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Time-Optimal Control of a Third-Order Plant

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In this paper the time-optimal control of a plant possessing one pole at the origin and two non-zero distinct real poles is studied. Cases corresponding to both non-zero poles in the left-half-plane and one or both non-zero poles in the right-half-plane are considered. In each case, the system is analyzed in the regulating or tracking modes and a controller which simultaneously reduces the error and its first and second derivatives to zero in minimum time is designed. For the tracking system, the class of admissible inputs is found which, after a minimum transient time, can be followed without any error. It is shown that in both regulating and tracking systems if the plant is stable the error and its first and second derivatives can be reduced to zero with at most two switching reversals of the control. The same is established when the plant is unstable, provided that the initial values of error and/or its derivatives fall in a "controllable region". It is shown that, with some linear transformations, the equations of the switch curve and the switch surface can be made independent of the plant's constant gain and dependent only on the ratio of the non-zero pole values.

1. INTRODUCTION

There are numerous papers in control literature on the time-optimal control problem. A comprehensive treatment of the subject as well as a bibliography of some of these papers appear in [1]. Most of these papers, however, deal with second-order stable systems with zero input. Control of unstable regulating or tracking systems, especially of orders larger than two, has received little attention. Higdon [2] presents a study of the control of unstable systems. Also, the class of input signals that an n-th order relay-controlled plant can track has been investigated in references [3] and [4].

In this paper time-optimal control of a third order plant is studied. The control is assumed to be limited in magnitude. The plant is assumed to have one pole at the origin of the complex plane and two non-zero distinct real poles. Three pole configurations are considered: both non-zero poles in the left-half of the complex plane;

one non-zero pole in the left-half plane and the other in the right-half plane; and both non-zero poles in the right-half plane.

It is well-known that the time-optimal control of a normal n-th order plant with real poles and with control of limited magnitude is of the bang-bang type. This control is unique and can be achieved by at most $n - 1$ switching reversals of the control (cf. [1]). If the plant is stable, any initial state in the state space can be driven to the origin in minimum time by the application of the time-optimal control. For unstable systems, however, the initial state must fall in a controllable region which is a sub-space of the state space.

The controllable region is determined for the unstable cases and it is established that this region reduces as the number of unstable poles increases. Explicit equations are obtained for the switch surface and the switch curve within the switch surface in each case. Through linear similarity transformations it is shown that the equations for the switch curve and the switch surface can be made independent of the constant gain of the plant. Further, it is shown that the projection of the switch curve on one of the two-dimensional planes in the state space is independent of the pole values, and that the equations of the switch curve and the switch surface depend only on the ratio of the values of the non-zero poles. Finally, the class of input signals that the plant can follow perfectly after a minimum transient time is determined.

2. PROBLEM FORMULATION

The block diagram of the system under consideration is shown in Fig. 1 where the plant transfer function is

$$G(s) = \frac{K}{s(s + a)(s + b)}, \qquad a \neq b. \tag{1}$$

The control $u(t)$ is assumed to be limited in magnitude. Eqn. (1) implies that

$$[D^3 + (a + b) D^2 + abD] c(t) = Ku(t) \tag{2}$$

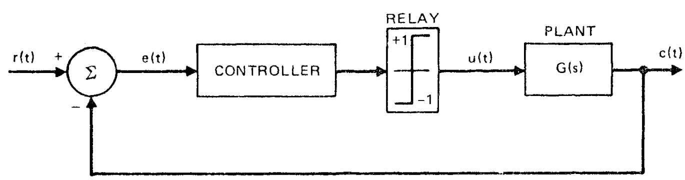

Fig. 1. Block diagram of a relay-controlled plant.

where $D = d/dt$. The plant output is

$$c(t) = r(t) - e(t) \tag{3}$$

where $r(t)$ is the input and $e(t)$ represents the error. From (2) note that one can assume $|u(t)| \leqq 1$ for all t with no loss of generality.

