KYBERNETIKA — VOLUME 16 (1980), NUMBER 5

On an Equivalence of System-Theoretical and Categorical Concepts

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Minimal reduction of systems is investigated in a general categorical setting. Considering a base-category $\mathscr{K}$ and a concrete category $\mathscr{D}$ of systems in $\mathscr{K}$, the existence and universality of minimal reductions is characterized in terms of the forgetful functor $\mathscr{D} \to \mathscr{K}$.

1. INTRODUCTION

A very general model of systems in a category $\mathscr{K}$ has been sketched by Arbib and Manes [5]: systems form an (abstract) category $\mathscr{D}$, endowed with a forgetful functor

$$U : \mathscr{D} \to \mathscr{K}$$

(which forgets the dynamics) and a factorization of $\mathscr{D}$-morphisms. The latter allows to study subsystems and reductions. This model was further developed by Ehrig and Kreowski [7] who gave general sufficient conditions on the functor U for the existence of reductions and minimal realizations. The aim of the present paper is to prove that these conditions are also necessary. Hence, the given categorical concepts are equivalent to those of system theory. Since the mentioned model is, in fact, not specific for system theory but has a much wider scope, our results reveal interconnections of other parts of structural mathematics to this theory.

A rough formulation of the main results:

(i) All systems have minimal reductions iff the functor U preserve cointersections.

(ii) Reduction is universal iff U preserves cointersections and co-preimages.

(iii) If U is a (right) adjoint and preserves cointersections then minimal realizations can be obtained via Nerode equivalence.

In case of Arbib-Manes machines, where $\mathscr{D}$ is the category of dynamics over some varietor (input process) $F : \mathscr{K} \to \mathscr{K}$, these results have been proved earlier:

(i) in [1], [13];

(ii) in [15];
(iii) in [2].

What is new is the generality in which these results hold, moreover, with a small number of side conditions. In contrast, various side conditions have been used in the previous papers – owing to the fact that the characterizations there concerned the varietor F, not only the forgetful functor U. New is also a solution of these problems in terms of factorization properties of output morphisms $UQ \to Y$, as explained in [7]. (These factorizations have been introduced by Herrlich [11].)

I. MINIMAL REDUCTION

I,1 A system is, roughly speaking, a dynamics on a set. To determine a system theory means to specify 1) what dynamics are considered 2) what are dynamorphisms, i.e. maps compatible with dynamics and 3) what are subsystems. In a more general setting we start with a structured set (e.g., a vector space or a topological space) and we specify dynamics with respect to this structure. Thus, we start with a "base" category $\mathscr{K}$ (of sets or vector spaces or topological spaces, etc.) and we form a system theory over $\mathscr{K}$. Here is the abstract concept.

I,2 Definition. *A system theory* **S** *in a category* $\mathscr{K}$ consists of

(a) a category $\mathscr{D}$, the object of which are called dynamics and morphisms are called dynamorphisms;

(b) a faithful (so-called forgetful) functor $U : \mathscr{D} \to \mathscr{K}$;

(c) a factorization system $(\mathscr{E}, \mathscr{M})$ for dynamorphisms.

A system is then a triple $S = (Q, Y, y)$ which consists of a dynamics Q, an output object Y in the category $\mathscr{K}$ and an output morphism $y : UQ \to Y$.

I,3 Remark. Recall that a factorization system $(\mathscr{E}, \mathscr{M})$ in a category $\mathscr{D}$ consists of a class $\mathscr{E}$ of epimorphisms and a class $\mathscr{M}$ of monomorphisms such that:

(a) $\mathscr{D} = \mathscr{M} . \mathscr{E}$, i.e. every morphism $f : Q \to \bar{Q}$ factorizes as $f = m . e$, where $e : Q \to Q_0$ is in $\mathscr{E}$ and $m : Q_0 \to \bar{Q}$ is in $\mathscr{M}$;

(b) $\mathscr{M} . \mathscr{M} \subseteq \mathscr{M}$ and $\mathscr{E} . \mathscr{E} \subseteq \mathscr{E}$, i.e. both classes are closed to composition;

(c) $\mathscr{E} \cap \mathscr{M}$ is the class of all isomorphisms;

(d) in every commutative square

$$\begin{tikzcd} \cdot \arrow[r, "e \in \mathscr{E}"] \arrow[d, "p"'] & \cdot \arrow[d, "q"] \\ \cdot \arrow[r, "m \in \mathscr{M}"'] & \cdot \end{tikzcd}$$

there exists a "diagonal" morphism d, making the following diagram

$$\begin{tikzcd} {} \arrow[r, "e"] \arrow[d, "p"'] & {} \arrow[d, "q"] \arrow[dl, "d"'] \\ {} \arrow[r, "m"'] & {} \end{tikzcd}$$

commutative.

The reason for considering this general notion is to specify what is a subobject and a quotient object: given an object Q in $\mathscr{D}$, each monomorphism $m : Q' \to Q$ in $\mathscr{M}$ represents a subobject of Q (informally denoted by Q') and each epimorphism $e : Q \to$ $\to Q'$ in $\mathscr{E}$ represents a quotient object (Q') of Q.

I,4 Example: sequential Σ-machines form a system theory in the category $\mathscr{K}$ of sets and mappings. Dynamics are pairs $Q = (Q_0, \delta)$ where Q_0 is the set (of states) and $\delta : Q_0 \times \Sigma \to Q_0$ is the (next-state) map. Dynamomorphisms

$$f : (Q_0, \delta) \to (Q_0', \delta')$$

are maps $f : Q_0 \to Q_0'$ subject to $f(q\sigma) = f(q)\,\sigma$, more precisely

$$f(\delta(q, \sigma)) = \delta'(f(q), \sigma) \quad \text{for each} \quad q \in Q_0, \sigma \in \Sigma .$$

Thus, $\mathscr{D}$ is the category of Medvedev machines (= sequential machines without output).

The forgetful functor $U : \mathscr{D} \to SET$ simply forgets the next-state map, thus $UQ =$ $= Q_0$ for objects; $Uf = f$ for morphisms.

Finally, the class $\mathscr{E}$ consists of all onto dynamorphisms and the class $\mathscr{M}$ of all one-to-one dynamorphisms.

Here, systems are precisely sequential machines, more specifically, non-initial Moore sequential Σ-machines.

Remark. With the above example it can be easily seen how more complex system theories fit in the general framework, e.g.

machines in a closed or pseudo-closed category [9];

Arbib-Manes machines in a category [5];

continuous-time systems [8].

I,5 A *system morphism* is a dynamorphism which respects the outputs. Thus, given systems

$$S = (Q, Y, y) \quad \text{and} \quad S' = (Q', Y', y')$$

 then a dynamorphism $f : Q \to Q'$ is a *system morphism* provided that

$$\begin{tikzcd} UQ \arrow[rr, "Uf"] \arrow[dr, "y"'] & & UQ' \arrow[dl, "y'"] \\ & Y & \end{tikzcd}$$

commutes.

