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Simultaneous Channels Decomposable into Memoryless Components II

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The problem of asymptotic behaviour of the maximum length of ε-codes is studied in this paper for a special class of channels, which we introduce in Definition 1, the Section 1. The main result of this paper is Theorem on ε-Capacity.

We remark that the double numbered references are related to [1] and most of the basic notations are defined in the Section 2 in [1].

1. BASIC PROPERTIES OF SIMULTANEOUS CHANNELS

Let W be the set of all probability matrices and $\mathscr{S}_W$ be the σ-algebra of Borel sets in W generated by the metric

$$(1) \qquad \varrho(w, w^*) = \|w - w^*\| = \max_{j,i} |w(j \mid i) - w^*(j \mid i)| .$$

Let $(G, \mathscr{S}, \xi)$ be a probability space and $t : G \to W$ be an $\mathscr{S} - \mathscr{S}_W$ measurable mapping, i.e. $t^{-1}(F) \in \mathscr{S}$ for each $F \in \mathscr{S}_W$. Similarly as in the Section 1 in [1] we shall use notations t_α or w_{t_α} instead of $t(\alpha)$. We see that if $y \in B^n$, then there exists unique probability $w_t(\cdot \mid y)$ on subsets of A^n satisfying

$$(2) \qquad w_t(x \mid y) = \int_G \prod_{k=1}^n w_{t_\alpha}(x_k \mid y_k) \,\mathrm{d}\xi(\alpha)$$

for each $x \in A^n$. Hence if $\eta \in B^I$, then there exists unique probability $w_t(\cdot \mid \eta)$ defined on $\mathscr{T}_A$ such that

$$(3) \qquad w_t(\{x \in A^I;\, x_k = j_k,\, k = 1, \ldots, n\} \mid \eta) = w_t(\{j_k\}_{k=1}^n \mid \{\eta_k\}_{k=1}^n) ,$$

where the right-hand side is defined by (2). In order to introduce the definition of a simultaneous channel put

(4) $$T_0 = \{t;\, t : G \to W \text{ and is } \mathscr{S} - \mathscr{S}_W \text{ measurable}\}$$

and denote $R_{t_\alpha}(p)$ the transmission rate of the matrix w_{t_α} with respect to the vector p (cf. (1.6)).

Definition 1. Let $T \subset T_0$ be a non-empty set satisfying for every vector $p \in P$ the following conditions.

1. The function $\varphi(\alpha, p) = \inf\{R_{t_\alpha}(p);\, t \in T\}$ is measurable.
2. If $\varepsilon > 0$, then there exists such $s \in T$ that the inequality

$$\varphi(\alpha, p) \leqq R_{s_\alpha}(p) \leqq \varphi(\alpha, p) + \varepsilon$$

holds for every $\alpha \in G$.

The family of probabilities $\{w_t(\cdot \mid \eta);\, \eta \in B^I,\, t \in T\}$ which are defined by (3) will be called a simultaneous channel decomposable into memoryless components and will be denoted $C = (B, A, \xi, T)$.

Definition 2. Let $C = (B, A, \xi, T)$ be a simultaneous channel. An n-dimensional code $\{Q(y)\}_{y \in Y}$ (cf. [2] p. 116) will be called (n, N, ε) code for C, if the length of this code is N and (cf. (2))

$$w_t(Q(y) \mid y) = \sum_{x \in Q(y)} w_t(x \mid y) > 1 - \varepsilon$$

for every $y \in Y$, $t \in T$.

If $T = \{t\}$ i.e. T contains only element t, then (B, A, ξ, T) is a channel decomposable into memoryless components (cf. [3]); a more complicated example of the simultaneous channel is given in Proposition 1.

We see that in the general case described in Definition 1 the simultaneous channel is a family of channels $\{C_t\}_{t \in T}$, $C_t = (B, A, \xi, \{t\})$. The reason for such a definition can be that we want to construct an ε-code for a channel C_{t_0} and our only knowledge about t_0 is that t_0 belongs to T.