For the regulating problem $r(t) = 0$ for all t and (2) becomes

$$[D^3 + (a + b) D^2 + abD] e(t) = -Ku(t)\,. \tag{4}$$

Defining $e(t) = e_1(t)$, $D[e(t)] = \dot{e}(t) = e_2(t)$ and $D^2[e(t)] = \ddot{e}(t) = e_3(t)$, (4) can be written in the matrix form as

$$\mathbf{E}(t) = \mathbf{A}\,\mathbf{E}(t) + \mathbf{B}\,u(t) \tag{5}$$

where

$$\mathbf{E}(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -ab & -(a+b) \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ -K \end{bmatrix}.$$

The eigenvalues of the matrix $\mathbf{A}$ are the poles of the plant transfer function, i.e., $\lambda_1 = 0$, $\lambda_2 = -a$ and $\lambda_3 = -b$. Define matrix Λ as $\Lambda = \operatorname{diag}[\lambda_1, \lambda_2, \lambda_3]$. Then (cf. [1]) there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \Lambda$ where $\mathbf{P}$ is the Vandermonde matrix of the eigenvalues of $\mathbf{A}$, i.e.

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -a & -b \\ 0 & a^2 & b^2 \end{bmatrix}. \tag{6}$$

Defining $\mathbf{Y}(t) = \mathbf{P}^{-1}\,\mathbf{E}(t)$, or

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \frac{1}{ab(a-b)} \begin{bmatrix} ab(a-b) & a^2 - b^2 & a - b \\ 0 & b^2 & b \\ 0 & -a^2 & -a \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}, \tag{7}$$

(5) yields

$$\dot{\mathbf{Y}}(t) = \Lambda\,\mathbf{Y}(t) + \bar{\mathbf{B}}\,u(t) \tag{8}$$

where

$$\bar{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B} = \frac{K}{ab(a-b)} \begin{bmatrix} b - a \\ -b \\ a \end{bmatrix}. \tag{9}$$

Now let

$$x_1(t) = \frac{a^2 b}{K}\,y_1(t)\,, \quad x_2(t) = \frac{a^2(b-a)}{K}\,y_2(t) \quad \text{and} \quad x_3(t) = \frac{b^2(b-a)}{K}\,y_3(t)\,. \tag{10}$$

Then from (7), (8) and (9) we have

$$\dot{\mathbf{X}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -b \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} -a \\ a \\ -b \end{bmatrix} u(t) \tag{11}$$

where $|u(t)| \leqq 1$ for all t. Let

$$x_1(0) = \xi_1\,, \quad x_2(0) = \xi_2 \quad \text{and} \quad x_3(0) = \xi_3\,. \tag{12}$$

Note that $\dot{\mathbf{X}}(t)$ does not depend on the plant's constant gain K. Further, note that $\mathbf{X}(t) = 0 \Leftrightarrow \mathbf{Y}(t) = 0 \Leftrightarrow \mathbf{E}(t) = 0$. Therefore, driving the state vector $\mathbf{X}(t)$ to the origin of the state space is equivalent to driving the error and its first and second derivatives to zero. Fig. 2 shows how $x_1(t)$, $x_2(t)$ and $x_3(t)$ can be obtained from $e_1(t)$, $e_2(t)$ and $e_3(t)$.

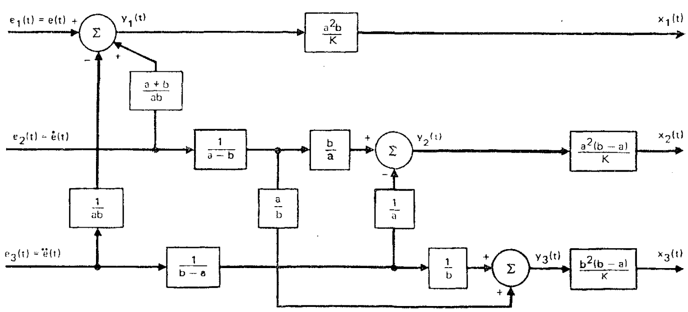

Fig. 2. Block diagram of the linear transformation generating $x_1(t)$, $x_2(t)$ and $x_3(t)$ from $e(t)$, $\dot{e}(t)$ and $\ddot{e}(t)$.

System (11) is normal if $a \neq b$ (cf. [1]). Then if a and b are real numbers, there exists a unique bang-bang control sequence with at most two switching reversals which drives any $\mathbf{X}(0)$ in the "controllable region" to the origin of the state space in minimum time. Hence, we can let $u(t) = u = \pm 1$ for all t in (11).

3. TIME-OPTIMAL CONTROL OF STABLE PLANT

Let $a > 0$ and $b > 0$. Then the plant will be stable (cf. [5]). We first obtain the equations of the switch curve and the switch surface. Then we determine the time-optimal control law and present its engineering realization.

