \text{for} \quad |z| \leqq 1 \tag{8}$$

is fulfilled, then $\Delta_X = \Delta_0$.

Proof. Denote

$$Q_s = \sum_{j=1}^{m} A_0^{-1} B_j Z_{s-j} - \sum_{k=1}^{n} A_0^{-1} A_k X_{s-k} - \hat{X}_s .$$

Because $\hat{X}_s \in H_{s-1}$, we have $\operatorname{Cov}(Q_s, Z_s) = 0$. From

$$X_s = \hat{X}_s + Q_s + A_0^{-1} B_0 Z_s$$

we obtain

$$\Delta_X - \Delta_0 = E Q_s Q_s'$$

and clearly $\Delta_X - \Delta_0$ must be a positive semidefinite matrix.

Let H_{s-1}^0 be the Hilbert space generated by all elements of the random vectors

$$\begin{array}{llll} A_0^{-1} B_0 Z_{s-1}, & & & \\ A_0^{-1} B_0 Z_{s-2}, & A_0^{-1} B_1 Z_{s-2}, & & \\ \ldots\ldots\ldots\ldots\ldots\ldots, & & & \\ A_0^{-1} B_0 Z_{s-m}, & A_0^{-1} B_1 Z_{s-m}, & , \ldots, A_0^{-1} B_{m-1} Z_{s-m} , & \end{array}$$

$$A_0^{-1}B_0Z_{s-m-1},\ A_0^{-1}B_1Z_{s-m-1}, \dots, A_0^{-1}B_mZ_{s-m-1},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$A_0^{-1}B_0Z_{s-2m},\quad A_0^{-1}B_1Z_{s-2m},\quad \dots, A_0^{-1}B_mZ_{s-2m},$$
$$X_{s-m-1},\ X_{s-m-2},\ X_{s-m-3},\quad \dots$$

It is clear that $H_{s-1} \subset H^0_{s-1}$. The equality $\Delta_X = \Delta_0$ holds if and only if

$$\sum_{j=1}^{m} A_0^{-1}B_jZ_{s-j} \in H_{s-1}.$$

Therefore, the condition

$$\sum_{j=1}^{m} A_0^{-1}B_jZ_{s-j} \in H^0_{s-1} \tag{9}$$

is necessary for $\Delta_X = \Delta_0$. Since the vectors Z_t are uncorrelated, (9) holds if and only if there exist $p \times p$ matrices E_{rs} such that

$$A_0^{-1}B_1 = E_{11}A_0^{-1}B_0,$$
$$A_0^{-1}B_2 = E_{21}A_0^{-1}B_0 + E_{22}A_0^{-1}B_1,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$A_0^{-1}B_m = E_{m1}A_0^{-1}B_0 + \dots + E_{mm}A_0^{-1}B_{m-1}.$$

If we put

$$E_0 = I,$$
$$E_1 = E_{11},$$
$$E_2 = E_{21} + E_{22}E_1,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$E_m = E_{m1} + E_{m2}E_1 + \dots + E_{mm}E_{m-1},$$

then

$$A_0^{-1}B_j = E_jA_0^{-1}B_0, \quad j = 0, 1, \dots, m. \tag{10}$$

Denote

$$C_j = A_0E_jA_0^{-1}, \quad j = 0, 1, \dots, m.$$

Then condition (10) is equivalent to

$$B_j = C_jB_0, \quad j = 0, 1, \dots, m. \tag{11}$$

It is proved that condition (7) is necessary for $\Delta_X = \Delta_0$.

Now, we shall assume that conditions (7) and (8) are fulfilled. Then (1) is equivalent to

$$\sum_{k=0}^{n} A_kX_{t-k} = \sum_{j=0}^{m} C_j\xi_{t-j}, \tag{12}$$

where

$$\xi_{t-j} = B_0Z_{t-j} \quad \text{for} \quad j = 0, 1, \dots, m.$$

Using the same method as in the proof of Theorem 2 we can derive from assumption (8) that there exist matrices S_h $(h = 0, 1, 2, \ldots)$ with elements s^h_{uv} such that

$$(13) \qquad \xi_s = \sum_{h=0}^{\infty} S_h X_{s-h}$$

and

$$\sum_{h=0}^{\infty} |s^h_{uv}| < \infty \quad \text{for all pairs } (u, v) .$$

From (12) we get

$$(14) \qquad X_s = X_s^0 + A_0^{-1} B_0 Z_s ,$$

where

$$X_s^0 = -\sum_{k=1}^{n} A_0^{-1} A_k X_{s-k} + \sum_{j=1}^{m} A_0^{-1} C_j \xi_{s-j} .$$

