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On Probability of First Order Formulas in a Given Model

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This paper is intended to define a probability measure for formulas of a first order language, interpreted in a given model. The definition of probability is based on the concept of satisfaction of the formulas by sequences of individuals of a given universe. Several results about satisfaction are mentioned, and many results about relations among probabilities are presented. Conditional probability and mutual independence of formulas are introduced and used to derive the Bayes' theorem and some other results of elementary meaning.

INTRODUCTION

One of the fundamental tasks of mathematical informatics is to build up logical foundations of information theory, in order to use it at design of automatized information systems. The first step of this work is to define elementary concepts of informatics in terms of semantic logic, further to describe properties of objects by logical formulas, and, if possible, to construct suitable algorithms in terms of first order logic. The most available link to connect semantic logic with mathematical informatics seems to be probability theory. The mission of probability theory to bridge the two disciplines mentioned as the first investigated Carnap and Bar-Hillel in their works [1], [2], [3].

This paper tries to be a positive contribution to the efforts of building up a probability apparatus over formulas of semantic logic, in order to use it then to form the base of information theory and to apply it in forming information systems theory. Nevertheless, only a few fundamental probabilistic concepts are defined here, and only some of their elementary properties and mutual relations are deduced. Therefore, the paper has an introductory character and results obtained are of elementary meaning only.

Let

$$\mathscr{L} = (X, \mathscr{P}, \mathscr{F}, C) \tag{1.1}$$

be a first order language with equality, where $X = \{x_1, x_2, \ldots\}$ is a denumerable set of (individual) variables and $\mathscr{P}, \mathscr{F}, C$ are sets of predicate, function and constant symbols, respectively. We construct formulas of the language $\mathscr{L}$ of atomic formulas in the common way, by means of logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, quantifiers $\forall, \exists$ and the symbol $\equiv$ of equality. We use also some other metasymbols such as $=, \sim, \models, \Rightarrow, \Leftrightarrow$, to denote identity, syntactic equivalence, satisfaction, consequence and semantic equivalence of formulas.

We write $\varphi\{x_1, x_2, \ldots, x_n\}$ iff all free variables of the formula φ are of the set $\{x_1, x_2, \ldots, x_n\}$. If φ has exactly the free variables $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$, then we sometimes write $\varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$. Furthermore, we denote by W the set of all formulas of the language $\mathscr{L}$ and by W_m the set of all those formulas of W which have exactly $m > 0$ free variables.

The language $\mathscr{L}$ is supposed to be built up as a formal system with given axioms and inference rules. The notion of the proof of a formula is used in the common sense, as a finite sequence of formulas derived from the axioms by the inference rules.

Clearly, the relation $\sim$ is an equivalence in the set W. It induces a partition $W/\sim$ of the set W into equivalence classes

$$[\varphi] = \{\psi \mid \psi \sim \varphi\}, \quad \varphi, \psi \in W.$$

1. SATISFACTION OF FIRST ORDER FORMULAS IN A MODEL

Let

$$\mathscr{A} = (A, \mathscr{P}', \mathscr{F}', C') \tag{1.2}$$

be a model of the language $\mathscr{L}$, where $A \neq \emptyset$ is a universe of individuals and $\mathscr{P}'$, $\mathscr{F}'$ and C' are sets of predicates, functions and constants, respectively, to express real relations available in the universe A.

There is a unique relation between the language (1.1) and its model (1.2); it can be expressed by interpretation $\mathscr{I}$, i.e. by such a mapping

$$\mathscr{I} : \mathscr{L} \rightarrow \mathscr{A}$$

at which

$$\mathscr{I}(\mathscr{P}) = \mathscr{P}', \quad \mathscr{I}(\mathscr{F}) = \mathscr{F}', \quad \mathscr{I}(C) = C',$$

and which is admissible in the sense that if $P \in \mathscr{P}$ or $F \in \mathscr{F}$ is a k-ary predicate or function symbol, respectively, then $P' = \mathscr{I}(P)$ or $F' = \mathscr{I}(F)$ is a k-ary predicate of the set $\mathscr{P}'$ or a function with k variables of the set $\mathscr{F}'$, respectively, and if $c \in C$ is a constant symbol then $c' = \mathscr{I}(c)$ is a constant of the set C'.

