KYBERNETIKA — VOLUME 16 (1980), NUMBER 5

Power Spectrum of the Periodic Group Pulse Process

KAREL VOKURKA*

Power spectrum of the periodic group pulse process is derived for two special cases, first, when the number of pulses in the groups is constant, and second, when the process studied is Poisson periodic. It is shown that the first model yields, as a special case, one-pulse periodic process known for example from pulse modulation systems. The second model (Poisson periodic) may be used for the description of such physical phenomena as cavitation noise, shot and photomultiplier noise under periodic modulation and others. The derived results are illustrated by an example.

1. INTRODUCTION

Many noises are impulsive in nature, that is they may be described as random sequences of pulse-like events. Shot noise, thermal noise, Barkhausen noise, cavitation noise and impulsive noise are but a few examples. Thus random pulse processes are natural to be used as mathematical models for these noises. Here the homogeneous Poisson pulse process plays a prominent role both for the variety of noises it is suitable to model and for the amenability to mathematical treatment. See for example Middleton [1] for a comprehensive survey and Richter and Smits [2] for further references. However, there are many types of noise, such as flicker noise [3], Barkhausen noise [3], [4], acoustically induced cavitation noise [5], [6] etc. where the homogeneous Poisson model is not applicable any more. Some of such noises require models that are taking into account the correlation among different pulse parameters [7]. In some other cases the events are grouped to form pulse clusters so that the mean pulse density is not constant in time. In this paper we derive the power spectrum of such a group pulse process in a special case when the groups follow one another periodically.

* This work was performed while the author was at the Department of Physics — Faculty of Electrical Engineering of the Czech Technical University.

The power spectrum of a periodic group pulse process will be derived for two distinctive distributions of pulses in groups. First, the number of pulses in groups will be taken constant and second, the number of pulses in groups will be governed by Poisson's law. We shall show that under supplementary conditions in this second case the process will be Poisson periodic.

The group pulse processes were studied first in communications [8], [9]. However, in communications the nonoverlapping of pulses is usually required. On the contrary, in physics the overlapping pulses are of primary interest. And this is just the case we shall study here.

2. PERIODIC GROUP PULSE PROCESS

We shall assume a random process $\xi(t)$ which is formed of periodically repeating groups of random pulses (Fig. 1). The pulses may overlap without any restriction. Then we may write for the process $\xi(t)$

$$\xi(t) = \sum_{k=-\infty}^{\infty} \sum_{n=1}^{N_k} f(t - kT_0 - \varepsilon_{kn}, \mathbf{a}_{kn}) . \tag{1}$$

Here $f(t, \mathbf{a})$ is a non-random time function defining the shape of a single pulse, N_k is a random number of pulses in the k-th group, T_0 is a period with which the groups

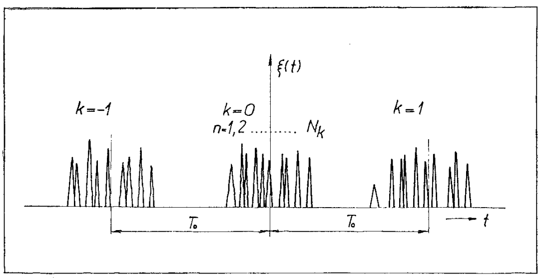

Fig. 1. A possible realization of the random periodic group pulse process.

follow one another, ε_{nk} is a random epoch of the n-th pulse in the k-th group with respect to the centre of the k-th group and finally $\mathbf{a}_{kn}$ is an m-dimensional random vector of m random parameters of the n-th pulse in the k-th group.

Here we shall limit ourselves to such processes where the individual pulses are all statistically independent of one another, the statistical characteristics of the single pulses are time invariant and random epochs ε are statistically independent on other random parameters $\boldsymbol{a}$. Futher, we shall suppose that discrete random variables N_k are mutually indepedent, do not depend anyhow on the pulses and have common distribution. The process $\xi(t)$ is then fully described by probability densities $w_1(\varepsilon)$, $w_1(N)$, $w_m(\boldsymbol{a})$, function $f(t, \boldsymbol{a})$ and period T_0.