Obviously $Z_s \perp H_{s-1}$. Further, $X_s^0 \in H_{s-1}$ with respect to (13). This gives $X_s^0 = \hat{X}_s$. Then, of course, we have from (14) that $\Delta_X = \Delta_0$. □

The real applications are based on the following modification of the two previous theorems.

Theorem 4. Let $\{\eta_t\}$ and $\{\zeta_t\}$ be uncorrelated white noises with r and v components, respectively. Let $A_0, \ldots, A_n$ be $p \times p$ matrices, $S_0, \ldots, S_m$ be $p \times r$ matrices and $T_0, \ldots, T_m$ be $p \times v$ matrices. Assume that

$$\operatorname{Det}\left(\sum_{k=0}^{n} A_k z^k\right) \neq 0 \quad \text{for} \quad |z| \leqq 1$$

and that at least one of the matrices S_0 and T_0 is different from the zero matrix. Then there exists uniquely a process $\{X_t\}$ such that

$$(15) \qquad \sum_{k=0}^{n} A_k X_{t-k} = \sum_{j=0}^{m} S_j \eta_{t-j} + \sum_{j=0}^{m} T_j \zeta_{t-j}$$

and that each element of X_t belongs to the Hilbert space H_t generated by all elements of η_s and ζ_s for $s \leqq t$. The process $\{X_t\}$ possesses the matrix of spectral densities

$$(16) \qquad f(\lambda) = (2\pi)^{-1} A^{-1}(SS^* + TT^*) A^{*-1} ,$$

where

$$A = \sum_{k=0}^{n} A_k \, e^{-ik\lambda} , \quad S = \sum_{j=0}^{m} S_j \, e^{-ij\lambda} , \quad T = \sum_{j=0}^{m} T_j \, e^{-ij\lambda} .$$

Let $\hat{X}_s$ be the best linear extrapolation of X_s based on $X_{s-1}, X_{s-2}, \ldots$ Denote

$$\Delta_X = \mathrm{E}(X_s - \hat{X}_s)(X_s - \hat{X}_s)' , \quad \Delta_0 = A_0^{-1}(S_0 S_0' + T_0 T_0') A_0'^{-1} .$$

Then $\Delta_X - \Delta_0$ is a positive semidefinite matrix. If $\Delta_X = \Delta_0$, then there exist $p \times p$ matrices $C_0, \ldots, C_m$ such that the conditions

$$(S_j, T_j) = C_j(S_0, T_0), \quad j = 0, 1, \ldots, m, \tag{17}$$

are fulfilled. If there exist $p \times p$ matrices $C_0, \ldots, C_m$ such that (17) holds and if

$$\operatorname{Det}\left(\sum_{j=0}^{m} C_j z^j\right) \neq 0 \quad \text{for} \quad |z| \leqq 1, \tag{18}$$

then $\Delta_X = \Delta_0$.

Proof. The assertion follows from Theorem 2 and Theorem 3, if we put

$$B_j = (S_j, T_j), \quad Z_t = (\eta_t', \zeta_t')'.$$

□

3. WHEN THE EXTRAPOLATION CANNOT BE IMPROVED

We shall consider a p-dimensional process $\{X_t\}$ and a q-dimensional process $\{Y_t\}$. Put $r = p + q$ and $W_t = (X_t', Y_t')'$.