Satisfaction of formulas of the language $\mathscr{L}$ in the model $\mathscr{A}$ is used here in the common sense. As usually we write

$$(1.3) \qquad \mathscr{A} \models \varphi\{x_1, x_2, \ldots, x_n\} [a_1, a_2, \ldots]$$

iff the formula φ is satisfied in the model $\mathscr{A}$ by the sequence $a_1, a_2, \ldots$ of individuals of the universe A.

It is well-known that satisfaction of the formula $\varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ by the sequence $a_1, a_2, \ldots$ depends on values of those members of the sequence which have the indices $i_1, i_2, \ldots, i_m$, i.e. which substitute exactly all free variables in the formula $\varphi\{x_1, x_2, \ldots \ldots, x_n\}$, $n \geqq m$. Thus, satisfaction of the formula φ depends on the values $a_{i_1}, a_{i_2}, \ldots \ldots, a_{i_m}$ and does not depend on other members of the sequence $a_1, a_2, \ldots$. Therefore, we say briefly that if the formula $\varphi(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ is satisfied by the sequence $a_1, a_2, \ldots$, then φ is satisfied by the m-tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$.

If φ is satisfied exactly for a set $B \subseteq A^m$ of m-tuples $(b_1, b_2, \ldots, b_m)$, where b_j substitutes the variables x_{i_j} then B is called a *satisfaction body* of the formula φ; we denote it by $|\varphi|$. Analogically, the cylinder

$$(1.4) \qquad C(\varphi) = \{(x_1, x_2, \ldots) \mid x_{i_1} = b_1, \ldots, x_{i_m} = b_m\}, \quad (b_1, b_2, \ldots, b_m) \in |\varphi|$$

with the base $|\varphi|$ is called a *satisfaction cylinder* of the formula φ. If $|\varphi| = A^m$ then $|\varphi|$ or $C(\varphi)$ is called a *maximal satisfaction body* or a *maximal satisfaction cylinder* of the formula φ, respectively. Obviously, the cylinder $C(\varphi)$ is maximal iff the body $|\varphi|$ is maximal.

The formula φ is said to be *true* (in a model $\mathscr{A}$) iff its body $|\varphi|$ is maximal. If φ has this property in all models then φ is said to be *valid* formula (of the language $\mathscr{L}$), symbolically $\models \varphi$. The satisfaction body or the satisfaction cylinder of a closed formula φ is either maximal (when φ is true), or empty (when φ is false).

For a true formula φ which is satisfied by any sequence $a_1, a_2, \ldots$ we omit the part $[a_1, a_2, \ldots]$ in (1.3) in case when φ is true and write $\mathscr{A} \models \varphi\{x_1, x_2, \ldots, x_n\}$ or $\mathscr{A} \models \varphi$ only instead of (1.3). Obviously, the relation $\mathscr{A} \models \varphi$ is fulfilled iff the body $|\varphi|$ is maximal.

In what follows we use some known facts:

(F 1) If $\mathscr{A} \models \varphi$ then

$$\mathscr{A} \models (Kx_1)(Kx_2)\ldots(Kx_n)\,\varphi\{x_1, x_2, \ldots, x_n\}, \quad (K \text{ is } \forall \text{ or } \exists).$$

(F 2) A formula φ is true in $\mathscr{A}$ iff its closure is true in $\mathscr{A}$, i.e. $\mathscr{A} \models \varphi$ iff

$$(1.5) \qquad \mathscr{A} \models (\forall x_1)(\forall x_2)\ldots(\forall x_n)\,\varphi\{x_1, x_2, \ldots, x_n\}.$$

(F 3) For any formula $\varphi \in W_m$,

$$(1.6) \qquad |\varphi| = A^m, \quad m > 0,$$

iff $\mathscr{A} \models \varphi$.