With $u(t) = u = \pm 1$ for all t and with initial conditions (12), the solution of (11) is

(13a) $$x_1(t) = -aut + \xi_1 \,,$$

(13b) $$x_2(t) = (\xi_2 - u)\,e^{-at} + u \,,$$

(13c) $$x_3(t) = (\xi_3 + u)\,e^{-bt} - u \,.$$

Eliminating time t in (13) yields

(14) $$x_2 = (\xi_2 - u)\exp\left[u(x_1 - \xi_1)\right] + u$$

(15) $$x_3 = (\xi_3 + u)\exp\left[\frac{b}{a}\,u(x_1 - \xi_1)\right] - u$$

where the dependence on t is dropped for convenience. Eqns. (14) and (15) express the trajectory of the system in the three-dimensional state space, which starts at $\mathbf{X}(0) = (\xi_1, \xi_2, \xi_3)^T$ and is generated by the constant control u. Eqn. (14) represents the projection of this trajectory onto the x_1x_2 plane. Note that this equation is independent of the pole values a and b. Eqn. (15) which depends on the ratio b/a, represents the projection of this trajectory onto the x_1x_3 plane. Figs. 3 and 4 show the curves generated by (14) and (15), respectively, for different values of ξ_1, ξ_2 and ξ_3.

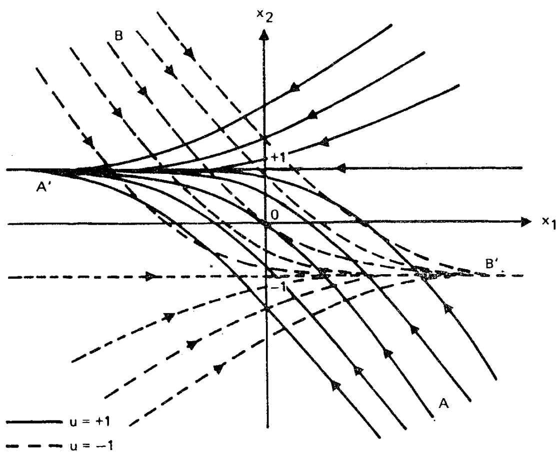

Fig. 3. The projections of the forced trajectories in the x_1x_2 plane for the case $a > 0$, $b > 0$ (The arrows indicate the direction of increasing time.).

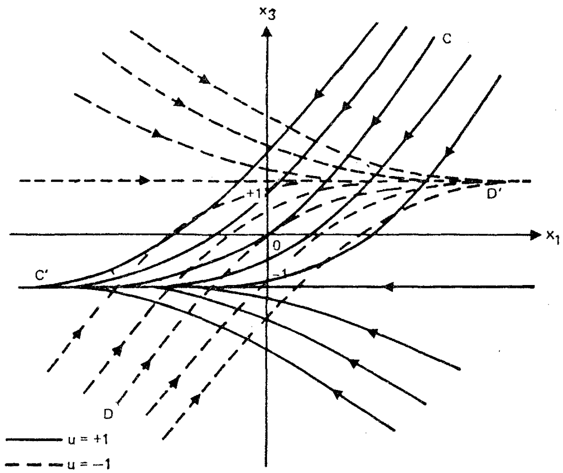

Fig. 4. The projections of the forced trajectories in the x_1x_3 plane for the case $a > 0$, $b > 0$ (The arrows indicate the direction of increasing time.).

The equations of the curves $A0A'$ and $B0B'$ which pass through the origin in Fig. 3 are $x_2 = 1 - \exp x_1$ and $x_2 = -1 + \exp(-x_1)$, respectively. These curves are tangent to each other at the origin. Let Γ_{12}^+ and Γ_{12}^- represent the curve sections $A0$ and $B0$, respectively, and let Γ_{12} be the union of Γ_{12}^+ and Γ_{12}^- (Fig. 5). The equation of the curve Γ_{12} is, then,

$$x_2 = \operatorname{sgn}(x_1)(1 - \exp|x_1|) \tag{16}$$

where sgn $(\cdot)$ indicates the signum function. Similarly, let Γ_{13}^+ and Γ_{13}^- represent the curve sections $C0$ and $D0$ in Fig. 4, respectively, and let Γ_{13} be the union of Γ_{13}^+ and Γ_{13}^- (Fig. 6). The equation of the curve Γ_{13} is

$$x_3 = \operatorname{sgn}(x_1)\left[-1 + \exp\left(\frac{b}{a}|x_1|\right)\right]. \tag{17}$$

The effect of variation of the ratio b/a on Γ_{13} is shown in Fig. 6.