In the following section we shall derive an expression for the power spectrum of the process (1). The autocorrelation function of this process could be determined from the power spectrum by applying the Wiener-Khinchine theorem.

3. POWER SPECTRUM

Let us consider a truncated realization $\xi_T(t)$ of the process $\xi(t)$ so that $\xi_T(t) = \xi(t)$ in the interval $(-T/2, +T/2)$ and $\xi_T(t) = 0$ for $|t| > T/2$. Let us futher suppose that the principal interval $(-T/2, +T/2)$ fully spans the $(2K + 1)$ groups of random pulses so that $T = (2K + 1)\, T_0$. Then

$$\xi_T(t) = \sum_{k=-K}^{K} \sum_{n=1}^{N_k} f(t - kT_0 - \varepsilon_{kn}, \boldsymbol{a}_{kn}) . \tag{2}$$

The spectrum of this truncated realization is given by the Fourier transform of (2), hence

$$S_T(\omega) = \sum_{k=-K}^{K} \sum_{n=1}^{N_k} s(\omega, \boldsymbol{a}_{kn})\, \mathrm{e}^{-\mathrm{j}\omega(kT_0 + \varepsilon_{kn})} . \tag{3}$$

Here $s(\omega, \boldsymbol{a}_{kn})$ is the spectrum of a single pulse $f(t, \boldsymbol{a}_{kn})$.

The power spectrum will be calculated from the definition formula

$$\mathcal{W}(\omega) = \lim_{K \to \infty} \frac{1}{(2K + 1)\, T_0} \overline{|S_T(\omega)|^2} . \tag{4}$$

The square of the modulus of $S_T(\omega)$ equals

$$|S_T(\omega)|^2 = S_T(\omega)\, S_T^*(\omega) = \sum_{k=-K}^{K} \sum_{n=1}^{N_k} |s(\omega, \boldsymbol{a}_{kn})|^2 + \tag{5}$$

$$+ \sum_{k=-K}^{K} \sum_{n=1}^{N_k} \sum_{l=-K}^{K} \sum_{m=1}^{N_l} s(\omega, \boldsymbol{a}_{kn})\, \mathrm{e}^{-\mathrm{j}\omega\varepsilon_{kn}} s^*(\omega, \boldsymbol{a}_{lm})\, \mathrm{e}^{+\mathrm{j}\omega\varepsilon_{lm}}\, \mathrm{e}^{-\mathrm{j}\omega(k-l)T_0} .$$

$k \neq l$ and $n \neq m$ simultaneously

Now we shall take the ensemble average of (5) over all possible values of random variables $\boldsymbol{a}_{kn}$, ε_{kn} and N_k. Here we shall limit ourselves to two interesting types of distribution N_k, namely $N_k = N_0 =$ const. and N_k has Poisson distribution. Since the random variables $\boldsymbol{a}_{kn}$, ε_{kn}, and N_k are independent (for different kn or k respectively) and with identical distribution for all kn and k, we may write for the first type:

1) The number of pulses in groups is constant, that is

$$w_1(N_k) = \delta(N_k - N_0) . \tag{6}$$

Then the double series on the right-hand side of (5) contains $(2K + 1) N_0$ equal terms of the type $\overline{|s(\omega, \boldsymbol{a})|^2}$. The four-fold series contains $(2K + 1) 2KN_0^2$ terms of the form $|\overline{s(\omega, \boldsymbol{a})}|^2 |\chi(\omega)|^2 \exp[-j\omega(k - l) T_0]$, where $k \neq l$, and $(2K + 1) . (N_0^2 - N_0)$ terms of the form $|\overline{s(\omega, \boldsymbol{a})}|^2 |\chi(\omega)|^2$. Here $\chi(\omega)$ is a characteristic function of random epoch ε. After rearranging we thus get

$$\overline{|S_T(\omega)|^2} = (2K + 1) N_0 \overline{|s(\omega, \boldsymbol{a})|^2} - (2K + 1) N_0 |\overline{s(\omega, \boldsymbol{a})}|^2 |\chi(\omega)|^2 + \\ + N_0^2 |\overline{s(\omega, \boldsymbol{a})}|^2 |\chi(\omega)|^2 \sum_{k=-K}^{K} \sum_{l=-K}^{K} e^{-j\omega(k-l)T_0} . \tag{7}$$