It is obvious, that any formula φ can be extended (e.g. by conjunction of φ with a suitable true formula) to a formula φ^*, which is satisfied iff φ is satisfied, and the set of free variables of which contains the set of all free variables of φ as a proper subset. In case of two formulas φ and ψ this extension can be chosen in such a way, that the extended formulas φ^* and ψ^* have the same free variables. The process of extension of formulas by this way is called a *unification of free variables* and yielded formulas φ^* and ψ^*-*unified formulas.*

The advantage of the unification of free variables is that it enables to suppose for yielding formulas the same domain of satisfaction as for their components.

(F 4) Let $\varphi, \psi \in W_m$ be any unified formulas. Then the following equations are fulfilled:

$$|\neg\varphi| \doteq \overline{|\varphi|}\,, \tag{1.7}$$

$$|\varphi \wedge \psi| = |\varphi| \cap |\psi|\,, \tag{1.8}$$

$$|\varphi \vee \psi| = |\varphi| \cup |\psi|\,, \tag{1.9}$$

$$|\varphi \rightarrow \psi| = \overline{|\varphi|} \cup |\psi|\,, \tag{1.10}$$

$$|\varphi \leftrightarrow \psi| = (|\varphi| \cap |\psi|) \cup (\overline{|\varphi|} \cap \overline{|\psi|})\,, \tag{1.11}$$

where $\overline{Y} = A^m - Y$ is the complement of the set Y with respect to A^m.

(F 5) Let $\varphi, \psi \in W_m$ be any unified formulas, φ being true. Then

$$|\neg\varphi| = \emptyset\,,$$
$$|\varphi \wedge \psi| = |\varphi \rightarrow \psi| = |\varphi \leftrightarrow \psi| = |\psi|\,,$$
$$|\varphi \vee \psi| = |\psi \rightarrow \varphi| = A^m\,.$$

(F 6) Under the assumptions of (F 5)

$$\mathscr{A} \models \varphi \vee \psi\,, \quad \mathscr{A} \models \psi \rightarrow \varphi\,.$$

(F 7) If $\mathscr{A} \models \psi$ then from the assumptions of (F 5) it follows

$$\mathscr{A} \models \varphi \wedge \psi\,, \quad \mathscr{A} \models \varphi \rightarrow \psi\,, \quad \mathscr{A} \models \varphi \leftrightarrow \psi\,.$$

Definition 1.1. For any two unified formulas $\varphi, \psi \in W_m$, ψ is called a consequent (or ψ follows) of φ (in a model $\mathscr{A}$), symbolically $\varphi \Rightarrow \psi$, iff for any sequence $a_1, a_2, \ldots$ of individuals of the universe A for which

$$\mathscr{A} \models \varphi\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}\,[a_1, a_2, \ldots] \tag{1.12}$$

also

$$\mathscr{A} \models \psi\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}\,[a_1, a_2, \ldots]\,. \tag{1.13}$$

If $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$ then φ and ψ are said to be *semantically equivalent* (in the model $\mathscr{A}$), symbolically $\varphi \Leftrightarrow \psi$. Formulas φ and ψ are semantically equivalent (in the model $\mathscr{A}$) iff for any sequence $a_1, a_2, \ldots$ (1.12) holds iff (1.13) holds.

(F 8) Let $\varphi, \psi \in W_m$ be any unified formulas. Then

1. $\varphi \Rightarrow \psi$ iff
$$|\varphi| \subseteq |\psi|$$
2. $\varphi \Leftrightarrow \psi$ iff
$$|\varphi| = |\psi| .$$

(F 9) Let $\varphi, \psi \in W_m$ be any unified formulas, ψ being true. Then $\varphi \Rightarrow \psi$.