To prove the next assertions, we shall need the following notations. If $\{F_k\}_{k=1}^m$ is a partition of a set F and $\{w_k\}_{k=1}^m$ are probability matrices, then by the symbol

(5) $$t = \sum_1^m w_k \chi_{F_k}$$

we shall mean the mapping $t : F \to W$ such that $t(\alpha) = w_k$ for every $\alpha \in F_k$, $k = 1, \ldots, m$; if C_1, C_2 are disjoint sets and $t_i : C_i \to W$, then

(6) $$t = t_1 \chi_{C_1} + t_2 \chi_{C_2}$$

is the mapping $t : C_1 \cup C_2 \to W$ satisfying $t \mid C_i = t_i$, $i = 1, 2$.

Proposition 1. Let $t_0 \in T_0$ and $h : G \to \langle 0, 1 \rangle$ be a measurable function. If we denote (cf. (1), (4))

$$T = \{t \in T_0;\ \varrho(t(\alpha), t_0(\alpha)) \leqq h(\alpha) \quad \text{for every} \quad \alpha \in G\}\,, \tag{7}$$

then the set T satisfies the assumptions of Definition 1.

Proof. At first we prove measurability of the function $\varphi(., p)$. Let us denote W_n the set of all matrices $w \in W$ such that $w(j \mid i)$ is an integer multiple of n^{-1} for each $i \in B$, $j \in A$ and put

$$\mathcal{O}(w^*) = \{w \in W;\ \varrho(w, w^*) < d^2 n^{-1}\}\,, \quad d = \max\{a, b\}\ ;$$

the sets $\{\mathcal{O}(w^*);\ w^* \in W_n\}$ cover W by Lemma 2.4. If $\theta \in (0, 1)$ then there exists a finite measurable partition $\mathscr{C}_n = \{C_k^{(n)}\}_{k=1}^{m_n}$ of the set $G_\theta = \{\alpha \in G;\ h(\alpha) \geqq \theta\}$ satisfying for $k = 1, \ldots, m_n$ the conditions

1. there exists $w_k \in W_n$ such that $\varrho(t_0(\alpha), w_k) < d^2 n^{-1}$ for every $\alpha \in C_k^{(n)}$;
2. if $\alpha, \beta \in C_k^{(n)}$, then $|h(\alpha) - h(\beta)| < d^2 n^{-1}$.

This means that for each $n \geqq n_1 = n_1(d, \theta)$ there exists a mapping s_n with values in W_n such that (cf. (5))

$$s_n = \sum_1^{m_n} w_k \chi_{C_k^{(n)}} \tag{8}$$

and $\varrho(t_0(\alpha), s_n(\alpha)) \leqq d^2/n < \theta$ whenever $\alpha \in G_\theta$. Hence if we denote

$$W(n, k) = \{w \in W_n;\ \varrho(w, t_0(\alpha)) < h(\alpha) \quad \text{for every} \quad \alpha \in C_k^{(n)}\}\,,$$

$$T_n = \{\sum_1^{m_n} w_k \chi_{C_k^{(n)}};\ w_k \in W(n, k)\}\,, \quad T^j = \bigcup_{n=n_1}^{\infty} T_n$$

then these sets are non-empty. Since the set T^j is countable, it is sufficient to prove the relation

$$\varphi(\alpha, p)\, \chi_{G_\theta}(\alpha) = \inf\{R_{t_\alpha}(p);\ t \in T^j\}\,. \tag{9}$$

Let $\alpha \in G_\theta$ and γ be a positive number. Since the set T is convex, the inequality (2.16) implies that there is a matrix $w^* \in W$ satisfying

$$\varphi(\alpha, p) \leqq R_{w^*}(p) < \varphi(\alpha, p) + \gamma\,,$$
$$\varrho(w^*, t_0(\alpha)) < h(\alpha)\,.$$