Let Γ^+ be the curve whose projections on the x_1x_2 and x_1x_3 planes are respectively, Γ_{12}^+ and Γ_{13}^+. The curve Γ^+ is the set of all states which can be forced to the origin of the state space in minimum time by the control $u = +1$. Curve Γ^- defined in a similar manner is the set of all states which can be forced to the origin in minimum

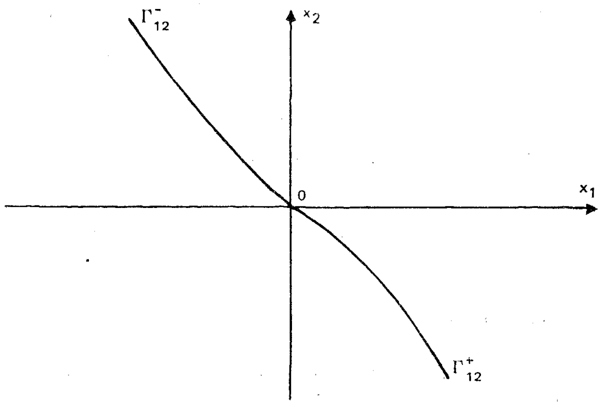

Fig. 5. Curve Γ_{12}, the projection of the switch curve Γ on the x_1x_2 plane when $a > 0$ and $b > 0$.

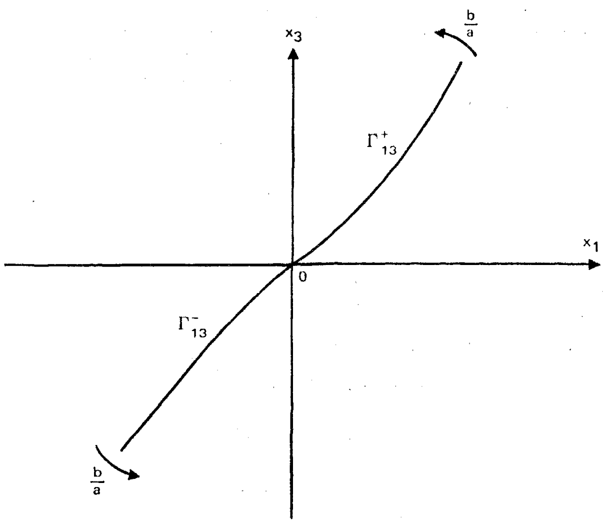

Fig. 6. Curve Γ_{13}, the projection of the switch curve Γ on the x_1x_3 plane when $a > 0$ and $b > 0$.

time by the control $u = -1$. Note that $x_1 > 0$ for Γ^+ and $x_1 < 0$ for Γ^-. Define curve Γ as the union of Γ^+ and Γ^-. We call Γ the switch curve. Thus any state on the switch curve Γ can be driven to the origin in minimum time by the application of the control

$$u_\Gamma^* = \operatorname{sgn}(x_1)\,. \tag{18}$$

Using (16), (17) and (18), the equation of the switch curve is

(19)
$$\Gamma = \left\{(x_1, x_2, x_3) : x_2 = u_\Gamma^* \left[1 - \exp\left(u_\Gamma^* x_1\right)\right],\ x_3 = u_\Gamma^* \left[-1 + \exp\left(\frac{b}{a} u_\Gamma^* x_1\right)\right]\right\}.$$

3.2. Equation of the Switch Surface

We define the switch surface, S, as the set of all states (x_1, x_2, x_3) which can be driven to the origin of the state space by one of the control sequences $\{+1, -1\}$, $\{-1, +1\}$, $\{+1\}$ or $\{-1\}$. Thus the switch surface S contains the switch curve Γ. To obtain the equation of the switch surface, consider a state (x_1, x_2, x_3) and suppose that it can be driven to the origin by the control sequence $\{u, -u\}$. This would mean that the state (x_1, x_2, x_3) can be driven to the switch curve Γ by the control u and subsequently to the origin by the control $-u$. Suppose that the state $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \Gamma$ lies on the trajectory starting at (x_1, x_2, x_3) and generated by the control u. Then from (14), (15) and (19) we have

$$\bar{x}_2 = (x_2 - u) \exp\left[u(\bar{x}_1 - x_1)\right] + u \tag{20a}$$

$$\bar{x}_3 = (x_3 + u) \exp\left[\frac{b}{a} u(\bar{x}_1 - x_1)\right] - u \tag{20b}$$

$$\bar{x}_2 = -u\left[1 - \exp\left(-u\bar{x}_1\right)\right] \tag{20c}$$

$$\bar{x}_3 = -u\left[-1 + \exp\left(-\frac{b}{a} u\bar{x}_1\right)\right]. \tag{20d}$$