(F 10) Let $\varphi, \psi \in W_m$ be any unified formulas, φ being true. Then $\varphi \Leftrightarrow \psi$ iff ψ is a true formula.

Comment 1.1. The relation $\Rightarrow$ defines a partial order in the set W_m; analogically, $\Leftrightarrow$ is an equivalence relation in the set W_m.

2. PROBABILITY OF SATISFACTION FORMULAS IN A GIVEN MODEL

Let W_m^* be a set of all unified formulas with m free variables. Let

$$f : W_m^* \to 2^{A^m} \tag{2.1}$$

be a mapping with the property

$$f(\varphi) = |\varphi| , \quad \varphi \in W_m^* .$$

We denote by $\mathscr{B}_m$ the set $f(W_m^*)$ of the images of all formulas $\varphi \in W_m^*$ in the mapping (2.1).

Lemma 2.1. The set system $\mathscr{B}_m$ forms an algebra in the space A^m, $m \geqq 1$.

Proof. By (2.1), for each element $V \in \mathscr{B}_m$ there exists a formula $\varphi \in W_m^*$ for which $f(\varphi) = V = |\varphi|$. For each $\varphi \in W_m^*$ holds $\neg\varphi \in W_m^*$ and, consequently, $f(\neg\varphi) \in \mathscr{B}_m$. Therefore, by (1.7)

$$f(\neg\varphi) = |\neg\varphi| = \overline{|\varphi|} = \overline{V} ,$$

i.e. $\overline{V} \in \mathscr{B}_m$. Further, if $V_1, V_2 \in \mathscr{B}_m$ then there exist formulas $\varphi_1, \varphi_2 \in W_m^*$, for which

$$f(\varphi_1) = V_1 , \quad f(\varphi_2) = V_2 .$$

For $\varphi_1 \vee \varphi_2 \in W_m$ and, by (1.9)

$$f(\varphi_1 \vee \varphi_2) = f(\varphi_1) \cup f(\varphi_2) = V_1 \cup V_2 ,$$

then $V_1 \cup V_2 \in \mathscr{B}_m$. Thus, $\mathscr{B}_m$ is an algebra in the space A^m.

Let P_m be a probability measure defined on the set $\mathscr{B}_m$. The number $\mathsf{P}_m(|\varphi|)$ for $|\varphi| \in \mathscr{B}_m$ which we rather write in the more simply way $\mathsf{P}_m(\varphi)$, is called a *probability of satisfaction* of the formula φ, shortly a *probability of the formula* φ.

It follows from definition of satisfaction that if $\varphi \in W_m$ then $|\varphi| \subseteq A^m$. If, in order to unify, we extend φ to φ^* by means of a true formula $\varphi_1 \in W_r$, then $\varphi^* \in W_{m+r}$ and $|\varphi^*| \subseteq A^{m+r}$. Moreover, $A \models \varphi_1$ implies

$$|\varphi^*| = |\varphi| \times A^r .$$

Therefore, for probabilities

$$\mathsf{P}_{m+r}(\varphi^*) = \mathsf{P}_m(\varphi) \, . \, \mathsf{P}_r(A^r) \, . \tag{2.2}$$

holds. But for every $r = 1, 2, \ldots$ is $\mathsf{P}_r(A^r) = 1$, consequently,

$$\mathsf{P}_{m+r}(\varphi^*) = \mathsf{P}_m(\varphi) \, .$$

This equation implies that probability of the formula φ^* with an extended set of variables is equal to probability of the former formula φ.

Let $\mathscr{B}_m^*$ be the minimal σ-algebra on the space A^m generated by the algebra $\mathscr{B}_m$. According to Caratheodory's extension theorem there exists a unique extension of the measure P_m defined on $\mathscr{B}_m$ to the probability measure P_m^* defined on $\mathscr{B}_m^*$, with the property

$$\mathsf{P}_m^*(\varphi) = \mathsf{P}_m(\varphi) \, , \quad \varphi \in W_m \, .$$

Now, we define a system $\mathscr{C}$ of the cylinders $C(\varphi)$ with the base $|\varphi|$, constructed according to (1.4) in the space A_0 of all sequences of individuals of the universe A. It can be easy shown [7] that the system $\mathscr{C}$ forms an algebra in the space A_0.

