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An Invariant for Continuous Mappings

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The purpose of this work is to show that the Topological Entropy given by Adler, Konheim and McAndrew [1] is not the only invariant for continuous mappings, but, there also exists another invariant which we call here the Topological δ-entropy.

1. δ-ENTROPY

In what follows, we shall assume that X is a compact topological space. For any open cover $\mathfrak{A}$ of X, let $N(\mathfrak{A})$ denote the number of sets in a subcover of minimal cardinality. A subcover of a cover is minimal if no other subcover contains fewer members. Since X is compact and $\mathfrak{A}$ is an open cover, therefore there always exists a finite subcover.

Definition 1.1. The expression $H_\delta(\mathfrak{A}) = [\log N(\mathfrak{A})]^\delta$, $0 < \delta \leqq 1$; is defined as the δ-entropy of the open cover $\mathfrak{A}$.

Definition 1.2. For any two open covers $\mathfrak{A}$ and $\mathfrak{B}$ of X, $\mathfrak{A} \vee \mathfrak{B} = \{A \cap B / A \in \mathfrak{A}, B \in \mathfrak{B}\}$ is defined as the join of $\mathfrak{A}$ and $\mathfrak{B}$.

Definition 1.3. An open cover $\mathfrak{B}$ is said to be a refinement of an open cover $\mathfrak{A}$; denoted as $\mathfrak{A} \prec \mathfrak{B}$, if every member of $\mathfrak{B}$ is a subset of some member of $\mathfrak{A}$.

The following theorem shows that δ-entropy is sub-additive.

Theorem 1.1. If $\mathfrak{A}$ and $\mathfrak{B}$ are open covers of X, then

$$H_\delta(\mathfrak{A} \vee \mathfrak{B}) \leqq H_\delta(\mathfrak{A}) + H_\delta(\mathfrak{B}).$$

Proof. Let $\{A_1, A_2, \ldots, A_{N(\mathscr{U})}\}$ be a minimal subcover of $\mathfrak{A}$ and $\{B_1, B_2, \ldots, B_{N(\mathscr{B})}\}$ (here from typographical reasons $N(\mathscr{U})$, resp. $N(\mathscr{B})$, is used instead of $N(\mathfrak{A})$, resp.

 $N(\mathfrak{B})$) be a minimal subcover of $\mathfrak{B}$. Now, $\{A_i \cap B_j / i = 1, 2, \ldots, N(\mathfrak{A}); j = 1, 2, \ldots \ldots, N(\mathfrak{B})\}$ is a subcover of $\mathfrak{A} \vee \mathfrak{B}$. Consequently,

$$N(\mathfrak{A} \vee \mathfrak{B}) \leqq N(\mathfrak{A})\, N(\mathfrak{B}) \Rightarrow \log N(\mathfrak{A} \vee \mathfrak{B}) \leqq \log N(\mathfrak{A}) + \log N(\mathfrak{B}) \Rightarrow$$

$$\Rightarrow [\log N(\mathfrak{A} \vee \mathfrak{B})]^\delta \leqq [\log N(\mathfrak{A})]^\delta + [\log N(\mathfrak{B})]^\delta \Rightarrow$$

(cf. Hardy [2] — p. 32)

$$\Rightarrow H_\delta(\mathfrak{A} \vee \mathfrak{B}) \leqq H_\delta(\mathfrak{A}) + H_\delta(\mathfrak{B}).$$

2. TOPOLOGICAL δ-ENTROPY

Let Φ be a continuous mapping of X into itself. If $\mathfrak{A}$ is an open cover of X, then, the family $\Phi^{-1}\mathfrak{A} = \{\Phi^{-1}A / A \in \mathfrak{A}\}$ is also an open cover.