Denote by $\mathbf{S}(Y)$ the category of all systems in $\mathbf{S}$ with the output object Y and all system morphisms.

I,6 A notion, fundamental for further development, is the reduction of a system. For systems over sets, reduction is an identification of indistinguishable states. Generally:

Definition. *A reduction of a system* S is any system morphism $f : S \to S'$ such that $f \in \mathscr{E}$.

A system S is *reduced* if it has no reductions other then isomorphisms.

Example. For each sequential machine

$$S : Q \times \Sigma \xrightarrow{\delta} Q \xrightarrow{y} Y$$

denote by $\delta^* : Q \times \Sigma^* \to Q$ the usual extension to input strings. The minimal reduction of S (= the one with the least number of states, in case S is finite) is obtained as a quotient under the *Nerode equivalence* $\approx$, defined on the state set Q by $q_1 \approx q_2$ iff for each input string $w \in \Sigma^*$

$$y(\delta^*(q_1, w)) = y(\delta^*(q_2, w)) .$$

Put $S/\approx\ = (Q/\approx, \bar{\delta}, Y, \bar{y})$ where

$$\bar{\delta}([q], \sigma) = [\delta(q, \sigma)] \quad \text{and} \quad \bar{y}([q]) = y(q) \quad \text{for each} \quad q \in Q\,, \quad \sigma \in \Sigma\,.$$

Then the canonical map $f : Q \to Q/\approx$ defines a reduction $f : S \to S/\approx$.

This is the only reduction of S which is itself a reduced system. In fact, this is the minimal reduction in the following sense:

I,7 Definition. A reduction $f : S \to S_0$ of a system S is *minimal* provided that any other reduction can be further reduced to S_0, i.e. for each reduction $g : S \to \bar{S}$ there exists a reduction $\bar{g} : \bar{S} \to S_0$ subject to $f = \bar{g} . g$.

A system theory is said to *have minimal reductions* if for each system there exists a minimal reduction.

$$\begin{tikzcd} S \arrow[r, dashed, "f"] \arrow[d, "g"'] & S_0 \\ \bar{S} \arrow[ur, "\bar{g}"'] & \end{tikzcd}$$

Fact: Minimal reduction is unique up-to isomorphism. I.e., given a minimal reduction $f : S \to S_0$ then

(i) for each isomorphism of systems $i : S_0 \to S_0'$ also

$$i \,.\, f : S \to S_0'$$

is a minimal reduction;

(ii) for each minimal reduction $f' : S \to S_0'$ there is a unique isomorphism of systems $i : S_0 \to S_0'$ with $f' = i \,.\, f$.

Remark. Minimal reduction is always reduced. (Proof. Given a minimal reduction $f : S \to S_0$ and a reduction $h : S_0 \to \bar{S}_0$ of S_0 we are to verify that h is an isomorphism. Since $g = h \,.\, f : S \to \bar{S}_0$ is a reduction of S, there exists a reduction $\bar{g} : \bar{S}_0 \to S_0$ subject to $f = \bar{g} \,.\, h \,.\, f$. Since f is an epi, there follows $1_{S_0} = \bar{g} \,.\, h$; thus h is a split mono as well as an epi – hence, an isomorphism.)

Conversely: in a system theory with minimal reductions every reduced reduction (i.e. every reduction $f : S \to S_0$ with S_0 reduced) is minimal. Indeed, besides the reduced reduction S_0, the system S has a minimal reduction $f' : S \to S_0'$ and there

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "f'"'] & S_0 \arrow[dl, "\bar{f}"] \\ S_0' & \end{tikzcd}$$

exists, by definition, a reduction $\bar{f} : S_0 \to S_0'$ subject to $f' = \bar{f} \,.\, f$. Since S_0 is reduced, $\bar{f}$ is an isomorphism. Hence $f : S \to S_0$ is also a minimal reduction.

I,8 We are going to state a necessary and sufficient condition for a system theory in a category to have minimal reductions. The sufficiency of this condition is proved in [7] under a different terminology: the $\mathcal{M}_{OUT}$-morphisms, studied there, are easily seen to coincide with the present reduced systems. The condition is stated in terms of cointersections of quotients (which is the dual to intersections of subobjects).

Given a collection (possibly large but non-void) of quotients of an object Q, i.e. a collection of epis

$$e_i : Q \to Q_i \quad (i \in I)$$

their cointersection is the multiple pushout $(*)$.

$$(*) \qquad \begin{tikzcd} & Q \arrow[dl, "e_i"'] \arrow[d] \arrow[dr] & \\ Q_i \arrow[dr, "\bar{e}_i"'] & {} & {} \cdots i \in I \, [e_i \in \mathscr{E}] \arrow[dl] \\ & \bar{Q} & \end{tikzcd}$$

(Remark: it follows from the axioms of factorization system that if each e_i belongs to $\mathscr{E}$ then so does each $\bar{e}_i$.) A system theory is said to *have $\mathscr{E}$-cointersections* if for each dynamics $Q \in \mathscr{D}$ and each collection of its $\mathscr{E}$-quotients (i.e., dynamorphisms $e_i : Q \to Q_i$ in $\mathscr{E}$) this multiple pushout exists. This is a weak requirement, indeed: every co-well powered category $\mathscr{D}$, which is lither complete or cocomplete, fulfils it.

Another weak requirement is that the forgetful functor $U : \mathscr{D} \to \mathscr{K}$ should *preserve $\mathscr{E}$-epis*, i.e., for each dynamorphism $e : Q \to Q'$ in $\mathscr{E}$ the morphism Ue is an epi in $\mathscr{K}$.

I,9 Theorem. For a system theory $\boldsymbol{S}$ with cointersections and such that the forgetful functor U preserves $\mathscr{E}$-epis, the following holds:

$\boldsymbol{S}$ has minimal reductions iff U preserves $\mathscr{E}$-cointersections (i.e. iff U maps each diagram $(*)$ to a cointersection in $\mathscr{K}$).

Proof. If U preserves $\mathscr{E}$-cointersections then the minimal reduction of any system S is obtained as the cointersection of all reductions of S – see [7].

