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Equations Associated with a Context-Free Grammar

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In this paper the equivalence between context-free grammars and systems of equations is studied. A uniqueness theorem for the solution of these equations is given. The reason of existence of parasitic solution is also found.

INTRODUCTION

Context-free languages may be described by context-free grammars, by push-down sequential machines or by sets of equations.

In this paper we deal with languages described by sets of equations. In the first section we recall the procedure to associate a set of equations with a context-free grammar. For further details see [4]. Conversely, given a set of equations of a special form, we may associate a context-free grammar with it. Such a set of equations is of the form $\bar{\mathscr{L}} = F(\bar{\mathscr{L}})$, where $F = (f_0, f_1, \ldots, f_n)$ and $\bar{\mathscr{L}}$ is a vector of $n + 1$ variables which range over $2^{V^*_T}$ (i.e. the set of all languages over a terminal vocabulary). Functions $f_i(\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n)$ are polynomials i.e. each of them is finite union of strings composed of variables and terminals.

The minimal solution of $\bar{\mathscr{L}} = F(\bar{\mathscr{L}})$ is a vector of languages; one component of this vector is the language generated by the original context-free grammar. If we start with a context-free grammar having no unproductive rules, the solution of the system $\bar{\mathscr{L}} = F(\bar{\mathscr{L}})$ is unique. The uniqueness of solution may be obtained in an elegant way for a class of equations using a topological method and this is done in [3]. It is easy to prove that a context-free grammar without unproductive rules exists for every context-free grammar in such a way that both generate the same language.

In Section 3 we consider the case in which the set of equations is associated with a context-free grammar which may have unproductive rules. In this case the associated system may have solutions which are not minimal. We call them parasitic solutions.

A natural question arises, namely: why the parasitic solutions do appear, and which are the associated grammars? The answer to this question is that these solutions appear if we enrich the grammar with some simple rules. To establish the reason of existence of parasitic solutions we introduce the notion of parasitic set of nonterminals, as a non-void set of nonterminals associated with a set of rules such that an infinite play with these nonterminals and rules is possible. We show that the reason of existence of parasitic solution is the existence of a parasitic set of nonterminals. Also in Section 3 following a method described in Blikle [1] we give a procedure to construct such parasitic solutions.

1. DEFINITIONS. EQUATIONS ASSOCIATED WITH A CONTEXT-FREE GRAMMAR

Let V be a finite set called vocabulary. The elements of V are called words. Let us consider the free monoid generated by V i.e. the set V^* of all finite strings of elements of V, including the null string. A language over V is a subset $L \subseteq V^*$. If $x \in V^*$ we denote by $|x|$ the length of the string x i.e. the number of occurrences of elements of V in this string.

A context-free grammar [5] is a quadruple $G = (V_T, V_N, S, R)$, where: V_T is called the vocabulary of terminal words, V_N is the vocabulary of nonterminal words, or the auxiliary vocabulary, $S \in V_N$ is the initial nonterminal or axiom, and R is a finite set of rules of the form $A \to \alpha$ with $A \in V_N$ and $\alpha \in (V_N \cup V_T)^*$, $|\alpha| > 0$. We say that x directly generates y iff there exist u, v in $(V_N \cup V_T)^*$ and a rule $A \to \alpha$ in R such that $x = uAv$ and $y = u\alpha v$. We write $x \Rightarrow y$. We say that x generates y and write $x \Rightarrow^* y$ iff there exists a sequence $z_0, z_1, \ldots, z_q$ such that $x = z_0$, $y = z_q$ and $z_i \Rightarrow z_{i+1}$ for $i = 0, 1, \ldots, q - 1$. The language generated by G is $L(G) = \{x \in V_T^*;\ S \Rightarrow^* x\}$.

We may assume that in a context-free grammar the axiom is unspecified. Thus if $V_N = \{A_0, A_1, \ldots, A_n\}$ we define:

$$L_i = \{x \in V_T^*;\ A_i \Rightarrow^* x\} .$$

Let us denote $\bar{L} = (L_0, L_1, \ldots, L_n)$. Of course if $S = A_0$ then $L(G) = L_0$.

If L and L' are two languages over V_T, the concatenation of L and L' taken in this order is defined as $LL' = \{xx';\ x \in L,\ x' \in L'\}$. The union of two languages is their union as sets.