If $n > n_1$ is such that the inequalities

$$R_{w^*}(p) < \varphi(\alpha, p) + \gamma - 8d^3 n^{-1/2}\,, \tag{10}$$
$$\varrho(w^*, t_0(\alpha)) < h(\alpha) - 4d^2/n$$

hold, we choose $w_n^* \in W_n$ so that

$$\varrho(w^*, w_n^*) < d^2/n \tag{11}$$

and put (cf. (8))

$$s_n^*(\beta) = \begin{cases} s_n(\beta) & \beta \notin C_k^{(n)}, \\ w_n^* & \beta \in C_k^{(n)}, \end{cases}$$

where $C_k^{(n)}$ is the set belonging to $\mathscr{C}_n$ which contains α. Taking into account (8), (10), (11) and the second property of $\mathscr{C}_n$, we see that $s_n^* \in T^j$ and (9) can be easily proved by means of (2.16).

The next step is to prove that the condition 2 in Definition 1 is fulfilled. For $\varepsilon > 0$ there is an $\theta > 0$ such that (cf. (2.16))

$$|R_w(p) - R_{w^*}(p)| < \varepsilon$$

whenever $\varrho(w, w^*) < \theta$. Let $\mathscr{D}_n = \{D_k^{(n)}\}_{k=1}^{m_n}$ be a measurable partition of the set $G_\theta = h^{-1}(\langle \theta, 1 \rangle)$ satisfying

$$|h(\alpha) - h(\beta)| < \theta/n\,, \quad |\varphi(\alpha, p) - \varphi(\beta, p)| < \varepsilon/4\,, \tag{12}$$
$$\varrho(t_0(\alpha), t_0(\beta)) < \theta/n$$

for each $\alpha, \beta \in D_k^{(n)}$, $k = 1, \ldots, m_n$. We choose $\Delta \in (2n^{-1}, d^2 n^{-1})$ and for every k we find $\alpha_k \in D_k^{(n)}$, $t_k \in T$ so that

$$|\varphi(\alpha_k, p) - R_{t_k(\alpha_k)}(p)| < \varepsilon/4\,. \tag{13}$$

Denoting $w_k = \Delta t_0(\alpha_k) + (1 - \Delta)\, t_k(\alpha_k)$ and putting

$$s_n = \sum_{k=1}^{m_n} w_k \chi_{D_k^{(n)}} + t_0 \chi_{G - G_\theta}\,,$$

we obtain a mapping belonging to T and (2.16), (13) and (12) complete the proof.

Let us define for a simultaneous channel $C = (B, A, \xi, T)$ quantile functions by the formulas

$$r(\varepsilon) = \sup_{p \in P} \inf \{y;\ \xi\{\alpha;\ \varphi(\alpha, p) \leqq y\} \geqq \varepsilon\}\,, \tag{14}$$
$$r'(\varepsilon) = \sup_{p \in P} \sup \{y;\ \xi\{\alpha;\ \varphi(\alpha, p) \geqq y\} \geqq 1 - \varepsilon\}\,.$$

These functions have properties similar to that proved in [2].

Lemma 1. Let us denote $\mathcal{S}_\varepsilon$ the family of all sets which have ξ measure at least ε.

(I)
$$r(\varepsilon) = \max_{p \in P} \inf_{\mathcal{A} \in \mathcal{S}_\varepsilon} (\operatorname{ess\,sup} \varphi(\cdot, p)\, \chi_{\mathcal{A}}(\cdot)),$$

$$r'(\varepsilon) = \max_{p \in P} \sup_{\mathcal{A} \in \mathcal{S}_{1-\varepsilon}} (\operatorname{ess\,inf} \varphi(\cdot, p)\, \chi_{\mathcal{A}}(\cdot)), \tag{15}$$

where $\operatorname{ess\,inf} f\chi_{\mathcal{A}} = \sup\{y; \xi\{\alpha \in \mathcal{A}; f(\alpha) < y\} = 0\}$, $\operatorname{ess\,sup}\ f = \|f\|_\infty = \inf\{y; \xi\{\alpha; f(\alpha) \geqq y\} = 0\}$.

(II) The functions r, r' are non-decreasing on $(0, 1)$, $r \leqq r'$ and $r'(\varepsilon_1) \leqq r(\varepsilon_2)$ whenever $\varepsilon_1 < \varepsilon_2$.