Now from (4) and (7) we get the power spectrum of the group pulse process with constant number of pulses in groups in the form

$$\mathscr{W}(\omega) = \nu_0 \overline{|s(\omega, \boldsymbol{a})|^2} - \nu_0 |\overline{s(\omega, \boldsymbol{a})}|^2 |\chi(\omega)|^2 + \\ + \nu_0^2 |\overline{s(\omega, \boldsymbol{a})}|^2 |\chi(\omega)|^2 \, 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) . \tag{8}$$

Here $\nu_0 = N_0/T_0$ is the pulse density, $\delta(\omega)$ is the delta function and in deriving (8) we used the identity [8], [9]

$$\lim_{K\to\infty} \frac{1}{(2K + 1)} \sum_{k=-K}^{K} \sum_{l=-K}^{K} e^{-j\omega(k-l)T_0} = \frac{2\pi}{T_0} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) .$$

The power spectrum (8) has both the continuous part (the first two terms on the right hand side of the equation (8)) and the line part (the last term). The second term is of some interest as it is due to restriction on the number of pulses in groups. As we shall see later it completely disappears for the Poisson process.

In a special case when $N_0 = 1$ we get the one-pulse periodic pulse process first studied by Macfarlane [10] and Lawson and Uhlenbeck [11] (see also Middleton [1] and Konovalov and Tarasenko [9]). For example when $f(t, \boldsymbol{a}) = a[U(t) - U(t - \vartheta_0)]$, where a is a random amplitude, $U(t)$ is a unit step function, ϑ_0 is fixed and $w(\varepsilon) = \delta(\varepsilon)$, we obtain the case of PAM (pulse amplitude modulation). Similar simplifications lead to the case of PPM (pulse position modulation) and PDM (pulse duration modulation).

At last, still for $N_0 = 1$, when $w(\varepsilon) = \delta(\varepsilon)$ and $w_m(\boldsymbol{a}) = w_m(\boldsymbol{a}_0) = \delta(a - a_0) \,.\, \delta(\vartheta - \vartheta_0) \ldots$ from (8) we get the power spectrum of a periodic deterministic process in the well known form

$$(9) \qquad \mathscr{W}(\omega) = \frac{1}{T_0^2} |s(\omega, \boldsymbol{a}_0)|^2 \, 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) = 2\pi \sum_{k=-\infty}^{\infty} |c_k|^2 \, \delta(\omega - k\omega_0) \,,$$

where $c_k = (1/T_0) \, s(k\omega_0, \boldsymbol{a}_0)$ are the Fourier series expansion coefficients.

2) Now we shall consider the second type of distribution of random variable N_k, namely when N_k is Poisson distributed with the mean $\bar{N}_0$

$$(10) \qquad P(N_k) = \frac{\bar{N}_0^{N_k}}{N_k!} \, e^{-\bar{N}_0} \quad N_k = 0, 1, 2, \ldots \,.$$

Then from (5) we readily obtain

$$(11) \qquad \overline{|S_T(\omega)|^2} = (2K + 1) \, \bar{N}_0 \overline{|s(\omega, \boldsymbol{a})|^2} + \\ + \bar{N}_0^2 |\overline{s(\omega, \boldsymbol{a})}|^2 \, |\chi(\omega)|^2 \sum_{k=-K}^{K} \sum_{l=-K}^{K} e^{-j\omega(k-l)T_0} \,.$$

Again, from (4) and (11) we get the required power spectrum of the group pulse process with a number of pulses in groups Poisson distributed in the form

$$(12) \qquad \mathscr{W}(\omega) = \bar{\nu}_0 \overline{|s(\omega, \boldsymbol{a})|^2} + \\ + \bar{\nu}_0^2 |\overline{s(\omega, \boldsymbol{a})}|^2 \, |\chi(\omega)|^2 \, 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \,.$$

Here $\bar{\nu}_0 = \bar{N}_0 / T_0$ is the mean pulse density.