In the system $\mathscr{C}$ of subsets of the set A_0 we define the probability measure P_0 by

$$\mathsf{P}_0[C(\varphi)] = \mathsf{P}_m(\varphi) \, , \quad \varphi \in W_m \, .$$

It is well-known [7], [8] that the algebra $\mathscr{C}$ can be extended to the minimal σ-algebra $\mathscr{C}^*$ containing $\mathscr{C}$, and that the measure P_0 in $\mathscr{C}$ can be extended to the probability measure P defined in the σ-algebra $\mathscr{C}^*$, in such a way that

$$\mathsf{P}[C(\varphi)] = \mathsf{P}_0[C(\varphi)] \, , \quad \varphi \in W_m \, .$$

Therefore, we may write

$$\mathsf{P}[C(\varphi)] = \mathsf{P}_m(\varphi) \, , \quad \varphi \in W_m \, , \quad m = 1, 2, \ldots \tag{2.3}$$

what defines a universal probability measure in the set of cylinders by means of probability measures in the particular algebras $\mathscr{B}_m$, $m = 1, 2, \ldots$.

Notice, that in general it is impossible to define the probabilities P_m, $m > 1$, as a product of the elementary probabilities P_1 analogically to (2.2), because the bodies of satisfaction of formulas may not be rectangular.

Thus, we have defined for a given model $\mathscr{A}$ the probability space

$$\prod(\mathscr{A}) = (A_0, \mathscr{C}^*, \mathsf{P})$$

with the measure P defined in the σ-algebra $\mathscr{C}^*$ of satisfaction cylinders of formulas of the language $\mathscr{L}$, those cylinders being defined in the space A_0 of sequences of individuals of the universe A.

Finally, we accept the agreement that, for simplicity, we shall speak only about probabilities of formulas instead of probabilities of satisfaction cylinders of formulas, and write only $\mathsf{P}(\varphi)$ instead of $\mathsf{P}[C(\varphi)]$, where $\mathsf{P}(\varphi)$ can be computed, according to (2.3), as

$$\mathsf{P}(\varphi) = \mathsf{P}_m(\varphi), \quad \varphi \in W_m, \quad m = 1, 2, \ldots.$$

It is easy to see that from the facts (F 1)–(F 10) in the preceeding section next propositions follow:

(P 1) Let φ, ψ be arbitrary unified formulas. Then for their probabilities hold:

$$0 \leqq \mathsf{P}(\varphi) \leqq 1,$$
$$\mathsf{P}(\neg\varphi) = 1 - \mathsf{P}(\varphi),$$
$$\mathsf{P}(\varphi \vee \psi) \leqq \mathsf{P}(\varphi) + \mathsf{P}(\psi).$$

(P 2) Let $\varphi \in W_m$ be an arbitrary true formula. Then $\mathsf{P}(\varphi) = 1$.

(P 3) Let $\psi \in W_m^*$ be an arbitrary formula and $\varphi \in W_m^*$ be a true formula. Then

$$\mathsf{P}(\varphi \vee \psi) = \mathsf{P}(\psi \to \varphi) = 1,$$
$$\mathsf{P}(\varphi \wedge \psi) = \mathsf{P}(\varphi \to \psi) = \mathsf{P}(\varphi \leftrightarrow \psi) = \mathsf{P}(\psi).$$

Definition 2.1. Formulas $\varphi, \psi \in W_m^*$ are said to be *disjoint* (in a model $\mathscr{A}$) iff for their satisfaction bodies holds

$$|\varphi \wedge \psi| = |\varphi| \cap |\psi| = \emptyset.$$

(P 4) Let $\varphi, \psi \in W_m^*$ be arbitrary disjoint formulas. Then

$$\mathsf{P}(\varphi \wedge \psi) = 0 \tag{2.4}$$
$$\mathsf{P}(\varphi \vee \psi) = \mathsf{P}(\varphi) + \mathsf{P}(\psi).$$