Theorem 5. Let $\{W_t\}$ be defined by

$$\sum_{k=0}^{n} A_k W_{t-k} = \sum_{j=0}^{m} B_j Z_{t-j} \tag{19}$$

where A_k are $r \times r$ matrices such that

$$\operatorname{Det}\left(\sum_{k=0}^{n} A_k z^k\right) \neq 0 \quad \text{for} \quad |z| \leqq 1 \tag{20}$$

and B_j are $r \times v$ matrices, $B_0 \neq 0$; $\{Z_t\}$ is a v-dimensional white noise. Let each element of W_t belong to the Hilbert space generated by elements of Z_s for $s \leqq t$. Assume that $p \leqq v$. Define matrices K, L, M, N, P, Q, R, S by

$$\sum_{k=0}^{n} A_k \, e^{-ik\lambda} = \begin{Vmatrix} K, & L \\ M, & N \end{Vmatrix}, \quad \sum_{j=0}^{n} B_j \, e^{-ij\lambda} = \begin{Vmatrix} P, & Q \\ R, & S \end{Vmatrix},$$

where K and P are $p \times p$ blocks. If N is regular for all $\lambda \in \langle -\pi, \pi \rangle$ then $\{X_t\}$ possesses the matrix of spectral densities

$$f_{XX}(\lambda) = (2\pi)^{-1} (K - LN^{-1}M)^{-1} \left[(P - LN^{-1}R)(P - LN^{-1}R)^* + (Q - LN^{-1}S)(Q - LN^{-1}S)^*\right] (K - LN^{-1}M)^{*-1}. \tag{21}$$

Proof. Condition (20) ensures that the matrix $A = \sum A_k \, e^{-ik\lambda}$ is regular. Because N is assumed to be also regular, the matrix $K - LN^{-1}M$ is regular (see Theorem 1). The matrix $f_{XX}(\lambda)$ is the left-hand upper corner in the matrix $f(\lambda)$ which is given in (2). We apply Theorem 1 to A^{-1} and A^{*-1} and it leads to (21). □

Theorem 6. Assume that the conditions of Theorem 5 are fulfilled. Denote $\nu = \operatorname{Det} N$, $N_0 = \operatorname{Adj} N$. Define matrices F_k, G_j and H_j (not depending on λ) of the type $p \times p$, $p \times p$ and $p \times q$, respectively, by formulas

$$\nu K - LN_0M = \sum_{k=0}^{n(q+1)} F_k \,\mathrm{e}^{-ik\lambda}\,, \tag{22}$$

$$\nu P - LN_0R = \sum_{j=0}^{nq+m} G_j \,\mathrm{e}^{-ij\lambda}\,, \tag{23}$$

$$\nu Q - LN_0S = \sum_{j=0}^{nq+m} H_j \,\mathrm{e}^{-ij\lambda}\,. \tag{24}$$

Introduce blocks $K(z)$, $L(z)$, $M(z)$ and $N(z)$ by

$$\sum_{k=0}^{n} A_k z^k = \left\| \begin{matrix} K(z), & L(z) \\ M(z), & N(z) \end{matrix} \right\|,$$

where $K(z)$ is of the type $p \times p$. Assume that $\operatorname{Det}[N(z)] \neq 0$ for $|z| \leqq 1$. Let $\{\eta_t\}$ and $\{\zeta_t\}$ be uncorrelated p-dimensional and q-dimensional white noises, respectively. Then

$$\operatorname{Det}\Big(\sum_{k=0}^{n(q+1)} F_k z^k\Big) \neq 0 \quad \text{for} \quad |z| \leqq 1 \tag{25}$$

and the process $\{X_t\}$ defined by

$$\sum_{k=0}^{n(q+1)} F_k X_{t-k} = \sum_{j=0}^{nq+m} G_j \eta_{t-j} + \sum_{j=0}^{nq+m} H_j \zeta_{t-j} \tag{26}$$

such that elements of X_t belong to the Hilbert space generated by elements of η_s and ζ_s for $s \leqq t$, possesses the matrix $f_{XX}(\lambda)$ of spectral densities which is given in (21).