(III) r is left continuous, r' is right continuous and $r(\varepsilon+) = r'(\varepsilon)$, $r'(\varepsilon-) = r(\varepsilon)$.

(IV) The number $\varepsilon \in (0, 1)$ is a point of continuity of functions r, r' if and only if it is a point of continuity of one of them, which is equivalent to $r(\varepsilon) = r'(\varepsilon)$.

Proof. (I) Making use of (2.17) we can check that the function

$$\beta(p) = \inf\{\|\varphi(\cdot, p)\, \chi_{\mathcal{A}}(\cdot)\|_\infty; \mathcal{A} \in \mathcal{S}_\varepsilon\}$$

is continuous on P, which together with

$$r(\varepsilon) = \sup\{\beta(p); p \in P\}$$

completes the proof of the first equality in (15). The second one can be proved similarly.

(II) If $p \in P$ and

$$F_p(y) = \xi\{\alpha; \varphi(\alpha, p) \leqq y\}, \tag{16}$$

then for $\varepsilon_1 < \varepsilon_2$ belonging to $(0, 1)$

$$r'(\varepsilon_1) = \sup_{p \in P} \sup\{y; F_p(y) \leqq \varepsilon_1\} \leqq \sup_{p \in P} \sup\{y; F_p(y) < \varepsilon_2\} = r(\varepsilon_2). \tag{17}$$

(III) Now we prove that r' is right continuous. If $\varepsilon_n \searrow \varepsilon$, by I and II of this lemma we obtain

$$r'(\varepsilon) \leqq \inf_{n \geqq 1} r'(\varepsilon_n). \tag{18}$$

Let y_0 be an arbitrary number smaller than the right hand side of (18). We can choose a sequence $\{p_n\}_{n=1}$ of elements of P such that (cf. (14))

$$y_0 < \sup\{y; \xi\{\alpha; \varphi(\alpha, p_n) \geqq y\} \geqq 1 - \varepsilon_n\}, \quad n = 1, 2, \ldots$$

and compactness of the set P allows us to assume that the sequence $\{p_n\}$ converges to a vector $p \in P$. Denoting

$$\psi_n = \sup\{|\varphi(\alpha, p) - \varphi(\alpha, p_n)|; \alpha \in G\}$$

we see that

$$(19) \qquad 1 - \varepsilon_k \leqq \xi[\bigcup_{n=k}^{\infty} \{\alpha; \varphi(\alpha, p) + \psi_n \geqq y_0\}] .$$

Since $\lim_{n\to\infty} \psi_n = 0$ by (2.17), letting k tend to infinity we obtain from (19)

$$y_0 \leqq r'(\varepsilon) .$$

This means that the sign of equality may be written in (18). The rest of the proof is obvious.

Continuity of the quantile functions plays an important role in studying the asymptotic behaviour of the maximum length of ε-codes. Sufficient conditions for this continuity are given in the following proposition concerning the channel determined by (7).

Proposition 2. Let G be a connected metric space, $\mathscr{S}$ be the σ-algebra of Borel sets, $\xi(U)$ be positive for every non-empty open set U and let T be the set described by (7). If the functions t_0, h are continuous, then the quantile functions (14) are also continuous.

Proof. Let us assume we know that for every $p \in P$

$$\varphi_p(\alpha) = \inf_{t \in T} R_{t_\alpha}(p)$$

is a continuous function of the variable α. If we denote (cf. (16))

$$c_0 = \sup \{y; F_p(y) < \varepsilon\} , \quad c_1 = \sup \{y; F_p(y) \leqq \varepsilon\} ,$$

then $c_0 = c_1$ whenever $\varepsilon \in (0, 1)$. Suppose this is false; then we can find numbers $d_0 < d_1$ such that

$$(20) \qquad F_p(d_0) = F_p(d_1) = \varepsilon .$$

If $d_0 < d_0^* < d_1^* < d_1$, then the set $\varphi_p^{-1}(d_0^*, d_1^*)$ is either empty or non-empty and open, both possibilities yielding a contradiction with the assumptions. Now when we know that $c_0 = c_1$, the expression of the functions r, r' in the way described in (17) yields $r(\varepsilon) = r'(\varepsilon)$. But the last equality, by (IV) of the preceding lemma, means that the functions r, r' are continuous.