(P 5) Let φ, ψ be arbitrary unified formulas for which $\varphi \Rightarrow \psi$ or $\varphi \Leftrightarrow \psi$. Then

$$\mathsf{P}(\varphi) \leqq \mathsf{P}(\psi), \quad \varphi \Rightarrow \psi,$$

or

$$\mathsf{P}(\varphi) = \mathsf{P}(\psi), \quad \varphi \Leftrightarrow \psi,$$

respectively.

(P 6) For any two unified formulas φ, ψ,

$$P(\varphi \wedge \psi) \leqq \min \{P(\varphi), P(\psi)\}.$$

(P 7) Let φ, ψ be arbitrary unified formulas with the property $\varphi \Rightarrow \psi$. Then

$$P(\varphi \wedge \psi) = P(\varphi), \quad P(\varphi \vee \psi) = P(\psi). \tag{2.5}$$

It is useful to define also the notion of conditional probability for unified formulas.

Definition 2.2. Let $\varphi, \psi \in W_m^*$ be arbitrary unified formulas. The *conditional probability* $P(\varphi \mid \psi)$ of satisfaction of a formula φ, provided the formula ψ is satisfied, is defined by

$$P(\varphi \mid \psi) = \frac{P(\varphi \wedge \psi)}{P(\psi)}, \quad P(\psi) > 0. \tag{2.6}$$

Lemma 2.2. Let φ, ψ be arbitrary disjoint unified formulas with $|\psi| \neq \emptyset$. Then

$$P(\varphi \mid \psi) = 0. \tag{2.7}$$

Proof. From the assumption $|\psi| \neq \emptyset$, it follows $P(\psi) > 0$, and from disjointness of φ, ψ, by (P 4), (2.4) follows. Then the definition (2.6) implies (2.7).

Lemma 2.3. Let φ, ψ be arbitrary unified formulas in the relation $\psi \Rightarrow \varphi$, $|\psi| \neq \emptyset$. Then

$$P(\varphi \mid \psi) = 1, \tag{2.8}$$

or

$$P(\psi \mid \varphi) = \frac{P(\psi)}{P(\varphi)}, \tag{2.9}$$

respectively.

Proof. From $\psi \Rightarrow \varphi$, by (P 7) the equality (2.5) follows in the form $P(\varphi \wedge \psi) = P(\psi)$. Further, from $|\psi| \neq \emptyset$ follows $P(\psi) > 0$. Therefore, the defining formula (2.6) implies (2.8) or (2.9), respectively.

Lemma 2.4. Let φ, ψ be arbitrary unified formulas with $|\varphi| \neq \emptyset$, $|\psi| \neq \emptyset$. Then

$$P(\varphi \mid \psi) = \frac{P(\psi \mid \varphi) \, . \, P(\varphi)}{P(\psi)}. \tag{2.10}$$

Proof follows from the formula (2.6).

We use the concept of conditional probability to define independence of formulas as follows.

Definition 2.3. Let φ, ψ be arbitrary unified formulas. Satisfaction of a formula φ is said *not to depend on* satisfaction of a formula ψ (shortly, φ does *not depend on* ψ) iff

$$\mathsf{P}(\varphi \mid \psi) = \mathsf{P}(\varphi) . \tag{2.11}$$

If φ does not depend on ψ and at the same time ψ does not depend on φ, then we say that φ and ψ are (mutually) *independent.*

Theorem 2.1. Unified formulas φ, ψ are independent iff

$$\mathsf{P}(\varphi \wedge \psi) = \mathsf{P}(\varphi) \, . \, \mathsf{P}(\psi) . \tag{2.12}$$