Conversely, assume the existence of minimal reductions. Given a cointersection in the category $\mathscr{D}$:

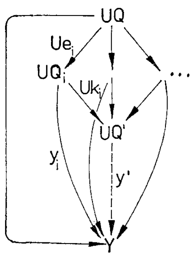
$$\begin{tikzcd} & Q \arrow[dl, "e_i"'] \arrow[d] \arrow[dr] & \\ Q_i \arrow[dr, "k_i"'] & {} & {} \cdots i \in I \arrow[dl] \\ & Q' & \end{tikzcd}$$

with each e_i in $\mathscr{E}$ $(i \in I)$ we shall prove that U maps it to a cointersection in $\mathscr{K}$. In other words, given a collection of morphisms

$$y_i : UQ_i \to Y \quad \text{in} \quad \mathscr{K} \quad (i \in I)$$

such that $y = y_i \,.\, Ue_i : UQ \to Y$ is independent of i, we shall prove that there exists $y' : UQ' \to Y$ subject to

$$y_i = y' \,.\, Uk_i \quad (i \in I)\,.$$



Remark: this y' is then unique because each k_i belongs to $\mathscr{E}$, hence each Uk_i is epi in $\mathscr{K}$.

The system $S = (Q,\ Y,\ y)$ (where $y = y_i \,.\, Ue_i$ for each i) has a minimal reduction

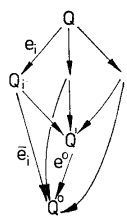
$$f : S \to S^0 = (Q^0,\ Y,\ y^0)\,.$$

For each $i \in I$ we clearly have a reduction of S:

$$e_i : S \to S_i = (Q_i,\ Y,\ y_i)\,.$$

By definition of minimal reduction there exist reduction

$$\bar{e}_i : S_i \to S^0 \quad \text{with} \quad f = \bar{e}_i \,.\, e_i \quad (i \in I)\,.$$



Since $\bar{e}_i \,.\, e_i$ is independent of $i \in I$, there exists a unique $e^0 : Q' \to Q^0$ subject to

$$\bar{e}_i = e^0 \,.\, k_i \quad (i \in I)\,.$$

Put $y' = y^0 \,.\, Ue^0 : UQ' \to Y$. Then for each $i \in I$ we have

$$y' \,.\, Uk_i = y^0 \,.\, U(e^0 \,.\, k_i) = y^0 \,.\, U\bar{e}_i\,.$$

Since $\bar{e}_i : S_i \to S^0$ is a system morphism, the proof is concluded: $y_i = y^0 . U\bar{e}_i = = y' . Uk_i$ $(i \in I)$.

I,10 Example: tree machines. Let $\Omega = \{n_i\}_{i \in I}$ be a type of algebras, i.e. a collection of (possibly infinite) cardinals n_i denoting the arity of the i-th operation. Then Ω-algebras with outputs are called Ω-tree machines. More precisely, form a system theory over $\mathscr{K} = SET$, denoting by $\mathscr{D}$ the category of Ω-algebras and homomorphism with the usual forgetful functor $U : \mathscr{D} \to SET$ and with the factorization system

$\mathscr{E}$ = all onto homomorphisms

$\mathscr{M}$ = all one-to-one homomorphisms.

Systems in this theory are just Ω-tree machines (cf. [4]). There the forgetful functor preserves cointersections iff the type Ω is finitary (i.e. each n_i is natural number).

Thus, for finitary tree machines (which is the case usually considered) each machine has a minimal reduction. And infinitary tree machines do not share this property.

Remark. Functors $U : SET \to SET$, preserving cointersections, are described in [13]: these are precisely all quotients of coproducts of finite hom-functors. More generally, Barr [6] exhibits simple side conditions under which each finitary functor $U : \mathscr{D} \to \mathscr{K}$ (i.e. a functor, preserving filtered colimits) preserves $\mathscr{E}$-cointersections for $\mathscr{E}$ = all coequalizers.

II. UNIVERSAL REDUCTION

II,1 Given a system theory with minimal reductions, several natural questions arise, e.g.:

a) Are minimal reductions $f : S \to S_0$ universal arrows?, i.e. does there exist, for each system morphism $g : S \to T$ with T reduced, a system morphism $g^* : S_0 \to T$ for which $g = g^* . f$?.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "g"'] & S_0 \arrow[dl, dashed, "g^*"] \\ T & \end{tikzcd}$$

b) Are reduced systems *hereditary*?, i.e., given a reduced system S and its subsystem $m : S_1 \to S$ $(m \in \mathscr{M})$, does there follow that S_1 is also reduced?

We shall show that these two problems are equivalent and the answers are often negative.

In the terminology of [7], the system theory *admits universal reduction* provided that each system has a minimal reduction which is a universal arrow. (In other words, for each fixed output object Y reduced systems form a reflective subcategory of the category of all systems.) The hereditarity of reduced systems is formulated in [7] as the condition that $(\mathscr{E}, \mathscr{M}_{OUT})$ is a factorization system such that $\mathscr{M}_{OUT} \,.\, \mathscr{M} = \mathscr{M}_{OUT}$.

II,2 We are going to state a necessary and sufficient condition on a system theory to admit universal reduction. We shall use, besides cointersections, also co-preimages (which are duals of preimages – pullbacks along a monomorphism).

Thus, a system theory is said to have $\mathscr{E}$-co-preimages if for arbitrary morphisms $e \in \mathscr{E}$ and f in $\mathscr{D}$ with a joint domain there exists a pushout:

$$
\begin{tikzcd}
{} \arrow[r, "e \in \mathscr{E}"] \arrow[d, "f"'] & {} \arrow[d, "f'"] \\
{} \arrow[r, "e'"'] & {}
\end{tikzcd}
$$

II,3 Theorem. The following conditions are equivalent for each system theory $\boldsymbol{S}$ with cointersections and co-preimages and with the forgetful functor U preserving $\mathscr{E}$-epis:

(i) $\boldsymbol{S}$ admits universal reduction;
(ii) $\boldsymbol{S}$ has minimal reductions and reduced systems are hereditary;
(iii) U preserves $\mathscr{E}$-cointersections and $\mathscr{E}$-co-preimages.

Proof. (i) → (iii) U preserves cointersections by I,2. Let

$$
\begin{tikzcd}
Q \arrow[r, "h"] \arrow[d, "(r \in \mathscr{E})\ r"'] & P \arrow[d, "r_1"] \\
Q_1 \arrow[r, "h_1"'] & P_1
\end{tikzcd}
$$

be a co-preimage and let β_1, β be arbitrary $\mathscr{K}$-morphisms with $\beta \,.\, Uh = \beta_1 \,.\, Ur$:

$$
\begin{tikzcd}
UQ \arrow[r, "Uh"] \arrow[d, "Ur"'] & UP \arrow[d, "Ur_1"] \arrow[dr, "\beta"] & \\
UQ_1 \arrow[r, "Uh_1"] \arrow[rr, bend right, "\beta_1"'] & UP_1 \arrow[r, dashed, "\lambda"] & Y
\end{tikzcd}
$$

 We are to show that there exists a (necessarily unique) $\lambda : UP_1 \to Y$ with $\beta = \lambda \,.\, Ur_1$ and $\beta_1 = \lambda \,.\, Uh_1$. Then Uh_1, Ur_1 is a pushout of Uh, Ur, which is a co-preimage (since $r \in \mathscr{E}$ implies Ur epi).