Proof. We have for $|z| \leqq 1$

$$\left| \begin{matrix} K(z), & L(z) \\ M(z), & N(z) \end{matrix} \right| = \operatorname{Det}[N(z)] \,.\, \operatorname{Det}\{K(z) - L(z)[N(z)]^{-1} M(z)\}\,. \tag{27}$$

The left-hand side of (27) is non-zero in view of (20) and thus

$$\operatorname{Det}\{K(z) - L(z)[N(z)]^{-1} M(z)\} \neq 0 \quad \text{for} \quad |z| \leqq 1\,.$$

Put

$$\nu(z) = \operatorname{Det}[N(z)]\,, \quad N_0(z) = \operatorname{Adj}[N(z)]\,.$$

From

$$[N(z)]^{-1} = [\nu(z)]^{-1} N_0(z)$$

we have

$$\operatorname{Det}[\nu(z) K(z) - L(z) N_0(z) M(z)] \neq 0 \quad \text{for} \quad |z| \leqq 1\,.$$

This is equivalent to (25). From formula (16) in Theorem 4 we obtain that the matrix $f_{XX}(\lambda)$ of spectral densities is

$$f_{XX}(\lambda) = (2\pi)^{-1} (vK - LN_0M)^{-1} [(vP - LN_0R)(vP - LN_0R)^* + + (vQ - LN_0S)(vQ - LN_0S)^*] (vK - LN_0M)^{*-1} ,$$

which can be arranged to form (21). □

Theorem 7. Let $\{W_t\}$ be an invertible r-dimensional ARMA process defined by

$$\sum_{k=0}^{n} A_k W_{t-k} = \sum_{j=0}^{m} B_j Z_{t-j} ;$$

therefore, A_k and B_j are $r \times r$ matrices such that

$$(29) \qquad \operatorname{Det} (\sum_{k=0}^{n} A_k z^k) \neq 0 , \quad \operatorname{Det} (\sum_{j=0}^{m} B_j z^j) \neq 0 \quad \text{for} \quad |z| \leqq 1 .$$

Assume that

$$(30) \qquad \operatorname{Det} [N(z)] \neq 0 \quad \text{for} \quad |z| \leqq 1 .$$

Let G_j and H_j be matrices defined in (23) and (24). Then the equality $\Delta_X = \Delta_I$ holds if and only if there exist $p \times p$ matrices $D_0, D_1, \ldots, D_{nq+m}$ such that

$$(31) \qquad (G_j, H_j) = D_j(G_0, H_0) \quad \text{for} \quad j = 0, 1, \ldots, nq + m .$$

Proof. Denote $\Delta_W = \mathrm{E}(W_s - \hat{W}_s)(W_s - \hat{W}_s)'$. At the beginning we shall prove that

$$(32) \qquad \Delta_W = A_0^{-1} B_0 B_0' A_0'^{-1} .$$

If we put $C_j = B_j B_0^{-1}$, we have $B_j = C_j B_0$ and (29) implies

$$\operatorname{Det} (\sum_{j=0}^{m} C_j z^j) \neq 0 \quad \text{for} \quad |z| \leqq 1 .$$

Formula (32) follows from Theorem 3.

The matrix Δ_I is the upper left-hand corner of the matrix Δ_W. Introduce matrices $P(z)$, $Q(z)$, $R(z)$ and $S(z)$ by

$$\sum_{j=0}^{m} B_j z^j = \begin{Vmatrix} P(z), & Q(z) \\ R(z), & S(z) \end{Vmatrix} ;$$

where $P(z)$ is a $p \times p$ block. We have

$$A_0 = \begin{Vmatrix} K(0), & L(0) \\ M(0), & N(0) \end{Vmatrix} , \quad B_0 = \begin{Vmatrix} P(0), & Q(0) \\ R(0), & S(0) \end{Vmatrix} ,$$

$$F_0 = v(0) K(0) - L(0) N_0(0) M(0) , \quad G_0 = v(0) P(0) - L(0) N_0(0) R(0) ,$$
$$H_0 = v(0) Q(0) - L(0) N_0(0) S(0) .$$

Using Theorem 1 we obtain

$$\Delta_I = F_0^{-1}(G_0 G_0' + H_0 H_0') F_0'^{-1} . \tag{33}$$

The process $\{X_t\}$ introduced in Theorem 5 has the same matrix of spectral densities as the process $\{X_t\}$ defined in (27). Both the processes must have the same properties concerning the linear extrapolation. Theorem 4 says that condition (31) is necessary for $\Delta_X = \Delta_I$. The same condition will be sufficient if we prove that

$$\operatorname{Det}\left(\sum_{j=0}^{nqm+m} D_j z^j\right) \neq 0 \quad \text{for} \quad |z| \leqq 1 .$$