Now we want to prove the mentioned continuity of $\varphi_p(\cdot)$. Before doing this we remark that

$$(21) \qquad \varphi_p(\alpha) = \inf \{R_w(p); \varrho(w, t_0(\alpha)) \leqq h(\alpha)\} .$$

If $h(\alpha)$ is positive and $\alpha_n \to \alpha$, then there is a sequence $\{w_n\}$ of matrices belonging to W such that

$$(22) \qquad \varrho(w_n, t_0(\alpha_n)) \leqq h(\alpha_n) , \quad R_{w_n}(p) \leqq \varphi(\alpha_n, p) + n^{-1} .$$

Since $w_{n_k} \to w$ for some increasing sequence of integers, Lemma 2.6 and relations (21), (22) imply

$$\varphi(\alpha, p) \leqq \liminf_{k \geqq 1} \varphi(\alpha_{n_k}, p) .$$

The rest of the proof is obvious.

2. THE ASYMPTOTIC BEHAVIOUR OF THE LENGTH OF ε-CODES

Let $C = (B, A, \xi, T)$ be a simultaneous channel, $t \in T$ and $\xi(\mathscr{A}) > 0$. In accordance with (2) and [2] we put

$$w_{\mathscr{A},t}(x \mid y) = \frac{1}{\xi(\mathscr{A})} \int_{\mathscr{A}} w_{t_\alpha}(x \mid y) \, \mathrm{d}\xi(\alpha) ,$$

where $w_{t_\alpha}(x \mid y)$ is defined by means of the matrix w_{t_α} by (1.1). If $y \in B^n$ and $Q(y) \subset A^n$ then we put

$$w_{\mathscr{A},t}(Q(y) \mid y) = \sum_{x \in Q(y)} w_{\mathscr{A},t}(x \mid y) . \tag{23}$$

If we denote for $p \in P$, $\varepsilon \in (0, 1)$ by $S_n^*(\varepsilon, C_{\mathscr{A}}, p)$ the maximum length of n-dimensional codes $(\{Q(y)\}_{y \in Y}$ (cf. [2], p. 116) which satisfy

$$Y \text{ contains only } p\text{-sequences} , \tag{24}$$

$$w_{\mathscr{A},t}(Q(y) \mid y) > 1 - \varepsilon \quad \text{for every} \quad y \in Y, t \in T$$

then the following assertion holds.

Lemma 2. (I) If $\varepsilon' \in (0, \varepsilon)$ and $n > 1024 d^6 (\varepsilon')^{-1}$ is such that (cf. (2.19)) $\varepsilon - \delta_n > \varepsilon'$, then (cf. (15))

$$\frac{1}{n} \log S_n^*(\varepsilon, C_{\mathscr{A}}, p) > \operatorname{ess\,inf} \varphi(\cdot, p) \, \chi_{\mathscr{A}}(\cdot) - n^{-1/2} f_1(n, \varepsilon, \varepsilon', d) , \tag{25}$$

where (cf. (3.4))

$$f_1(n, \varepsilon, \varepsilon', d) = f(n, \varepsilon - \delta_n, \varepsilon', d) + 8 d^3 n^{-3/2} .$$

(II) If $\varepsilon'' \in (0, 1 - \varepsilon)$ and $n > d$ is such that $1 - (\varepsilon + n^{-d^2} + 2\delta_n)) > \varepsilon''$, then

$$\frac{1}{n} \log S_n^*(\varepsilon, C_{\mathscr{A}}, p) < \operatorname{ess\,sup} \varphi(\cdot, p) \, \chi_{\mathscr{A}}(\cdot) + n^{-1/2} g_1(n, \varepsilon, \varepsilon'', \xi(\mathscr{A}), d) , \tag{26}$$

where (cf. (3.13))

$$g_1(n, \varepsilon, \varepsilon'', \xi(\mathscr{A}), d) = g(n, \varepsilon + \delta_n + n^{-d^2}, \varepsilon'', \xi(\mathscr{A}) \, n^{-5d^2}, d) + 9 d^3 n^{-3/2} .$$