If we further assume that the probability that during $(t, t + \mathrm{d}t)$ a pulse occurs is $\nu(t) \, \mathrm{d}t + o(\mathrm{d}t)$, and the probability that more than one pulse occurs is $o(\mathrm{d}t)$ the term $o(\mathrm{d}t)$ denoting a quantity which is of smaller order of magnitude than $\mathrm{d}t$ and $\nu(t) = \nu(t + T_0) \geqq 0$ is a periodic function of time, then it may be easily verified that the number of pulses in an interval $(0, T)$ will be Poisson distributed with the mean $\bar{N}_T = \sum \bar{N}_0$. This is due to the stability and infinite divisibility of Poisson distribution. Let us suppose that the interval $(0, T)$ fully spans the K groups and splits partially the two border groups (Fig. 2). The distribution of the sum of the K inner groups will be a K-fold convolution of the basic distribution (10). Because of the stability of Poisson distribution this will be Poisson distribution again. The two border groups will be also partitioned into four sets with Poisson distribution each. This follows from the infinite divisibility of Poisson distribution. At last, after the convolution of the inner parts of the two border groups with the sum of the K inner groups, where all terms entering into convolution are Poisson distributed, Poisson distribution will be generated again. Hence the process 2) is the Poisson periodic pulse process. Note that $\bar{\nu}(t) = \bar{N}_0\, w(\varepsilon)$ within each period T_0.

The expression (12) has only two terms on its right hand side. These two terms are characteristic of Poisson pulse processes, other types of pulse processes usually

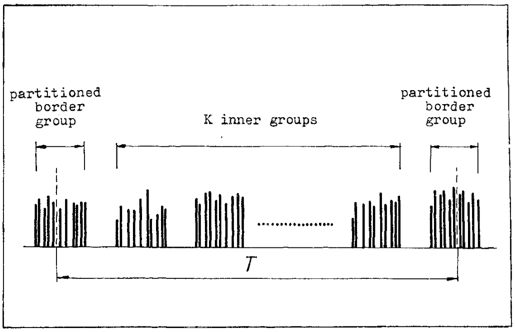

Fig. 2. Determination of the number of the pulses in an interval (0, T).

have – even if not necessarilly – the expression for power spectrum consisting of more than two terms. The first term represents the continuous part and is determined by the shape of a typical pulse and by the statistics of random parameters $\boldsymbol{a}$ (except for ε). On the contrary, the form of the line spectrum, which is represented by the second term, is determined not only by $f(t, \boldsymbol{a})$ and $w_m(\boldsymbol{a})$ but also by the pulse spreading in groups, that is by $w(\varepsilon)$.

Let us consider a special case, when $w(\varepsilon) = 1/T_0$. Because now $|\chi(\omega)|^2 = 0$ for $\omega = k\omega_0$, $k = \pm 1, \pm 2, \ldots$ and $|\chi(\omega)|^2 = 1$ for $\omega = k\omega_0$, $k = 0$, we get from (12)

$$\mathscr{W}(\omega) = \bar{\nu}_0 \overline{|s(\omega, \boldsymbol{a})|^2} + \bar{\nu}_0^2 |\overline{s(\omega, \boldsymbol{a})}|^2\, 2\pi\, \delta(\omega)\,, \tag{13}$$

which is a well known result for the power spectrum of the homogeneous Poisson process [1], [12] ($\bar{\nu}(t) = \bar{\nu}_0 = \text{const.}$).

The periodic Poisson pulse process described here may be used as a model for the cavitation noise induced acoustically (on the contrary the homogeneous process (13) may be used in the case of hydrodynamic caviation), shot noise and photomultiplier noise in the case of periodic modulation and in other cases where the generating physical mechanismus periodically varies with time and where, if steady conditions are established, the homogeneous Poisson process results.