Definition 2.1. The Topological δ-entropy $h_\delta(\Phi)$ of a continuous mapping Φ is defined as

$$h_\delta(\Phi) = \operatorname{Sup} h_\delta(\Phi, \mathfrak{A})$$

where Sup is taken over all open covers $\mathfrak{A}$ of X and $h_\delta(\Phi, \mathfrak{A})$ is given by

$$h_\delta(\Phi, \mathfrak{A}) = \lim_{n \to \infty} H_\delta(\mathfrak{A} \vee \Phi^{-1}\mathfrak{A} \vee \ldots \vee \Phi^{-(n-1)}\mathfrak{A}) \mid n^\delta .$$

In the following note we justify that this limit exists and is finite.

Note 2.1. Let the number of members in a minimal subcover of $\mathfrak{A} \vee \Phi^{-1}\mathfrak{A} \vee \ldots \ldots \vee \Phi^{-(n-1)}\mathfrak{A}$ be denoted by $N_n(\mathfrak{A})$. Therefore,

$$h_\delta(\Phi, \mathfrak{A}) = \lim_{n \to \infty} H_\delta(\mathfrak{A} \vee \Phi^{-1}\mathfrak{A} \vee \ldots \vee \Phi^{(n-1)}\mathfrak{A}) \mid n^\delta =$$

$$= \lim_{n \to \infty} \frac{[\log N_n(\mathfrak{A})]^\delta}{n^\delta} = \lim_{n \to \infty} \left[\frac{\log N_n(\mathfrak{A})}{n}\right]^\delta = \left[\lim_{n \to \infty} \frac{\log N_n(\mathfrak{A})}{n}\right]^\delta .$$

From [1]

$$\lim_{n \to \infty} \frac{1}{n} \log N_n(\mathfrak{A})$$

exists and is finite. Hence,

$$\lim_{n \to \infty} \frac{H_\delta(\mathfrak{A} \vee \Phi^{-1}\mathfrak{A} \vee \ldots \vee \Phi^{-(n-1)}\mathfrak{A})}{n^\delta}$$

exists and is finite.

Theorem 2.1. Topological δ-entropy is an invariant in the sense that $h_\delta(\Psi\Phi\Psi^{-1}) = h_\delta(\Phi)$ where Φ is a continuous mapping of X into itself and Ψ is a homeomorphism of X onto some X'; where X and X' both are compact topological spaces.

Proof. For an open cover $\mathfrak{A}$ of X, we have

$$h_\delta(\Psi\Phi\Psi^{-1}, \Psi\mathfrak{A}) =$$

$$= \lim_{n\to\infty} H_\delta(\Psi\mathfrak{A} \vee (\Psi\Phi\Psi^{-1})^{-1} \Psi\mathfrak{A} \vee \ldots \vee (\Psi\Phi\Psi^{-1})^{-(n-1)} \Psi\mathfrak{A})/n^\delta =$$

$$= \lim_{n\to\infty} H_\delta(\Psi\mathfrak{A} \vee \Psi\Phi^{-1}\Psi^{-1}\Psi\mathfrak{A} \vee \ldots \vee \Psi\Phi^{-(n-1)}\Psi^{-1}\Psi\mathfrak{A})/n^\delta =$$

$$= \lim_{n\to\infty} H_\delta(\Psi\mathfrak{A} \vee \Psi\Phi^{-1}\mathfrak{A} \vee \ldots \vee \Psi\Phi^{-(n-1)}\mathfrak{A})/n^\delta =$$

$$= \lim_{n\to\infty} H_\delta(\mathfrak{A} \vee \Phi^{-1}\mathfrak{A} \vee \ldots \vee \Phi^{-(n-1)}\mathfrak{A})/n^\delta = h_\delta(\Phi, \mathfrak{A}).$$

Since Ψ is a homeomorphism; therefore, as $\mathfrak{A}$ ranges over all open covers of X, $\Psi\mathfrak{A}$ ranges over all open covers of X'. Hence,

$$h_\delta(\Psi\Phi\Psi^{-1}) = h_\delta(\Phi).$$

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REFERENCES

[1] R. L. Adler, A. G. Konheim, M. H. McAndrew: Topological entropy. Trans. Amer. Math. Soc. *114* (1965), 2, 309—319.

[2] G. H. Hardy, J. E. Littelwood, G. Polya: Inequalities. 2nd edition, Cambridge University Press, Cambridge 1956.

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