Proof. Let $n > d$ and W_n^* be the set described by (3.1) and (3.2). If $t \in T$, then by Lemma 2.4 there are matrices $\{w_k\}_{k=1}^m$ and a measurable partition $\{C_k\}_{k=1}^m$ of $\mathscr{A}$ into sets of positive measure such that the mapping (cf. (5))

$$t^* = \sum_1^m w_k \chi_{C_k} \tag{27}$$

satisfies the inequality $m \leqq \exp_n(4d^2)$ and the relation

$$\varrho(t^*(\alpha), t(\alpha)) < d^2 n^{-4}, \tag{28}$$

ξ amost everywhere on $\mathscr{A}$.

(I) If we denote $W = \bigcup_{t \in T} t^*(\mathscr{A})$, then by Theorem 3.1 there is an n-dimensional code $\{Q(y)\}_{y \in Y}$ such that Y contains only p-sequences, the inequality (cf. (2.19))

$$w(Q(y) \mid y) < 1 - (\varepsilon - \delta_n) \tag{29}$$

is satisfied for all $y \in Y$, $w \in V$ and

$$\frac{1}{n} \log \operatorname{card} Y > \inf_{w \in V} R_w(p) - n^{-1/2} f(n, \varepsilon - \delta_n, \varepsilon', d). \tag{30}$$

Taking into account (29), Lemma 2.7 and (27) we see that

$$w_{\mathscr{A},t}(Q(y) \mid y) > 1 - \varepsilon$$

for every $t \in T$, $y \in Y$. Since the numbers $\{\xi(C_k)\}$ are positive, Lemma 2.6, the way in which t^* was constructed and (30) imply (25).

(II) Let $\{Q(y)\}_{y \in Y}$ be an n-dimensional code satisfying (24). By the second assumption in Definition 1 there is a mapping $t \in T$ such that the inequality

$$R_{t_\alpha}(p) < \varphi(\alpha, p) + d^3 n^{-2} \tag{31}$$

holds for every $\alpha \in G$. To prove (26) we shall modify the mapping t^* (cf. (27), (28)). Let $L = \{k \in \{1, \ldots, m\};\ \xi(C_k) < \exp_n(-5d^2)\ \xi(\mathscr{A})\}$ and $C = \bigcup_{k \in L} C_k$. Since

$$\xi(C) < n^{-d^2}\, \xi(\mathscr{A}), \tag{32}$$

we can choose $k_0 \in \{1, \ldots, m\} - L$. Let us denote $D = C_{k_0} \cup C$ and put

$$\tilde{t} = \sum_{k \in L'} w_k \chi_{C_k} + w_{k_0} \chi_D,$$

where $L' = \{1, \ldots, m\} - (\{k_0\} \cup L)$. Making use of (32) and Lemma 2.7 we obtain

$$w_{\mathscr{A},\tilde{t}}(Q(y) \mid y) = \xi(\mathscr{A})^{-1} \left[\int_{\mathscr{A}} w_{t_\alpha}(Q(y) \mid y)\, d\xi(\alpha) - \xi(C) \right] > \tag{33}$$

$$> 1 - (\varepsilon + \delta_n + n^{-d^2}).$$

Further, taking into account that the numbers $\{\xi(C_k)\}$ are positive, we easily obtain from the definition of t^*, (2.16) and (31) that

$$\max_k R_{w_k}(p) \leqq 9d^3 n^{-2} + \text{ess sup}\, \varphi(\cdot, p)\, \lambda_{\mathscr{A}}(\cdot)\,,$$

which together with (3.13) completes the proof.

The following theorem gives bounds for the length of ε-codes for channels, described in Definitions 1 and 2.