Eliminate $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ in (20) to obtain

$$1 + u\left[1 + (ux_2 - 1)\exp\left(-ux_1\right)\right]^{1/2} = \left\{1 - u\left[1 - (ux_3 + 1)\exp\left(-\frac{b}{a} ux_1\right)\right]^{1/2}\right\}^{a/b} \tag{21}$$

which is the equation of the switch surface S. It can be verified that S is a single-valued continuous and real surface which separates the state space into two parts. For $u = +1$, (21) yields the equation of a surface, S^+, which is the set of all states

(x_1, x_2, x_3) that can be driven to the switch curve Γ by the control $u = +1$. For $u = -1$, (21) yields the equation of a surface, S^-, which is the set of all states (x_1, x_2, x_3) that can be driven to Γ by the control $u = -1$. Note that S^+ and S^- are disjoint and $S = S^+ \cup S^-$. The switch surface S contains the switch curve Γ. In fact Γ separates S into two parts, $\Gamma^- \subset S^+$ and $\Gamma^+ \subset S^-$.

Control u in (21) can be determined as a function of x_1, x_2 and x_3. From Fig. 3 and Eqn. (16) observe that $(x_1, x_2, x_3) \in S^+$ if $x_2 > \operatorname{sgn}(x_1)(1 - \exp|x_1|)$, and $(x_1, x_2, x_3) \in S^-$ if $x_2 < \operatorname{sgn}(x_1)(1 - \exp|x_1|)$. It follows that

$$u = u_S^* = \operatorname{sgn}\left[x_2 - \operatorname{sgn}(x_1)(1 - \exp|x_1|)\right]. \tag{22}$$

3.3. Time-Optimal Control Law

Consider a state $(x_1, x_2, x_3) \notin S$. Let (x_1, x_2, x_{3p}) be the projection of the point represented by (x_1, x_2, x_3) in the state space onto S parallel to the x_3 axis. From (14), (15) and (21), it can be verified that if $x_3 > x_{3p}$ only the control $u = +1$, and if $x_3 < x_{3p}$ only the control $u = -1$ generates a trajectory which starts from (x_1, x_2, x_3) and intersects the switch surface S. Therefore, the time-optimal control law when $a > 0$ and $b > 0$ is as follows:

1. If (x_1, x_2, x_3) does not lie on the switch surface S given by (21) and (22), the time-optimal control is

$$u^* = \operatorname{sgn}(x_3 - x_{3p}) \tag{23}$$

   where (x_1, x_2, x_{3p}) is the projection of (x_1, x_2, x_3) onto S parallel to the x_3 axis.
2. If (x_1, x_2, x_3) lies on S but not the switch curve Γ described by (18) and (19), the control u_S^* given by (22) is time-optimal.
3. If (x_1, x_2, x_3) lies on Γ, the control u_Γ^* given by (18) is time-optimal.

Thus, the control sequence $\{u^*, u_S^*, u_\Gamma^*\}$ drives any state (x_1, x_2, x_3) which does not lie on S to the origin of the state space in minimum time. Note that this sequence is either $\{+1, -1, +1\}$ or $\{-1, +1, -1\}$. Also note that a maximum of two switching reversals of control is required to drive any state to the origin in minimum time.

3.4. Realization of the Time-Optimal Control

An engineering realization of the time-optimal control law is shown in Fig. 7. For each state (x_1, x_2, x_3), the value of x_{3p} is calculated, compared with x_3 and applied to the relay R to obtain u^* given by (23). The block indicated by "Switch