4. EXAMPLE

We shall illustrate the derived expressions by an example. Let the single pulse have the form of a two-sided symmetrical exponential (Fig. 3). This form is used for example as a first approximation for cavitation pulses [6]. Hence

$$f(t, \boldsymbol{a}) = a \exp\left(-|t|/\Theta\right).$$

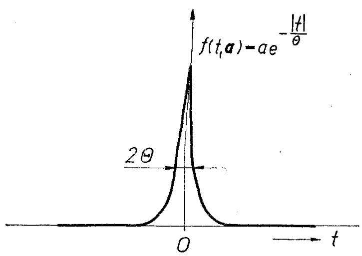

Fig.3. A two-sided symmetrical exponential pulse.

Here a is a random amplitude and Θ is a random time constant. We shall assume that the two random parameters are statistically independent. The spectrum of the pulse equals

$$s(\omega, \boldsymbol{a}) = \frac{2a\Theta}{1 + \omega^2\Theta^2}.$$

For $\omega\bar{\Theta} \ll 1$ and $\omega\bar{\Theta} \gg 1$ we may find approximate expressions. If $\omega\bar{\Theta} \ll 1$, then

$$\overline{|s(\omega, \boldsymbol{a})|^2} \approx 4\overline{a^2}\,\overline{\Theta^2},$$

$$|\overline{s(\omega, \boldsymbol{a})}|^2 \approx 4\bar{a}^2\bar{\Theta}^2,$$

and if $\omega\bar{\Theta} \gg 1$, then

$$\overline{|s(\omega, \boldsymbol{a})|^2} \approx 4\overline{a^2}\,\overline{\Theta^{-2}}\,\frac{1}{\omega^4},$$

$$|\overline{s(\omega, \boldsymbol{a})}|^2 \approx 4\overline{a^2\Theta^{-1}}\,\frac{1}{\omega^4}.$$

In applications one is usually interested in the power spectrum $W(\omega)$, which is defined only for positive values of ω. Thus $W(\omega) = 2\,\mathscr{W}(\omega)$, $\omega \geqq 0$. If ε is normally distributed, then the square of the modulus of the characteristic function equals

$$|\chi(\omega)|^2 = \exp\left(-\sigma_\varepsilon^2\omega^2\right).$$

Usually $\sigma_\varepsilon \ll T_0$. Introducing the variation quotient $V_a = \sigma_a/\bar{a}$ and $V_\Theta = \sigma_\Theta/\bar{\Theta}$ and changing ω for f we may finally write the aproximate expression for the low frequency continuous part of the power spectrum:

1)
$$N_k = N_0 = \text{const.}$$
$$\frac{W(f)}{8\nu_0\overline{a^2}\overline{\Theta^2}} \approx 1 - \frac{1}{(1 + V_a^2)(1 + V_\theta^2)}\,e^{-\sigma_\varepsilon^2(2\pi f)^2}\,.$$

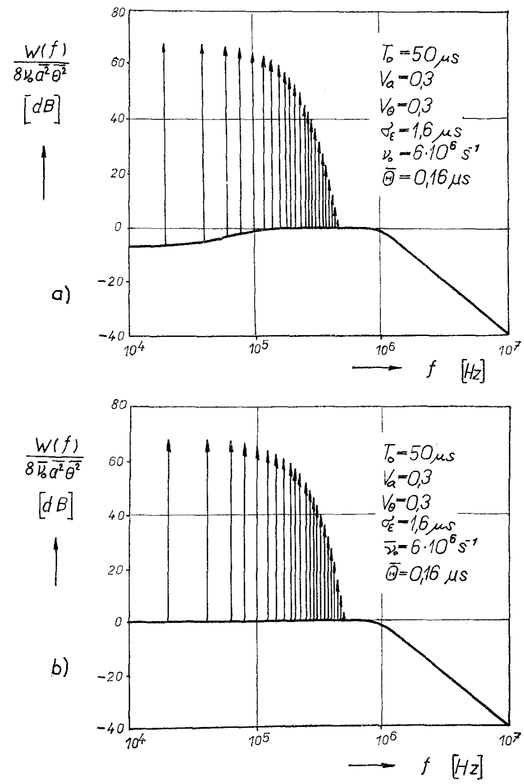

Fig. 4. The calculated power spectra of the periodic group pulse process. a) The number of pulses in groups is constant. b) The number of pulses in groups is Poisson distributed.