Consider the following systems in $\boldsymbol{S}(Y)$:

$$S = (Q, Y, \beta_1 \,.\, Ur) \quad \text{and} \quad S' = (P, Y, \beta)\,;$$

they both have a reduced reflection, say

$$\begin{aligned} f\ &: S \to \bar{S} = (\bar{Q}, Y, \bar{y}) \\ f' &: S' \to \bar{S}' = (\bar{P}, Y, \bar{y}')\,. \end{aligned}$$

Furthermore, $r : S \to S_1 = (Q_1, Y, \beta_1)$ is clearly a reduction, which can be further reduced to the minimal reduction $\bar{S}$: we have $g : S_1 \to \bar{S}$ with $f = g \,.\, r$. Since $h :$ $: S \to S'$ is a morphism in $\boldsymbol{S}(Y)$ (because of $\beta \,.\, Uh = \beta_1 \,.\, Ur$), we have a corresponding morphism of reflections, say $\bar{h} = \bar{S} \to \bar{S}'$ with $\bar{h} \,.\, f = f' \,.\, h$. In particular, since $f = g \,.\, r$,

$$f' \,.\, h = (\bar{h} \,.\, g) \,.\, r\,.$$

Now we use the fact that $h_1 \,.\, r = r_1 \,.\, h$ is a pushout to obtain $t : P_1 \to \bar{P}$ with $f' = t \,.\, r_1$ and $\bar{h} \,.\, g = t \,.\, h_1$.

$$\begin{tikzcd} Q \arrow[rr, "h"] \arrow[d, "r"'] \arrow[ddr, "f"] & & P \arrow[d, "r_1"'] \arrow[ddr, "f'"] & \\ Q_1 \arrow[rr, "h_1"] \arrow[dr, "g"'] & & P_1 \arrow[dr, dashed, "t"] & \\ & \bar{Q} \arrow[rr, "\bar{h}"] & & \bar{P} \end{tikzcd}$$

Put $\lambda = \bar{y}' \,.\, Ut$. Since $f' : S' \to \bar{S}'$ is a system morphism, we have $\beta = \bar{y}' \,.\, Uf'$, therefore

$$\begin{aligned} \lambda \,.\, Ur_1 &= (\bar{y}' \,.\, Ut) \,.\, Ur_1 \\ &= \bar{y}' \,.\, Uf' \\ &= \beta\,. \end{aligned}$$

Since $\bar{h} : \bar{S} \to \bar{S}'$ and $g : S_1 \to S$ are system morphisms, we have $\bar{y} = \bar{y}' \,.\, U\bar{h}$ and $\beta_1 = \bar{y} \,.\, Ug$, therefore

$$\begin{aligned} \lambda \,.\, Uh_1 &= (\bar{y}' \,.\, Ut) \,.\, Uh_1 \\ &= \bar{y}' \,.\, U\bar{h} \,.\, Ug \\ &= \bar{y} \,.\, Ug \\ &= \beta_1\,. \end{aligned}$$

(iii) → (ii) By I,2, we know that **S** admits minimal reduction. Let $S = (Q, Y, y)$ be a reduced system and let $m : S' = (Q', Y, y') \to S$ be its subsystem $(m \in \mathcal{M})$. We are to prove that the minimal reduction $f : S' \to S'' = (Q'', Y, y'')$ of S' is an isomorphism. (Then S' is reduced.) Consider the co-preimage:

$$\begin{tikzcd}
Q' \arrow[r, "m"] \arrow[d, "f"'] & Q \arrow[d, "f_1"] \\
Q'' \arrow[r, "m_1"'] & Q_0
\end{tikzcd}$$

This is preserved by U, hence $y . Um = y' = y''$. Uf implies that there exists $y_0 : UQ_0 \to Y$ with $y = y_0 . Uf_1$.

$$\begin{tikzcd}
UQ' \arrow[rr, "Um"] \arrow[dd, "Uf"'] \arrow[dr, "y'"] & & UQ \arrow[dd, "Uf_1"] \arrow[dl, "y"'] \\
& Y & \\
UQ'' \arrow[rr, "Um_1"'] \arrow[ur, "y''"] & & UQ_0 \arrow[ul, dashed, "y_0"']
\end{tikzcd}$$

Then $f_1 : S \to S_0 = (Q_0, Y, y_0)$ is a system morphism. Since f_1 is opposite to $f \in \mathcal{E}$ in a pushout, we have $f_1 \in \mathcal{E}$, i.e. $f_1 : S \to S_0$ is a reduction. Thus, f_1 is an isomorphism, for S is reduced. Thus, $m_1 . f = f_1 . m \in \mathcal{M}$, which implies $f \in \mathcal{E} \cap \mathcal{M}$ — – hence, f is an isomorphism.

(ii) → (i) Given a system morphism $h : S \to S'$ and given minimal reductions $f : S \to S_0$ and $f' : S' \to S'_0$, we are to exhibit a system morphism $h_0 : S_0 \to S'_0$ with $h_0 . f = f' . h$.

Let $f' . h = m . e$ be an image factorization of $f' . h$, say $e : Q \to P$ and $m : P \to$ $\to Q'_0$ where $S = (Q, Y, y)$, $S_0 = (Q_0, Y, y_0)$ and $S' = (Q', Y', y')$, $S'_0 = (Q'_0, Y, y'_0)$. Then we have a system morphism $m : \tilde{S} = (P, Y, y'_0 . Um) \to S'_0$. Since S'_0 is reduced, so is $\tilde{S}$.

$$\begin{tikzcd}
& Q \arrow[r, "h"] \arrow[dl, "f"'] \arrow[d, "e"] & Q' \arrow[d, "f'"] \\
Q_0 & P \arrow[l, dashed, "\bar{e}"'] \arrow[r, "m"] & Q'_0
\end{tikzcd}$$

Furthermore, $e : S \to \tilde{S}$ is a reduction of S, thus there is a reduction $\bar{e} : \tilde{S} \to S_0$ with $f = \bar{e} . e$. Since $\tilde{S}$ is reduced, $\bar{e}$ is an isomorphism. Put

$$h_0 = m . \bar{e}^{-1} : S_0 \to S'_0 .$$

II,4 Examples. Whenever the forgetful functor $U : \mathscr{D} \to \mathscr{K}$ preserves colimits (particularly, whenever U is a left adjoint) then it preserves cointersections and co-preimages, of course. This is the case e.g. for

a) automata in a closed cocomplete category, particularly, for sequential machines in $\mathscr{K} = SET$ and bilinear machines in $\mathscr{K}$ = vector spaces;

b) continuous-time systems in a closed cocomplete category, studied in [8].