Proof. From (2.6), it follows the equation

$$\mathsf{P}(\varphi \mid \psi) \, \mathsf{P}(\psi) = \mathsf{P}(\varphi \wedge \psi) = \mathsf{P}(\psi \mid \varphi) \, \mathsf{P}(\varphi) , \tag{2.13}$$

from which we obtain, according to (2.11),

$$\mathsf{P}(\psi \mid \varphi) = \mathsf{P}(\psi) .$$

Thus, if φ does not depend on ψ, then ψ does not depend on φ, i.e. φ and ψ are (mutually) independent. Obviously, then (2.13) implies (2.12).

Theorem 2.2. Let φ, ψ be arbitrary unified formulas, φ being true. Then φ and ψ are independent.

Proof. By (P 2) from $\mathscr{A} \models \varphi$ $\mathsf{P}(\varphi) = 1$ follows, what by (2.9) in Lemma 2.3 implies that ψ does not depend on φ. Thus, φ and ψ are independent.

Theorem 2.3. Let (φ, ψ) be a pair of independent unified formulas. Then $(\varphi, \neg\psi)$, $(\neg\varphi, \psi)$, $(\neg\varphi, \neg\psi)$ also are pairs of independent formulas.

Proof. The formulas $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ are clearly disjoint, and

$$|\varphi| = |\varphi \wedge \psi| \cup |\varphi \wedge \neg\psi| ,$$

i.e.

$$\varphi \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi) .$$

Consequently, by (P 5)

$$\mathsf{P}(\varphi) = \mathsf{P}(\varphi \wedge \psi) + \mathsf{P}(\varphi \wedge \neg\psi)$$

and with the regard to independence of φ and ψ,

$$\mathsf{P}(\varphi \wedge \neg\psi) = \mathsf{P}(\varphi) \,.\, \mathsf{P}(\neg\psi)\,,$$

what implies, by the Theorem 2.1, that φ and $\neg\psi$ are independent.

Analogically, it can be also proven that the pairs $(\neg\varphi, \psi)$ and $(\neg\varphi, \neg\psi)$ are independent.

Definition 2.4. Let S be a denumerable set of unified formulas of the set W_m. The set S is said to form a *complete set* of formulas (in a model $\mathscr{A}$) if

(a) for each $\varphi \in S$, $|\varphi| \neq \emptyset$,

(b) each two formulas $\varphi_j, \varphi_k \in S$, $j \neq k$, are disjoint,

(c) $\bigcup_S |\varphi| = A^m$.

The set S is said to be *nontrivial* if it contains at least two elements. If the set S has the properties (a) and (b) but not (c), then S is called an *incomplete set* of formulas.

Definition 2.4 implies that for a complete set S of formulas,

$$\mathsf{P}(\bigcup_k |\varphi_k|) = \sum_k \mathsf{P}(\varphi_k) = 1\,, \quad \varphi_k \in S\,. \tag{2.14}$$

Lemma 2.5. A nontrivial set of formulas does not contain any true formula.

Proof. Let S be a complete set of formulas containing φ and ψ, $\mathscr{A} \models \varphi$. Then, by (a) of the Definition 2.4, $|\varphi| \neq \emptyset$, $|\psi| \neq \emptyset$. Thus, $\mathsf{P}(\varphi) > 0$, $\mathsf{P}(\psi) > 0$. The assumption $\mathscr{A} \models \varphi$, by (P 2), implies $\mathsf{P}(\varphi) = 1$. Therefore, $\mathsf{P}(\varphi) + \mathsf{P}(\psi) > 1$, what is a contradiction to (2.14). For $\mathsf{P}(\psi) > 0$, it should be $\mathsf{P}(\varphi) < 1$; thus, φ cannot be true.

Theorem 2.4. Let S' be a nontrivial incomplete set of formulas of the set W_m. Then there exists such a formula $\psi \in W_m^*$ that $S = S' \cup \{\psi\}$ is a complete set.