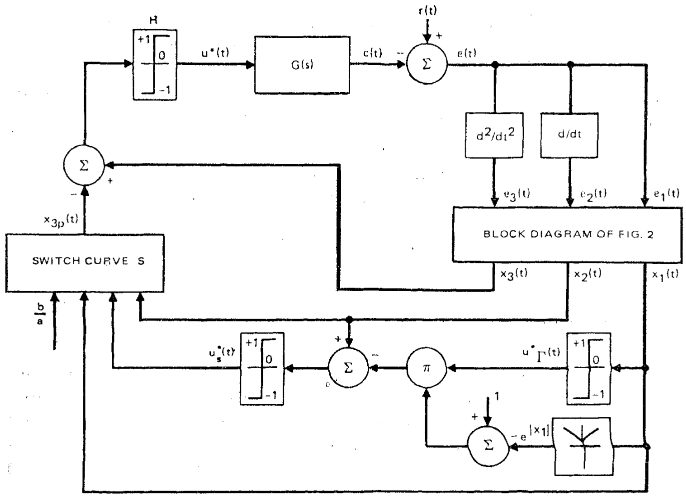

Fig. 7. Block diagram of the time-optimal control system.

Curve S" in Fig. 7 receives x_1, x_2, u_S^* and b/a as input and calculates x_{3p} which by (21) is given by

(24a) $$x_{3p} = -u_S^* + u_S^* M^{b/a}(2 - M^{b/a}) \exp\left(\frac{b}{a} u_S^* x_1\right)$$

where

(24b) $$M = 1 + u_S^*[1 + (u_S^* x_2 - 1)\exp(-u_S^* x_1)]^{1/2} .$$

An analog simulation of (24) can be used for the corresponding block in Fig. 7, or a digital computer can be used to calculate x_{3p}. Note that only the ratio b/a of the non-zero pole values plays a role in determining the time-optimal control.

4. TIME-OPTIMAL CONTROL OF UNSTABLE PLANT

When the plant is unstable, the initial state $\mathbf{X}(0)$ can no longer be arbitrary. Only initial states that fall within a certain sub-space of the three-dimensional state space can be driven to the origin by the application of a bang-bang control. We refer to this sub-space as the controllable region. We consider the cases where one or both of the non-zero poles are in the right half of the complex plane. The controllable region and the time-optimal control law will be determined in each case.

4.1. The Case of One Unstable Pole

Let $a < 0$ and $b > 0$. Then Eqns. (13), (14), and (15) still hold. Suppose that the initial state (ξ_1, ξ_2, ξ_3) can be driven to the origin by the application of the constant control u and that the required time is T. Then $x_2(T) = 0$ and (13b) yields $\xi_2 = u[1 - \exp(aT)]$.

Now since $|u| \leqq 1$, $a < 0$ and $T > 0$, we have

$$|\xi_2| \leqq 1 - \exp(aT) < 1 . \tag{25}$$

Note that no restrictions on ξ_1 and ξ_3 are required in order for T to be a finite positive real number. Therefore, the controllable region in this case is

$$-\infty < \xi_1 < \infty , \quad -1 < \xi_2 < +1 , \quad -\infty < \xi_3 < +\infty . \tag{26}$$

The curves generated by (14) for different values of ξ_1 and ξ_2 in this case are similar to those shown in Fig. 3 when the directions of the arrows are changed. The curves generated by (15) for different values of ξ_1 and ξ_3 in this case are similar to those shown in Fig. 3 except the dashed curves correspond to $u = +1$ and the solid curves correspond to $u = -1$. The equations for the curves Γ_{12} and Γ_{13} in this case are as follows:

$$\Gamma_{12}\colon\ x_2 = -\operatorname{sgn}(x_1)\,[1 - \exp(-|x_1|)] \tag{27}$$

$$\Gamma_{13}\colon\ x_3 = -\operatorname{sgn}(x_1)\left[-1 + \exp\left(-\frac{b}{a}|x_1|\right)\right]. \tag{28}$$

Note that $b/a < 0$ in this case. The curves Γ_{12} and Γ_{13} as well as the effect of variation of the ratio b/a on Γ_{13} are shown in Figs. 8 and 9.

Again define the curves Γ^+ and Γ^- as in the previous case. In this case, any state on the curve $\Gamma = \Gamma^+ \cup \Gamma^-$ can be driven to the origin of the state space in minimum time by the application of the control

$$u_\Gamma^* = -\operatorname{sgn}(x_1) . \tag{29}$$

The equation of the switch curve Γ in terms of u_Γ^* in this case is identical to that in the previous case [Eqn. (19)]. Since the system of equations (20) have not changed, the equation of the switch surface S in terms of u_S^* in this case is also the same as before [Eqn. (21)]. However, in the present case $a < 0$, $|x_2| < 1$ and

$$u_S^* = \operatorname{sgn}[x_2 + \operatorname{sgn}(x_1)(1 - \exp(-|x_1|))] . \tag{30}$$