Put

$$G(z) = \sum_{j=0}^{nq+m} G_j z^j , \quad H(z) = \sum_{j=0}^{nq+m} H_j z^j , \quad D(z) = \sum_{j=0}^{nq+m} D_j z^j .$$

With respect to (23) and (24) condition (31) is equivalent to

$$\nu(z) P(z) - L(z) N_0(z) R(z) = D(z) G_0 , \tag{34}$$

$$\nu(z) Q(z) - L(z) N_0(z) S(z) = D(z) H_0 . \tag{35}$$

Now, for brevity, we shall not write the argument z. From Theorem 1 we get

$$\begin{Vmatrix} K, & L \\ M, & N \end{Vmatrix}^{-1} \begin{Vmatrix} P, & Q \\ R, & S \end{Vmatrix} = \tag{36}$$

$$= \begin{Vmatrix} (K - LN^{-1}M)^{-1} (P - LN^{-1}R), & (K - LN^{-1}M)^{-1} (Q - LN^{-1}S) \\ * & * \end{Vmatrix}$$

where $*$ denotes a block which is of no interest for us. Both matrices on the left-hand side of (36) are regular for $|z| \leqq 1$ according to assumption (29). Both of them are of type $(p + q) \times (p + q)$. The first p rows of their product must form a matrix of rank p. Using (34) and (35) we can write this matrix in the form

$$(K - LN^{-1}M)^{-1} (P - LN^{-1}R, Q - LN^{-1}S) =$$
$$= \nu^{-1}(K - LN^{-1}M)^{-1} D(G_0, H_0) .$$

Because $D = D(z)$ is of the type $p \times p$, we see that it must be regular for $|z| \leqq 1$. □

The result will be applied to some special cases.

4. AR(1)

Consider a $(p + q)$-dimensional autoregressive process $\{W_t\}$ defined by

$$A_0 W_t + A_1 W_{t-1} = Z_t , \tag{37}$$

where

$$\operatorname{Det}(A_0 + A_1 z) \neq 0 \quad \text{for} \quad |z| \leqq 1 .$$

Put

$$A_0 = \begin{Vmatrix} A_0^{11}, & A_0^{12} \\ A_0^{21}, & A_0^{22} \end{Vmatrix}, \qquad A_1 = \begin{Vmatrix} A_1^{11}, & A_1^{12} \\ A_1^{21}, & A_1^{22} \end{Vmatrix},$$

$$U = -A_0^{-1}A_1 = \begin{Vmatrix} U_{11}, & U_{12} \\ U_{21}, & U_{22} \end{Vmatrix}, \quad M = A_0^{-1} = \begin{Vmatrix} M_{11}, & M_{12} \\ M_{21}, & M_{22} \end{Vmatrix},$$

where A_0^{11}, A_1^{11}, U_{11} and M_{11} are $p \times p$ blocks. Relation (37) is equivalent to

(38) $$W_t = UW_{t-1} + MZ_t .$$

Theorem 8. Let the matrix $N(z) = A_0^{22} + A_1^{22}z$ be regular for $|z| \leqq 1$. Then $\Delta_X = \Delta_I$ holds if and only if

(39) $$U_{12} = 0 .$$

Condition (39) is equivalent to

(40) $$A_1^{12} - A_0^{12}(A_0^{22})^{-1} A_1^{22} = 0 .$$

Proof. First, we show that (39) and (40) are equivalent. According to Theorem 1 the upper left-hand corner of the matrix $U = -A_0^{-1}A_1$ is

$$U_{12} = -[A_0^{11} - A_0^{12}(A_0^{22})^{-1} A_0^{21}] [A_1^{12} - A_0^{12}(A_0^{22})^{-1} A_1^{22}] .$$

The assumptions imply that A_0 and A_0^{22} are regular. Then $A_0^{11} - A_0^{12}(A_0^{22})^{-1} A_0^{21}$ must be also regular and the equivalence is clear.