On the other hand, Ω-tree machines (I,10) do not have universal reduction unless all arities are unary or nullary (in which case these machines are sequential), see [15].

Functors $U : SET \to SET$, preserving cointersections and co-preimages, are described in [14]: these are, up-to natural equivalence, precisely the functors $F_{\Sigma_1\Sigma_0}$ (where Σ_0 and Σ_1 are fixed sets) defined by

$$F_{\Sigma_1\Sigma_0}X = X \times \Sigma_1 + \Sigma_0 \qquad \text{on objects}$$

$$F_{\Sigma_1\Sigma_0}f = f \times \mathrm{id}_{\Sigma_1} + \mathrm{id}_{\Sigma_0} \quad \text{on morphisms.}$$

Observe that left adjoints $U : SET \to SET$ are just $U \cong F_{\Sigma_1\Sigma_0}$ with $\Sigma_0 = \emptyset$.

III. MINIMAL REALIZATION AND NERODE EQUIVALENCE

III,1 So far we have worked with systems not considering any initialization. Now we approach the fundamental concept of a behavior of an (initial) system.

We start with an (output) object $Y \in \mathscr{K}$ and an (initialization) object $I \in \mathscr{K}$. An initial system is a tuple $S = (Q, Y, y, I, i)$, where (Q, Y, y) is a system and $i : I \to UQ$ is a morphism in $\mathscr{K}$. A system morphism (of initial systems) $f : (Q, Y, y, I, i) \to$ $\to (Q', Y, y', I, i')$ is a morphism $f : Q \to Q'$ in $\mathscr{D}$ subject to

$$y = y' \,.\, Uf \quad \text{and} \quad i' = Uf \,.\, i \,.$$

$$\begin{tikzcd} & UQ \arrow[dd, "Uf"] \arrow[dr, "y"] & \\ I \arrow[ur, "i"] \arrow[dr, "i'"'] & & Y \\ & UQ' \arrow[ur, "y'"'] & \end{tikzcd}$$

This gives rise to a category $\mathbf{S}(Y, I)$ of initial systems (for each pair of objects $Y, I \in \mathscr{K}^0$).

III,2 Definition. A system theory $\boldsymbol{S}$ is *standard* provided that

(i) The forgetful functor U has a left adjoint. Explicitly, provided that for each object $X \in \mathscr{K}$ there exists a dynamics $X^\# \in \mathscr{D}$, freely generated by a morphism

$\eta : X \to UX^{\#}$ in the sense that, for each dynamics Q and each morphism $f : X \to UQ$ in $\mathscr{K}$ there exists a unique dynamorphism $f^{\#} : X^{\#} \to Q$ subject to $f = Uf^{\#} . \eta$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta"'] & UQ \\ UX^{\#} \arrow[ur, "Uf^{\#}"'] & \end{tikzcd}$$

(ii) For each dynamics Q the morphism $1^{\#}_{UQ} : (UQ)^{\#} \to Q$ belongs to $\mathscr{E}$.

Remark. The latter condition (ii) is satisfied e.g. whenever there exists a factorization system $(\mathscr{E}_0, \mathscr{M}_0)$ in $\mathscr{K}$ such that

$$\mathscr{E} = \{e \in \mathscr{D}^{mor};\ Ue \in \mathscr{E}_0\}\ .$$

(This is usually the situation in the current system theories.)

Indeed, since $U\, 1^{\#}_{UQ} . \eta = 1_{UQ}$, we see that $U\, 1^{\#}_{UQ}$ is a split epi, thus an element of $\mathscr{E}_0$, and so $1^{\#}_{UQ} \in \mathscr{E}$.

III,3 For each initial system $S = (Q, Y, y, I, i)$ in a standard system theory we have a dynamorphism $i^{\#} : I^{\#} \to Q$ and we define the *behavior* morphism

$$b_S = y . Ui^{\#} : UI^{\#} \to Y .$$

The system S is *reachable* in case $i^{\#} \in \mathscr{E}$.

$$\begin{tikzcd} I \arrow[r, "i"] \arrow[d, "\eta"'] & UQ \arrow[r, "y"] & Y \\ UI^{\#} \arrow[ur, "Ui^{\#}"'] & & \end{tikzcd}$$

III,4 Example. The free dynamics for sequential Σ-machines is

$$I^{\#} = (I \times \Sigma^*, \varphi)$$

where Σ^* denotes the free monoid of strings in Σ and

$$\varphi : (I \times \Sigma^*) \times \Sigma \to I \times \Sigma^*$$

is the concatenation: $\varphi(i, \sigma_1 \dots \sigma_n; \sigma) = (i, \sigma_1 \dots \sigma_n \sigma)$.

In the usual situation, I is a singleton set $I = \{\Lambda\}$ and $i(\Lambda) = q_0$ is the initial state of the machine. Then $I^{\#} = (\Sigma^*, \varphi)$ and the map $i^{\#} : \Sigma^* \to Q$ assigns to each string

$\sigma_1 \ldots \sigma_n \in \Sigma^*$ the state $i^\#(\sigma_1 \ldots \sigma_n) = q_n$, reached from q_0 when the inputs $\sigma_1, \ldots$ $\ldots, \sigma_n$ have been applied. Thus, $i^\#$ is onto iff each state is reachable from q_0.

III,5 Remarks. (i) For two systems S_1 and S_2 the existence of a system morphism $f : S_1 \to S_2$ guarantees that their behaviors are equal: $b_{S_1} = b_{S_2}$. Indeed, if $S_1 = (Q_1, Y, y_1, I, i_1)$ and $S_2 = (Q_2, Y, y_2, I, i_2)$, then

$$f \,.\, i_1^\# = i_2^\#$$

because $i_2^\#$ is the only morphism with $i_2 = Ui_2^\# \,.\, \eta$ and we have

$$i_2 = Uf \,.\, i_1 = Uf \,.\, Ui_1^\# \,.\, \eta = U(f \,.\, i_1^\#) \,.\, \eta \,.$$

Therefore,

$$b_{S_2} = y_2 \,.\, Ui_2^\# = y_2 \,.\, Uf \,.\, Ui_1^\# = y_1 \,.\, Ui_1^\# = b_{S_1} \,.$$

$$\begin{tikzcd}
 & & UQ_1 \arrow[dd, "Uf"] \arrow[dr, "y_1"] & \\
I \arrow[urr, "i_1"] \arrow[r, "\eta"] \arrow[drr, "i_2"'] & UI^\# \arrow[ur, "Ui_1^\#"] \arrow[dr, "Ui_2^\#"'] & & Y \\
 & & UQ_2 \arrow[ur, "y_2"'] &
\end{tikzcd}$$

(ii) Any reduction of a reachable system is reachable. Indeed, in the above equality $f \,.\, i_1^\# = i_2^\#$: if $i_1^\# \in \mathscr{E}$ (i.e., if S_1 is reachable) and $f \in \mathscr{E}$ (i.e., S_2 is a reduction) then $i_2^\# \in \mathscr{E}$.