1. **Theorem on Estimation of the Maximum Length of ε-Codes.** Let $C = (B, A, \xi, T)$ be a simultaneous channel decomposable into memoryless components, $\varepsilon \in (0, 1)$ and $S_n(\varepsilon, C)$ be the maximum length of n-dimensional ε-codes for C (cf. also (1.7)). If $\theta' \in (0, \varepsilon)$ and $\theta'' \in (\varepsilon, 1)$, then there exist positive constants $M(\varepsilon, \theta')$, $L(\varepsilon, \theta'')$ such that the inequalities (cf. (14))

$$\frac{1}{n} \log S_n(\varepsilon, C) > r'(\theta') - M(\varepsilon, \theta')\, n^{-1/2}\,,$$

$$\frac{1}{n} \log S_n(\varepsilon, C) < r(\theta'') + L(\varepsilon, \theta'')\, n^{-1/2} \log n$$

hold for every n. These constants can be chosen so that they would depend on the channel C only through $d = \max\{a, b\}$.

Proof. The first step is to prove the first inequality. We shall use the fact that if $\xi(\mathscr{A}) \geqq 1 - \theta'$ and $\{Q(y)\}_{y \in Y}$ is an n-dimensional code satisfying for every $t \in T$ the inequality (cf. (23))

$$w_{\mathscr{A},t}(Q(y) \mid y) > 1 - \varepsilon + \theta'\,,$$

then (cf. (2))

$$w_t(Q(y) \mid y) \geqq \int_{\mathscr{A}} w_{t_\alpha}(Q(y) \mid y)\, \mathrm{d}\xi(\alpha) > 1 - \varepsilon\,,$$

which means that (cf. (24))

$$\frac{1}{n} \log S_n(\varepsilon, C) \geqq \frac{1}{n} \log S_n^*(\varepsilon - \theta', C_{\mathscr{A}}, p)\,. \tag{34}$$

Hence if we choose $p \in P$, $\mathscr{A} \in \mathscr{S}_{1-\theta'}$ such that (cf. (15))

$$r'(\theta') < n^{-1} + \text{ess inf}\, \varphi(\cdot, p)\, \chi_{\mathscr{A}}(\cdot)\,, \tag{35}$$

then the preceding Lemma, (34) and (35) imply the existence of the sought constant M.

(II) Let $\{Q(y)\}_{y \in Y}$ be an (n, N, ε) code for C. Let us denote P_n the set of all vectors whose coordinates are integer multiples of n^{-1} and put (cf. (2.1) in [2])

$$Y_p = \{y \in Y; N(i \mid y) = np_i \quad i = 1, \ldots, b\}.$$

If $\xi(\mathscr{A}) \geqq \theta''$, then similarly as in [2], p. 127

$$w_{\mathscr{A},t}(Q(y) \mid y) > 1 - \varepsilon/\theta''$$

and (cf. (24))

$$\frac{1}{n} \log \operatorname{card} Y_p \leqq \frac{1}{n} \log S_n^*(\varepsilon/\theta'', C_{\mathscr{A}}, p). \tag{36}$$

Denoting $\varepsilon'' = 2^{-1}(1 - \varepsilon/\theta'')$ and choosing $\mathscr{A} \in \mathscr{S}_{\theta''}$ such that (cf. (15))

$$r(\theta'') > -n^{-1} + \operatorname{ess\,sup} \varphi(\cdot, p)\, \chi_{\mathscr{A}}(\cdot)$$

we see that (36) together with Lemma 2 implies existence of the sought constant L.

The results of the last theorem may be strengthened in some special cases. This is done in the following theorem.