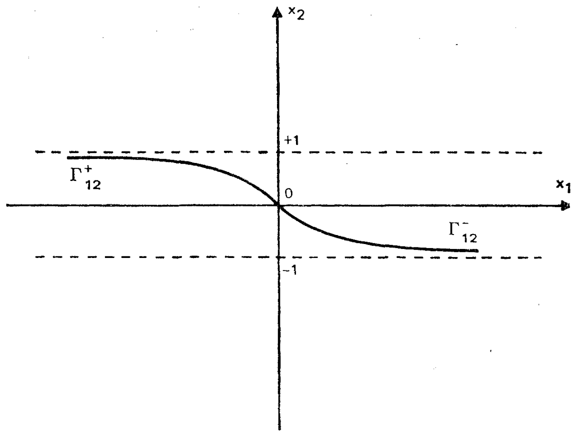

Fig. 8. Curve Γ_{12}, the projection of the switch curve Γ on the x_1x_2 plane when $a < 0$ and $b > 0$.

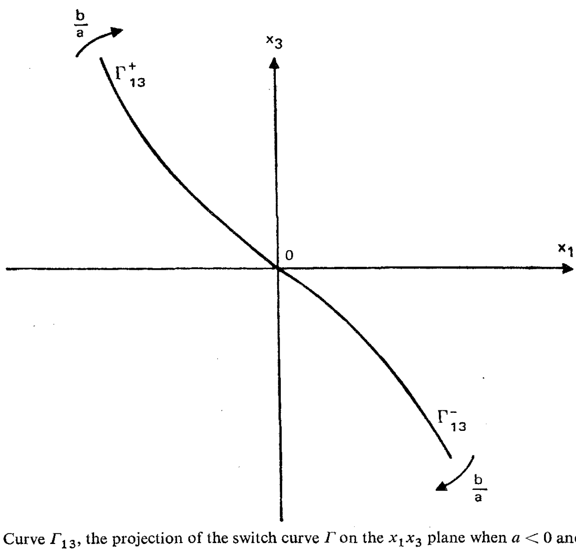

Fig. 9. Curve Γ_{13}, the projection of the switch curve Γ on the x_1x_3 plane when $a < 0$ and $b > 0$.

The time-optimal control law in this case is as follows:

1. If (x_1, x_2, x_3) does not lie on the switch surface S given by (21) and (30), the time-optimal control u^* is given by (23).
2. If (x_1, x_2, x_3) lies on S but not the switch curve Γ described by (19) and (29), the control u_S^* given by (30) is time-optimal.
3. If (x_1, x_2, x_3) lies on Γ, the control u_Γ^* given by (29) is time-optimal.

Thus, the control sequence $\{u^*, u_S^*, u_\Gamma^*\}$ drives any state $(x_1, x_2, x_3) \notin S$ in the controllable region (26) to the origin of the state space in minimum time. Note that a maximum of two switching reversals of control is required to drive any state in the controllable region to the origin. An engineering realization of this control law can be obtained by modifying the block diagram of Fig. 7 according to (29) and (30).

4.2. The Case of Two Unstable Poles

Let $a < 0$ and $b < 0$. Again Eqns. (13), (14), and (15) hold. Since $b < 0$, from (13c) with an analysis similar to that which led to (25) we obtain $|\xi_3| \leqq 1 - \exp(bT) < 1$. Thus, the controllable region (26) is further restricted to

(31) $$-\infty < \xi_1 < +\infty, \quad -1 < \xi_2 < +1, \quad -1 < \xi_3 < +1.$$

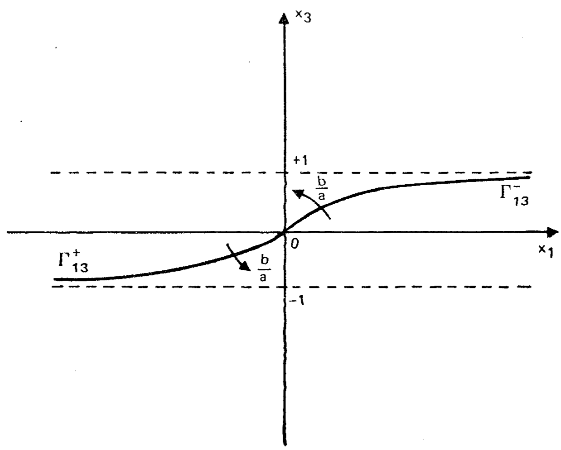

Fig. 10. Curve Γ_{13}, the projection of the switch curve Γ on the $x_1 x_3$ plane when $a < 0$ and $b < 0$.