Proof. Put $\psi = \neg(\bigvee\varphi)$, $\varphi \in S'$. Then $|\psi| \neq \emptyset$ and for each $\varphi \in S'$ φ and ψ are disjoint. Consequently, the formulas $\bigvee\varphi = \neg\psi$ and ψ are disjoint, and

$$|\bigvee\varphi \vee \psi| = |\neg\psi \vee \psi| = A^m\,, \quad \varphi \in S'\,.$$

Thus, by Definition 2.4, $S = S' \cup \{\psi\}$ is a complete set of formulas.

Surely, for the set S holds (2.14) because from the given assumptions, it follows

$$\mathsf{P}(\psi) = \mathsf{P}(\neg\bigvee\varphi) = 1 - \mathsf{P}(\bigvee\varphi) = 1 - \sum\mathsf{P}(\varphi)\,, \quad \varphi \in S'\,,$$

what implies

$$\mathsf{P}(\bigvee\varphi \vee \psi) = \mathsf{P}(\bigvee\varphi) + \mathsf{P}(\psi) = \sum\mathsf{P}(\varphi) + (1 - \sum\mathsf{P}(\varphi)) = 1\,.$$

Theorem 2.5. Let S be a nontrivial complete set of unified formulas of the set W_m, and $\psi \in W_m$ is an arbitrary formula unified with formulas of S, with $|\psi| \neq \emptyset$. Then for each $\varphi \in S$,

$$\mathsf{P}(\varphi \mid \psi) = \frac{\mathsf{P}(\psi \mid \varphi)\, \mathsf{P}(\varphi)}{\sum_{\varphi \in S} \mathsf{P}(\psi \mid \varphi)\, \mathsf{P}(\varphi)}, \quad \varphi \in S. \tag{2.15}$$

Proof. For each $\varphi \in S$ and $\psi \in W_m$, $\mathsf{P}(\varphi) > 0$ and

$$|\psi| = \bigcup_S |\varphi \wedge \psi|,$$

where for each two $\varphi_1, \varphi_2 \in S$ the sets $|\varphi_1 \wedge \psi|$ and $|\varphi_2 \wedge \psi|$ are disjoint. Therefore,

$$\mathsf{P}(\psi) = \sum_{\varphi \in S} \mathsf{P}(\varphi \wedge \psi),$$

what, according to (2.6), implies

$$\mathsf{P}(\psi) = \sum_{\varphi \in S} \mathsf{P}(\psi \mid \varphi)\, \mathsf{P}(\varphi). \tag{2.16}$$

Under the given assumptions holds the Lemma 2.4 what enables us to use (2.10). Putting the value (2.16) for $\mathsf{P}(\psi)$ into (2.10) we obtain (2.15).

The Theorem 2.5 is called a *Bayes' theorem* on probabilities of formulas of the complete set S, conditioned by the formula ψ; on the other hand (2.16) is called a *formula of complete probability* of formulas.

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REFERENCES

[1] R. Carnap: Logical Foundations of Probability. Chicago 1950.

[2] R. Carnap, Y. Bar-Hillel: Semantic Information. British Journal for the Philosophy of Science *4* (1953).

[3] R. Carnap, Y. Bar-Hillel: An Outline of the Theory of Semantic Information. Research Laboratory of Electronics, Massachusetts Institute of Technology, Report No. 247, 1952.

[4] C. C. Chang, H. J. Keisler: Model Theory. North-Holland Publishing Company, Amsterdam—London 1974.

[5] J. L. Bell, A. B. Slomson: Models and Ultraproducts. North-Holland Publishing Company, Amsterdam—London 1974.

[6] L. Rieger: Algebraic Methods of Mathematical Logic. Academic Press, New York—London 1967.

[7] A. Rényi: Probability Theory. North-Holland Publishing Company, Amsterdam—London 1970.

[8] M. Loève: Probability Theory. D. van Nostrand Company Inc., Princeton—New York—London 1961.

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