III,6 Given an abstract *behavior* $b : UI^\# \to Y$ we study its *realizations*, i.e.

$$\begin{tikzcd}
I \arrow[r, "\eta"] \arrow[dr, "i"'] & UI^\# \arrow[r, "b"] \arrow[d] & Y \\
 & UQ \arrow[ur, "y"'] &
\end{tikzcd}$$

systems S with behavior $b_S = b$. Each behavior b has a "free realization" $S^{(b)} = (I^\#, Y, b, I, \eta)$:

(i) $S^{(b)}$ is a reachable realization of b because $\eta^\# = 1_{I^\#} : I^\# \to I^\#$ belongs to $\mathscr{E}$ and fulfills $b \,.\, U\eta^\# = b$.

(ii) Each reachable realization of b is a reduction of $S^{(b)}$. Indeed, for each reachable realization $S = (Q, Y, y, I, i)$ of b the morphism $i^\# : I^\# \to Q$ (in $\mathscr{E}$) is a system morphism $i^\# : S^{(b)} \to S$.

Dually, the *minimal realization* of a behavior b is its reachable realization $S_{(b)}$ such that any reachable realization has $S_{(b)}$ as its reduction. Minimal realization is unique up-to an isomorphism of systems (whenever it exists). E.g., for finite sequential machines minimal realizations are characterized as the realizations with a minimum number of states. If each behavior has a minimal realization then we say that the system theory has minimal realizations. This is no new concept:

III,7 Theorem. A standard system theory has minimal realizations iff it has minimal reductions.

Proof. Using minimal reductions, the minimal realization of each behavior $b : UI^{\#} \to Y$ is obtained as the minimal reduction S_0 of the free realization $S^{(b)}$. Indeed, each reachable realization S of b is a reduction of $S^{(b)}$, hence it can be further reduced to S_0.

Conversely, in a system theory with minimal realizations each system S has a minimal reduction. This is clear for reachable systems: the minimal realization $S_{(b)}$ of the behavior $b = b_S$ is a minimal reduction of S since

(i) S is a reachable realization of b and hence it has $S_{(b)}$ as its reduction and

(ii) every reduction of S is also a reachable realization of b.

If S is not reachable, we can change its initialization (playing no role with respect to reductions) to obtain a reachable system $\bar{S}$ with corresponding reductions. More in detail, for each system $S = (Q, Y, y, I, i)$ put $\bar{S} = (Q, Y, y, UQ, 1_{UQ})$. Then $\bar{S}$ is reachable by (ii) in III,2.
Moreover

$$\begin{tikzcd} & UQ \arrow[dr, "y"] \arrow[dd, squiggly, "f"] & \\ I \arrow[ur, "i"] \arrow[dr, "i_0"'] & & Y \\ & UQ_0 \arrow[ur, "y_0"'] & \end{tikzcd} \qquad \begin{tikzcd} & UQ \arrow[dr, "y"] \arrow[dd, squiggly, "f"] & \\ UQ \arrow[ur, "1"] \arrow[dr, "f"'] & & Y \\ & UQ_0 \arrow[ur, "y_0"'] & \end{tikzcd}$$

(a) for each reduction $f : S \to S_0$ where $S_0 = (Q_0, Y, y_0, I, i_0)$ we have a reduction $f : \bar{S} \to (Q_0, Y, y_0, UQ, f)$;

(b) for each reduction $f : \bar{S} \to (Q_0, Y, y_0, UQ, i_0)$ we have a reduction $f : S \to$ $\to (Q_0, Y, y_0, I, i_0 . i)$.

This shows that the minimal reduction (Q_0, Y, y_0, UQ, i_0) of the reachable system $\bar{S}$ yields a minimal reduction $(Q_0, Y, y_0, I, i_0 . i)$ of S.

III,8 Example. For sequential machines, the minimal realization of a behavoir

$$f : \Sigma^* \to Y$$

 is obtained via the *Nerode equivalence* on Σ^* (cf. I,6)

$$u_1 \approx u_2 \quad \text{iff for each string} \quad w \in \Sigma^* : f(u_1 w) = f(u_2 w) .$$

Then $S_{(b)}$ has state set $\Sigma^*/\approx$ and next-state map is

$$\delta([u], \sigma) = [u\sigma]$$

while the output map is

$$y([u]) = f(u) .$$

We shall present a general notion of Nerode equivalence, based on the ideas of [4].

A relation on an object X of a category can be viewed as a morphism pair $p_1, p_2 : : E \to X$ (e.g., in *SET*, $E \subset X \times X$ and p_1, p_2 are the two projections, restricted to E). Of the three properties, characterizing equivalences in *SET*, reflexivity is easy to state generally: a pair $p_1, p_2 : E \to X$ is *reflexive* if there exists a morphism $d : X \to E$, subject to $p_1 . d = p_2 . d = 1_X$.

III,9 Definition. Let $b : UI^\# \to Y$ be a behavior in a standard system theory. A *b-equivalent pair* is a pair of morphisms in $\mathscr{K}$

$$p_1, p_2 : E \to UI^\#$$

such that the corresponding pair of dynamorphisms $p_1^\#, p_2^\# : E^\# \to I^\#$ satisfies $b . Up_1^\# = b . Up_2^\#$.

The *Nerode equivalence* of a behavior b is the largest reflexive, b-equivalent pair. Explicitly, it is a reflexive, b-equivalent pair $p_1, p_2 : E \to UI^\#$ such that for every other such pair $q_1, q_2 : E' \to UI^\#$ there exists a unique morphism $h : E' \to E$ subject to $q_1 = h . p_1$ and $q_2 = h . p_2$.

$$\begin{tikzcd}
E \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & UI^\# \\
 & E' \arrow[ul, dashed, "h"] \arrow[u, shift left, "q_1"] \arrow[u, shift right, "q_2"']
\end{tikzcd}$$

III,10 The construction of minimal realization as the quotient $\Sigma^*/\approx$ for sequential machines (III,8) corresponds to a coequalizer of the Nerode equivalence. Thus, assume that the Nerode equivalence $p_1, p_2 : E \to UI^\#$ has a coequalizer of the form Uf, where $f : I^\# \to Q$ is a dynamorphism. Then $b . Up_1^\# = b . Up_2^\#$ implies

$$b . p_1 = b . Up_1^\# . \eta_E = b . Up_2^\# . \eta_E = b . p_2 ;$$

$$\begin{tikzcd}
& I \arrow[d, "\eta"] & \\
E \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & UI^{\#} \arrow[r, "Uf"] \arrow[dr, "b"'] & UQ \arrow[d, dashed, "y"] \\
& & Y
\end{tikzcd}$$

hence there exists a unique morphism $y : UQ \to Y$ subject to $y \, . \, Uf = b$. The system

$$S = (Q, Y, y, I, Uf \, . \, \eta)$$

is called the *Nerode realization* of the behavior b. We say that a system theory *has Nerode realizations* if

(i) each behavior has a Nerode equivalence and

(ii) each Nerode equivalence $p_1, p_2 : E \to UI^{\#}$ has a coequalizer of the form Uf, where f is a dynamorphism.