Assume that $\Delta_X = \Delta_I$. Then conditions (34) and (35) must be fulfilled. In our case they read

(41) $$\operatorname{Det}(A_0^{22} + A_1^{22}z) I = (\textstyle\sum D_j z^j) G_0 ,$$

(42) $$-(A_0^{12} + A_1^{12}z) \operatorname{Adj}(A_0^{22} + A_1^{22}z) = (\textstyle\sum D_j z^j) H_0 .$$

Because

(43) $$G_0 = (\operatorname{Det} A_0^{22}) I , \quad H_0 = -A_0^{12} \operatorname{Adj} A_0^{22} ,$$

we have from (41)

(44) $$\textstyle\sum D_j z^j = (\operatorname{Det} A_0^{22})^{-1} [\operatorname{Det}(A_0^{22} + A_1^{22}z)] I .$$

Inserting from (44) into (42) we obtain

$$(A_0^{12} + A_1^{12}z) [\operatorname{Det}(A_0^{22} + A_1^{22}z)]^{-1} \operatorname{Adj}(A_0^{22} + A_1^{22}z) =$$
$$= A_0^{12}(\operatorname{Det} A_0^{22})^{-1} \operatorname{Adj} A_0^{22} ,$$

so that

$$A_1^{12} = A_0^{12}(A_0^{22})^{-1} A_1^{22} .$$

Hence, we proved that condition (40) is necessary. It remains to show that it is also sufficient. Define matrices D_j by (44). It ensures that (31) and (32) hold. From here (21) follows. □

5. MA(1)

Let $\{W_t\}$ be defined by

$$W_t = B_0 Z_t + B_1 Z_{t-1}, \tag{45}$$

where

$$\operatorname{Det}(B_0 + B_1 z) \neq 0 \quad \text{for} \quad |z| \leqq 1. \tag{46}$$

Put

$$B_0 = \begin{Vmatrix} B_0^{11}, & B_0^{12} \\ B_0^{21}, & B_0^{22} \end{Vmatrix}, \quad B_1 = \begin{Vmatrix} B_1^{11}, & B_1^{12} \\ B_1^{21}, & B_1^{22} \end{Vmatrix},$$

where B_0^{11} and B_1^{11} are $p \times p$ blocks.

Theorem 9. Let B_0^{11} be regular. Then $\Delta_X = \Delta_I$ holds if and only if the condition

$$B_1^{12} - B_1^{11}(B_0^{11})^{-1} B_0^{12} = 0 \tag{47}$$

is fulfilled.

Proof. Theorem 7 gives that $\Delta_X = \Delta_I$ holds if and only if there exist a $q \times q$ matrix D_1 such that

$$B_1^{11} = D_1 B_0^{11}, \quad B_1^{12} = D_1 B_0^{12}.$$

We assume that B_0^{11} is regular. Then $D_1 = B_1^{11}(B_0^{11})^{-1}$ and $B_1^{12} = D_1 B_0^{12}$ in the case that (47) holds. □

6. ARMA (1,1)

Consider an ARMA (1,1) process $\{W_t\}$ given by

$$A_0 W_t + A_1 W_{t-1} = B_0 Z_t + B_1 Z_{t-1}. \tag{48}$$

We assume that

$$\operatorname{Det}(A_0 + A_1 z) \neq 0, \quad \operatorname{Det}(B_0 + B_1 z) \neq 0 \quad \text{for} \quad |z| \leqq 1. \tag{49}$$

The matrices A_k and B_j will be written in the same block form as above.

Model (48) is overparametrized. Without any loss of generality we shall assume that $A_0 = I$.

Theorem 10. Let $N(z) = I + A_1^{22}z$ be regular for $|z| \leqq 1$. Assume that B_0^{11} is regular. Then $\Delta_X = \Delta_I$ holds if and only if

$$(50) \qquad A_1^{12}[B_0^{22} - B_0^{21}(B_0^{11})^{-1} B_0^{12}] = B_1^{12} - B_1^{11}(B_0^{11})^{-1} B_0^{12},$$

$$(51) \quad A_1^{12}(A_1^{22})^{k-1} [A_1^{22}B_0^{22} - A_1^{22}B_0^{21}(B_0^{11})^{-1} B_0^{12} - B_1^{22} + B_1^{21}(B_0^{11})^{-1} B_0^{12}] = 0$$

for $k = 1, 2, \ldots, q$.