The curves generated by (14) for different values of ξ_1 and ξ_2 in this case are exactly the same as those for the case $a < 0$, $b > 0$. Thus, the equation for the curve Γ_{12} is given by (27) which is illustrated in Fig. 8. The curves generated by (15) for different values of ξ_1 and ξ_3 in this case are similar to those shown in Fig. 4 when the directions of the arrows are changed. The equation for the curve Γ_{13} in this case is given by (28). Note that $b/a > 0$ in this case. The curve Γ_{13} as well as the effect of variation of the ratio b/a on it is shown in Fig. 10.

The equations of the switch curve Γ and the switch surface S in this case are identical to those for the case $a < 0$ and $b > 0$. That is, Γ is expressed by (19) and (29) and S is expressed by (21) and (30). Note that in these equations $a < 0$, $b < 0$, $|x_2| < 1$ and $|x_3| < 1$. With these restrictions, the time-optimal control law in this case is the same as that in the previous unstable case.

5. ADMISSIBLE INITIAL CONDITIONS

There are no restrictions on the initial conditions $e_1(0) = e(0)$, $e_2(0) = \dot{e}(0)$ and $e_3(0) = \ddot{e}(0)$ when the plant is stable. Restrictions are placed on these initial conditions by (26) and (31) when the plant is unstable. In order to determine these restrictions note that from (10) and (7) we have

$$x_2(t) = -\frac{a}{K}\left[b\,\dot{e}(t) + \ddot{e}(t)\right] \quad \text{and} \quad x_3(t) = -\frac{b}{K}\left[a\,\dot{e}(t) + \ddot{e}(t)\right].$$

Therefore $-1 < \xi_2 < +1$ implies that

$$|b\,\dot{e}(0) + \ddot{e}(0)| < \frac{|K|}{-a}. \tag{32}$$

Similarly, $-1 < \xi_3 < +1$ implies that

$$|a\,\dot{e}(0) + \ddot{e}(0)| < \frac{|K|}{-b}. \tag{33}$$

Thus, from (26), (32) must hold when $a < 0$. Also, from (31), (32) and (33) must hold when $a < 0$ and $b < 0$.

6. ADMISSIBLE INPUT FOR TRACKING SYSTEM

Thus far the input $r(t)$ has been assumed to be identically zero (regulating system). From (2) and (3), the inputs $r(t)$ that the system can track with no steady-state error must satisfy

$$\left[\mathrm{D}^3 + (a + b)\,\mathrm{D}^2 + ab\mathrm{D}\right] r(t) = 0. \tag{34}$$

If (34) were satisfied, Eqn. (4) would still hold and the analysis would be exactly the same as when $r(t) = 0$ for all t. The class of inputs $r(t)$ that satisfy (34) are of the form

(35) $$r(t) = r_0 + r_1 e^{-at} + r_2 e^{-bt}$$

where r_0, r_1 and r_2 are arbitrary constants. The plant output $c(t)$ can follow inputs of the form given in (35) with no steady-state error with at most two switching reversals of the control. Note that (35) has the same form as the impulse response of the plant $G(s)$.

7. CONCLUSIONS

A third order relay-controlled plant possessing one pole at the origin and two non-zero distinct real poles was analyzed. Cases corresponding to both non-zero poles in the left-half plane or in the right-half plane as well as the case with one pole in each half plane were studied. By simple linear transformations, the system was transformed to a canonical form which is independent of the plant's constant gain. The equations of the switch curve and the switch surface were derived in each case and it was shown that these equations are independent of the constant gain of the plant and that they depend only on the ratio of the non-zero pole values. The controllable regions were determined for the cases where the plant is unstable. It was shown that when the plant is stable, the controllable region is the whole three-dimensional state space. It was established that with a maximum of two switching reversals of the control any error and its first and second derivatives can be simultaneously brought to zero in minimum time provided that they are initially in the controllable region. The control law and its engineering realization were obtained in each case. Finally, the class of inputs which the plant can perfectly track with at most two switching reversals of the control were obtained. It was shown that this class of inputs has the same form as the impulse response of the plant.

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