III,11 We are going to prove that minimal realizations, whenever they exist, coincide with Nerode realizations. We shall need some more assumptions on the system theory.

Recall that the *kernel pair* of a morphism $k : A \to B$ is a pair $p_1, p_2 : E \to A$ which is largest with respect to the property $k \, . \, p_1 = k \, . \, p_2$, i.e. which constitutes a pullback square:

$$\begin{tikzcd}
E \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "k"] \\
A \arrow[r, "k"'] & B
\end{tikzcd}$$

Thus, to assume that a category has kernel pairs (of all of its morphisms k) is weaker than to assume it finitely complete. Each kernel pairs is reflexive, because the pair 1_A, 1_A fulfils $k \, . \, 1_A = k \, . \, 1_A$, whence there exists a unique $d : A \to E$ such that $1_A = p_1 \, . \, d$ and $1_A = p_2 \, . \, d$.

Conversely, given a reflexive pair $p_1, p_2 : E \to A$ its coequalizer $k : A \to B$ makes the above square a pushout.

Proof: we have a morphism $d : A \to E$ subject to $p_1 \, . \, d = p_2 \, . \, d = 1_A$; for arbitrary morphisms g_1, $g_2 : A \to C$ with $g_1 \, . \, p_1 = g_2 \, . \, p_2$ we have

$$g_1 = g_1 \, . \, p_1 \, . \, d = g_2 \, . \, p_2 \, . \, d = g_2 \, ,$$

hence $g_1 \, . \, p_1 = g_1 \, . \, p_2$ and the morphism $g_1 = g_2$ factorizes through k.

For the theorem below we assume that a standard system theory $\boldsymbol{S}$ is given such that

(a) $\boldsymbol{S}$ has $\mathscr{E}$-cointersections;

(b) both categories $\mathscr{D}$ and $\mathscr{K}$ have kernel pairs;

(c) The class $\mathscr{E}$ is the class of all regular epis (i.e. epis $e : Q \to Q'$ in $\mathscr{D}$ for which there exists a pair $p_1, p_2 = E \to Q$ such that e is its coequalizer).

Remark: in category with kernel pairs every regular epi is a coequalizer of its kernel pair.

(d) The forgetful functor preserves regular epis: $e \in \mathscr{E}$ implies that Ue is a regular epi in $\mathscr{K}$.

III,12 Theorem. A system theory as above has minimal realizations iff it has Nerode realizations. If so then the Nerode realization of any behavior is its minimal realization.

Proof. (i) Assume the existence of minimal realizations. First, let us observe that the forgetful functor U preserves coequalizers of reflexive pairs: indeed, given a reflexive pair $p_1, p_2 : A \to B$ in $\mathscr{D}$ then their pushout

$$\begin{tikzcd} {} \arrow[r, "p_1"] \arrow[d, "p_2"'] & {} \arrow[d, "k_2"] \\ {} \arrow[r, "k_1"'] & {} \end{tikzcd}$$

is an $\mathscr{E}$-cointersection (by definition of reflexivity both p_1 and p_2 are split, hence regular epis) such that $k_1 = k_2$ is a coequalizer. Since U preserves $\mathscr{E}$-cointesections (I,10) there follows that

$$\begin{tikzcd} {} \arrow[r, "Up_1"] \arrow[d, "Up_2"'] & {} \arrow[d, "Uk_2"] \\ {} \arrow[r, "Uk_1"'] & {} \end{tikzcd}$$

is a cointersection, i.e. pushout. Since $Uk_1 = Uk_2$, this is the coequalizer of Up_1 and Up_2.

Now, let $b : UI^{\#} \to Y$ be an arbitrary behavior. In its minimal realization $S_{(b)} = (Q_0, Y, y_0, I, i_0)$ denote, for short,

$$f = i_0^{\#} : I^{\#} \to Q_0 .$$

This is a regular epi, since $S_{(b)}$ is reachable; by hypothesis also Uf is a regular epi. We shall prove that the kernel pair of Uf:

$$p_1, p_2 : E \to UI^{\#}$$

is a Nerode equivalence. That will conclude the proof that our system theory has Nerode realizations: the coequalizer of p_1 and p_2 is Uf, because Uf is a regular epi.

(i_1) The pair p_1, p_2 is reflexive (since it is a kernel pair) and b-equivalent. Indeed, since $S_{(b)}$ is a realization of b, we have

$$b = y_0 \,.\, UI_0^{\#} = y_0 \,.\, Uf \,.$$

Further, $Uf \,.\, p_1 = Uf \,.\, p_2$ implies

$$f \,.\, p_1^{\#} = f \,.\, p_2^{\#} : E^{\#} \to Q_0$$

because, denoting $g = Uf \,.\, p_1 = Uf \,.\, p_2$, we have

$$g = Uf \,.\, Up_1^{\#} \,.\, \eta = U(f \,.\, p_1^{\#}) \,.\, \eta$$

hence $g^{\#} = f \,.\, p_1^{\#}$ – analogously $g^{\#} = f \,.\, p_2^{\#}$.