2. **Theorem on Estimation of the Maximum Length of ε-Codes for Channels Decomposable into Finitely Many Components.** Let $C = (B, A, \xi, T)$ be the simultaneous channel described in Definition 1.1. If $\varepsilon \in (0, 1)$, then there exist positive constants $M(\varepsilon)$, $L(\varepsilon)$ such that the inequalities (cf. (1.7))

$$\frac{1}{n} \log S_n(\varepsilon, C) > r(\varepsilon) - M(\varepsilon)\, n^{-1/2},$$

$$\frac{1}{n} \log S_n(\varepsilon, C) < r'(\varepsilon) + L(\varepsilon)\, n^{-1/2} \log n$$

hold for every n. These constants can be chosen so that they would depend on C only through ξ and the number $d = \max\{a, b\}$. Moreover, if the channel C is non-singular (cf. (3.10)), we can find a positive constant $L_1(\varepsilon)$ such that the inequality

$$\frac{1}{n} \log S_n(\varepsilon, C) < r'(\varepsilon) + L_1(\varepsilon)\, n^{-1/2}$$

holds for every n and $L_1(\varepsilon)$ depends on C only through d, $\varkappa(C)$ and the vector ξ.

Proof. Since the functions r, r' are constant on intervals containing no points of the set $\{\xi(\mathscr{A}); \mathscr{A} \subset \{1, \ldots, m\}\}$, existence of constants M, L follows from Theorem 1 and Lemma 1. Further, let us suppose that the channel is non-singular. We shall use notations from II of the proof of the preceding theorem and suppose $p \in P_n$ is such that the set Y_p is non-empty. If $\mathscr{A} \in \mathscr{S}_{\theta''}$ satisfies (cf. (15))

$$r'(\varepsilon) = r(\theta'') > -n^{-1} + \max_{\alpha \in \mathscr{A}} \inf_{t \in T} R_{t_\alpha}(p) ,$$

then we see that the existence follows from (36) and Theorem 3.2 (II).

3. Theorem on Codes with Errors Tending to Zero. If C is a simultaneous channel and $\gamma \in (0, 1)$, then

$$\lim_{n \to \infty} \frac{1}{n} \log S_n(n^{-\gamma}, C) = \bar{r} ,$$

where (cf. Definition 1 and (15))

$$\bar{r} = \sup_{p \in P} \text{ess inf}\, \varphi(., p) .$$

Proof. Similarly as in the first part of the proof of Theorem 1 one can prove by means of (25) that

$$\liminf_{n \geq 1} \frac{1}{n} \log S_n(n^{-\gamma}, C) \geqq \text{ess inf}\, \varphi(., p)$$

for every vector $p \in P$. Further, the inequality (cf. Lemma 1 (II))

$$\limsup_{n \geq 1} \frac{1}{n} \log S_n(n^{-\alpha}, C) \leqq \lim_{\varepsilon \to 0} r(\varepsilon)$$

can be again proved analogically as in Theorem 1 and we see that it is sufficient to show that

$$\lim_{\varepsilon \to 0} r(\varepsilon) = \bar{r} .$$

The last relation can be proved by applying the method of the proof of the continuity of r' from the right (cf. the proof of Lemma 1 (III)).

If ε is a continuity point of the function r, then by Lemma 1 the equality $r(\varepsilon) = r'(\varepsilon)$ holds. This means that the Theorem on Estimation of the Maximum Length of ε-Codes implies

4. Theorem on ε-Capacity. Let $C = (B, A, \xi, T)$ be a simultaneous channel decomposable into memoryless components. If $\varepsilon \in (0, 1)$ is a continuity point of the function $r(\varepsilon)$, then

$$\lim_{n \to \infty} \frac{1}{n} \log S_n(\varepsilon, C) = r(\varepsilon) . \tag{37}$$

Thus, if the assumptions in Proposition 2 are fulfilled, T is the set described by (7), $C = (B, A, \xi, T)$ and $\varepsilon \in (0, 1)$, then the limit (37) exists. If C is a simultaneous channel decomposable into finitely many memoryless components and $\varepsilon \in (0, 1)$ does not belong to the finite set $\{\xi(\mathscr{A}); \mathscr{A} \subset \{1, \ldots, m\}\}$, then ε is a continuity point of r by Lemma 1 and the limit in (37) exists. In the general case, the quantile functions are monotone i.e. the set of their discontinuity points is countable, so we have obtained for such a channel that limit (37) exists at least except for this countable set.

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