2) N_k is Poisson distributed

$$\frac{W(f)}{8\bar{v}_0 \overline{a^2} \overline{\Theta^2}} \approx 1 \, .$$

The approximate expression for the high frequency continuous power spectrum equals in both cases

$$\frac{W(f)}{8\bar{v}_0 \overline{a^2} \overline{\Theta^2}} \approx \frac{\overline{\Theta^{-2}}}{\bar{\Theta}^2} \frac{1}{(1 + V_\theta^2)} \frac{1}{(2\pi f)^4} \, .$$

The expression for the discrete part of the power spectrum can be also written in both cases in the form

$$\frac{W(f)}{8\bar{v}_0 \overline{a^2} \overline{\Theta^2}} \approx \bar{v}_0 \frac{1}{(1 + V_a^2)(1 + V_\theta^2)} \, e^{-\sigma_\varepsilon^2 (2\pi f)^2} \sum_{k=0}^{\infty} \delta(f - kf_0) \, .$$

The calculated spectra are given in Fig. 4. As we may see, the difference between the two cases appears at the low frequency part of the continuous spectrum.

5. CONCLUSION

In the paper the new type of the random pulse process has been introduced. This is the periodic group pulse process with overlapping among pulses. The expression for the power spectrum of this process has been derived and it has been shown that several more simple well known pulse processes may be regarded as special cases of this group pulse process.

The studied process may be used as a phenomenological model for several kinds of noise. For this purpose it is necessary to specify the shape of the typical pulse and the pulse statistical characteristics. However, this may often be a difficult task requiring thorough both theoretical as experimental analysis of the generating physical phenomena.

(Received September 5, 1979.)

REFERENCES

[1] D. Middleton: On the Theory of Random Noise: Phenomenological Models I, II. J. Appl. Physics *22* (1951), 9, 1143—1163, also in D. Middleton: An Introduction to Statistical Communication Theory. McGraw-Hill, New York 1960.

[2] W. J. Richter Jr., T. J. Smits: Numerical Evaluation of Rice's Integral Representation of the Probability Density Function for Poisson Impulsive Noise. J. Acoust. Soc. Am. *56* (1974), 2, 481—496.

[3] H. Bittel, L. Storm: Rauschen. Springer, Berlin 1971.

[4] P. Mazzetti: Study of Nonindependent Random Pulse Trains, with Application to the Barkhausen Noise. Nuovo Cimento *25* (1962), 6, 1322—1341.

[5] V. A. Akuličev, V. V. Olševskij: Svjaz statističeskich charakteristik akustičeskoj kavitacii i kavitacionnovo šuma. Akustičeskij žurnal *14* (1968), 1, 30—36.

[6] K. Vokurka: On the Cavitation Noise Theory. The 5th Conf. of Czechoslovac Physicists, Košice 1977, p. 918.

[7] Ch. Heiden: Power Spectrum of Stochastic Pulse Sequences with Correlation between the Pulse Parameters. Physical Review *188* (1969), 1, 319—326.

[8] B. R. Levin: Nekotorye obobščenija v teorii impulsnych slučajnych procesov. Elektrosvjaz (1963), 2, 21—28, also in B. R. Levin: Teoretičeskie osnovy statističeskoj radiotechniki. Sovetskoe Radio, Moskva 1969.

[9] G. V. Konovalov, E. M. Tarasenko: Impulsnye slučajnye procesy v elektrosvjazi. Svjaz, Moskva 1973.

[10] G. G. Macfarlane: On the Energy-Spectrum of an Almost Periodic Succession of Pulses. Proc. I.R.E. *37* (1949), 1139—1143.

[11] J. L. Lawson, G. E. Uhlenbeck: Treshold Signals. McGraw-Hill, New York 1950.

[12] S. O. Rice: Mathematical Analysis of Random Noise. Bell System Techn. J. *23* (1944), 3, 282—332.

Ing. Karel Vokurka, CSc., katedra elektroniky a měření Elektrotechnické fakulty Vysoké školy strojní a elektrotechnické (Department of Electronics and Measurement — College of Mechanical and Electrical Engineering), Nejedlého sady 14, 306 14 Plzeň. Czechoslovakia.