Proof. The equality $\Delta_X = \Delta_I$ holds if and only if conditions (34) and (35) are fulfilled. In our case we have

$$(52) \quad [\mathrm{Det}\,(I + A_1^{22}z)]\,(B_0^{11} + B_1^{11}z) - A_1^{12}z[\mathrm{Adj}\,(I + A_1^{22}z)]\,[B_0^{21} + B_1^{21}z) = \\ = D(z)\,G_0,$$

$$(53) \quad [\mathrm{Det}\,(I + A_1^{22}z)]\,(B_0^{11} + B_1^{11}z) - A_1^{12}z[\mathrm{Adj}\,(I + A_1^{22}z)]\,(B_0^{22} + B_1^{22}z) = \\ = D(z)\,H_0,$$

where $G_0 = B_0^{11}$, $H_0 = B_0^{12}$. From here we have for $z \neq 0$

$$(54) \qquad A_1^{12}(I + A_1^{22}z)^{-1}\,\{[B_0^{22} - B_0^{21}(B_0^{11})^{-1} B_0^{12}] + \\ + [B_1^{22} - B_1^{21}(B_0^{11})^{-1} B_0^{12}]\,z\} = B_1^{12} - B_1^{11}(B_0^{11})^{-1} B_0^{12}.$$

If a square matrix A has all its roots inside the unit circle, then

$$(55) \qquad (I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

(see [6], p. 118). There exists $\varepsilon > 0$ that for $0 < |z| < \varepsilon$ all the roots of $A_1^{22}z$ are inside the unit circle. This follows from the Gershgorin's theorem ([6], p. 415). For $0 < |z| < \varepsilon$ we have from (54) and (55)

$$A_1^{12} \sum_{k=0}^{\infty} (-1)^k\,(A_1^{22})^k\, z^k\{[B_0^{22} - B_0^{21}(B_0^{11})^{-1} B_0^{12}] + \\ + [B_1^{22} - B_1^{21}(B_0^{11})^{-1} B_0^{12}]\,z\} = B_1^{12} - B_1^{11}(B_0^{11})^{-1} B_0^{12}.$$

We compare the coefficients with z^k. For $k = 0$ we get formula (50) and for $k \geqq 1$ formula (51). It remains to prove that if (51) holds for $k = 1, 2, \ldots, q$, then it holds also for $k \geqq q + 1$. Let

$$\varrho(\lambda) = \mathrm{Det}\,(\lambda I - A_1^{22}) = \lambda^q + a_1\lambda^{q-1} + \ldots + a_q$$

be the characteristic polynomial of the matrix A_1^{22}. According to Hamilton-Calley theorem we have

$$(A_1^{22})^q + a_1(A_1^{22})^{q-1} + \ldots + a_q I = 0\,.$$

Multiplying by $(A_1^{22})^j$ for $j \geqq 1$ we see that $(A_1^{22})^{q+j}$ is a linear combination of the matrices $(A_1^{22})^{q+j-1}, \ldots, (A_1^{22})^j$. If (51) holds for $k = 1, 2, \ldots, q$, then by induction it holds also for $k \geqq q + 1$. □

In the case $p = q = 1$ the result can be considerably simplified.

Theorem 11. Let $\{W_t\}$ be a two-dimensional invertible ARMA (1,1) process defined by

$$W_t + BW_{t-1} = CZ_t + DZ_{t-1},$$

where B, C and D are 2×2 matrices with elements b_{ij}, c_{ij} and d_{ij}, respectively. Then $\Delta_X = \Delta_I$ holds if and only if

$$c_{11}d_{12} - c_{12}d_{11} - b_{12}(c_{11}c_{22} - c_{12}c_{21}) = 0,$$

$$b_{12}(c_{11}d_{22} - c_{12}d_{21}) + b_{22}(c_{12}d_{11} - c_{11}d_{12}) = 0.$$

Proof. Theorem 11 follows from Theorem 10. This result was also obtained in [7], Theorem 9. □

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Doc. RNDr. Jiří Anděl, CSc., Matematicko-fyzikální fakulta UK (Faculty of Mathematics and Physics - Charles University), Sokolovská 83, 186 00 Praha 8. Czechoslovakia.