Thus

$$b \,.\, Up_1^{\#} = y_0 \,.\, U(f \,.\, p_1^{\#}) = y_0 \,.\, U(f \,.\, p_2^{\#}) = b \,.\, Up_2^{\#} \,.$$

(i_2) For each reflexive b-equivalent pair $q_1, q_2 : \tilde{E} \to UI^{\#}$ we are going to verify that $Uf \,.\, q_1 = Uf \,.\, q_2$ – then, by definition by kernel pairs – there exists a unique morphisms $h \,.\, \tilde{E} \to E$ with $q_1 = p_1 \,.\, h$ and $q_2 = p_2 \,.\, h$.

$$\begin{tikzcd}
 & \tilde{E} \arrow[dl, dashed, "h"'] \arrow[d, shift right, "q_1"'] \arrow[d, shift left, "q_2"] & & \\
E \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & UI^{\#} \arrow[r, "Uf"] \arrow[d, "Uk"'] & UQ_0 \arrow[r, "y_0"] \arrow[dl, dashed, "Ug"'] & Y \arrow[dll, "y"] \\
 & UQ & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \tilde{E}^{\#} \arrow[d, shift right, "q_1^{\#}"'] \arrow[d, shift left, "q_2^{\#}"] & \\
E^{\#} \arrow[r, shift left, "p_1^{\#}"] \arrow[r, shift right, "p_2^{\#}"'] & I^{\#} \arrow[r, "f"] \arrow[d, "k"] & Q_0 \\
 & Q &
\end{tikzcd}$$

Since q_1, q_2 is a reflexive pair, there exists a morphism $\tilde{d} : UI^{\#} \to \tilde{E}$ with $q_1 \,.\, \tilde{d} = q_2 \,.\, \tilde{d} = 1_{UI^{\#}}$. There follows that also the pair $q_1^{\#}$, $q_2^{\#} : \tilde{E}^{\#} \to I^{\#}$ is reflexive (in $\mathscr{D}$): put

$$d_0 : \eta_{\tilde{E}} \,.\, \tilde{d} \,.\, \eta_I : I \to U\tilde{E}^{\#}$$

then the morphism $d_0^{\#} : I^{\#} \to \tilde{E}^{\#}$ fulfills $q_1^{\#} \,.\, d_0^{\#} = q_2^{\#} \,.\, d_0^{\#} = 1_{I^{\#}}$. (Proof: it suffices to verify that $U(q_1^{\#} \,.\, d_0^{\#}) \,.\, \eta_I = U(q_2^{\#} \,.\, d_0^{\#}) \,.\, \eta_I = U1_{I^{\#}} \,.\, \eta_I$. This is easy, for

$$Uq_1^{\#} \,.\, Ud_0^{\#} \,.\, \eta_I = Uq_1^{\#} \,.\, d_0 = Uq_1^{\#} \,.\, \eta_{\tilde{E}} \,.\, \tilde{d} \,.\, \eta_I = q_1 \,.\, \tilde{d} \,.\, \eta_I =$$
$$= 1_{UI^{\#}} \,.\, \eta_I = \eta_I$$

and analogously $Uq_2^{\#} \,.\, Ud_0^{\#} \,.\, \eta_I = \eta_I$.)

Therefore, as remarked at the start of this proof, there exists a coequalizer $k : I^{\#} \to \to Q$ of $q_1^{\#}$, $q_2^{\#}$, preserved by U. Now, the pair q_1, q_2 is b-equivalent, thus

$$b \,.\, Uq_1^{\#} = b \,.\, Uq_2^{\#} \,.$$

Because Uk is the coequalizer of $Uq_1^{\#}$ and $Uq_2^{\#}$, there exists a unique morphism $y : UQ \to Y$ with

$$b = y \,.\, Uk \,.$$

Now we can define a system

$$S = (Q, Y, y, I, Uk \,.\, \eta_I) \,.$$

Since $(Uk \,.\, \eta_I)^{\#} = k$, this is a reachable system with behavior $b_S = y \,.\, Uk = b$.

Thus, S is a reachable realization of the behavior b. There follows that there exists a system morphism $g = S \to S_{(b)}$ in $\mathscr{E}_0$. Now

$$i_0 = Ug \,.\, (Uk \,.\, \eta_I)$$

implies

$$f = i_0^{\#} = g \,.\, k$$

(proof: $i_0^{\#}$ is the only morphism with $i_0 = Ui_0^{\#} \,.\, \eta_I$ and we have $i_0 = U(g \,.\, k) \,.\, \eta_I$). We get

$$Uf \,.\, q_1 = Ug \,.\, (Uk \,.\, q_1) = Ug \,.\, (Uk \,.\, q_2) = Uf \,.\, q_2 \,.$$

That concludes the proof of (i).

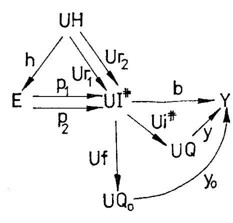
(ii) Assume that the system theory has Nerode realizations. Thus, for each behavior $b : UI^{\#} \to Y$ we have a Nerode equivalence $p_1, p_2 : E \to UI^{\#}$ with a coequalizer $Uf : UI^{\#} \to UQ_0$. We shall prove that the Nerode realization

$$S_0 = (Q_0, Y, y_0, I, Uf \,.\, \eta_I) \,,$$

where y_0 fulfils

$$b = y_0 \,.\, Uf \,,$$

is the minimal realization of b.



First, $(Uf \,.\, \eta_I)^{\#} = f \in \mathscr{E}$, thus, S_0 is reachable system with behavior $b_{S_0} = y_0 \,.\, Uf = b$.

Second, given a reachable realization $S = (Q, Y, y, I, i)$ of b, we shall verify that S_0 is its reduction. Let $r_1, r_2 : H \to I^{\#}$ be a kernel pair of $i^{\#} : I^{\#} \to Q$ in the category $\mathscr{D}$. This is a reflexive pair, hence so is (obviously) the pair Ur_1, Ur_2 in $\mathscr{K}$. Since S realizes b, we have $b = y \,.\, Ui^{\#}$ and so

$$b \,.\, Ur_1 = y \,.\, U(i^{\#} \,.\, r_1) = y \,.\, U(i^{\#} \,.\, r_2) = b \,.\, Ur_2 \,.$$

Thus, Ur_1 and Ur_2 is a reflexive, b-equivalent pair. This implies that there exists a unique morphism $h : UH \to E$ with $Ur_1 = p_1 \,.\, h$ and $Ur_2 = p_2 \,.\, h$.

There follows

$$U(f \,.\, r_1) = Uf \,.\, p_1 \,.\, h = Uf \,.\, p_2 \,.\, h = U(f \,.\, r_2)$$

and, since U is a faithful functor (I,2), we get

$$f \,.\, r_1 = f \,.\, r_2 \,.$$

Now $i^{\#}$ is a regular epi (since S is reachable system), hence a coequalizer of r_1 and r_2. This proves that there exists a unique morphism $g : Q \to Q_0$ subject to

$$f = g \,.\, i^{\#} \,.$$

Since $g \,.\, i^{\#} \in \mathscr{E}$ implies $g \in \mathscr{E}$, it suffices to verify that $g : S \to S_0$ is a system morphism to conclude the proof. By $f = g \,.\, i^{\#}$ we have

$$Uf \,.\, \eta_I = Ug \,.\, Ui^{\#} \,.\, \eta_I = Ug \,.\, i$$

and also

$$(y_0 \,.\, Ug) \,.\, Ui^{\#} = y_0 \,.\, Uf = b = y \,.\, Ui^{\#} \,,$$

which implies

$$y_0 \,.\, Ug = y$$

because $Ui^{\#}$ is epi.

(Received November 6, 1979.)

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