It is possible to associate a system of equations with a context-free grammar in the following way [4]: with each nonterminal A_i we associate a variable $\mathscr{L}_i$ over $2^{V^*_T}$, then to each A_k we associate an equation of the form:

$$\mathscr{L}_k = f_k(\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n) ,$$

where $f_k(\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n)$ is the union of the right parts in all rules of the form $A_k \to \alpha$, where nonterminals are replaced by the corresponding variables.

So we obtained the system associated with the context-free grammar G:

$$\begin{aligned} \mathscr{L}_0 &= f_0(\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n), \\ \mathscr{L}_1 &= f_1(\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathscr{L}_n &= f_n(\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n) \end{aligned} \tag{1}$$

or briefly:

$$\bar{\mathscr{L}} = F(\bar{\mathscr{L}}) \tag{2}$$

with $F = (f_0, f_1, \ldots, f_n)$, $\bar{\mathscr{L}} = (\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n)$,

$$F : (2^{V^*_T})^{n+1} \to (2^{V^*_T})^{n+1} . \tag{3}$$

Blikle [1] proved that F is monotonically σ-distributive i.e. if $\bar{A}_1, \bar{A}_2, \ldots$ is a sequence in $(2^{V^*_T})^{n+1}$, $\bar{A}_1 \subseteq \bar{A}_2 \subseteq \ldots$ then $F(\bigcup_{i=1}^{\infty} \bar{A}_i) = \bigcup_{i=1}^{\infty} F(\bar{A}_i)$.

By a solution of (2) we shall mean any vector $\bar{M} = (M_0, M_1, \ldots, M_n)$, where M_i, $i = 0, 1, \ldots, n$ are languages over V_T, such that $\bar{M} = F(\bar{M})$.

Proposition 1. $\bar{L} = (L_0, L_1, \ldots, L_n)$ is a solution of (2).

Proposition 2. $\bar{L} = (L_0, L_1, \ldots, L_n)$ is the minimal solution of the system (1).
A proof for these propositions can be found in [4].

2. THE UNIQUENESS OF THE SOLUTION

We assume in this section that the grammar G has no unproductive rules i.e. there are no rules of the form $A \to \alpha$ with $\alpha \in V_N^*$. Note that this fact may be assumed without loss of generality.

Proposition 3. If G is a context-free grammar without unproductive rules then the associated system (1) has a unique solution.

Proof. By Proposition 1 and 2 $\bar{L} = (L_0, L_1, \ldots, L_n)$ is the minimal solution for (1) so that it remains to prove that for an arbitrary solution $\bar{M} = (M_0, M_1, \ldots, M_n)$ we have $M_0 \subseteq L_0, \ldots, M_n \subseteq L_n$.

Let us consider the following property depending on k:

P(k): For any i such that $0 \leqq i \leqq n$ and for any $x \in M_i$ with $|x| \leqq k$ we have $x \in L_i$.

P(0) is true because, for every $0 \leqq i \leqq n$, there is no $x \in M_i$ such that $|x| = 0$. In fact if $\varepsilon \in M_i = f_i(M_0, M_1, \ldots, M_n)$ then the null string belongs to a term in

 $f_i(M_0, M_1, \ldots, M_n)$ which corresponds to a productive rule $A_i \to \alpha_1 A_{i_1}\alpha_2 \ldots \alpha_p A_{i_p}\alpha_{p+1}$ which leads to a contradiction.

If P(k) is true, let us prove that P($k+1$) is true. If $x \in M_i$ for some $0 \leqq i \leqq n$ and $|x| = k + 1$ we recall that $x \in \alpha_1 M_{i_1}\alpha_2 \ldots \alpha_p M_{i_p}\alpha_{p+1}$. Thus $x = \alpha_1 x_{i_1}\alpha_2 x_{i_2} \ldots \ldots \alpha_p x_{i_p}\alpha_{p+1}$. The above term corresponds to a productive rule such that $|x_{i_q}| \leqq k$ for $q = 1, 2, \ldots, p$. But by the induction hypothesis $x_{i_p} \in L_{i_p}$ and $x \in \alpha_1 L_{i_1} \ldots \ldots \alpha_p L_{i_p}\alpha_{p+1} \subseteq f_i(L_0, L_1, \ldots, L_n)$.

We proved that P(k) is true for any nonnegative integer k. This implies Proposition 3.

If G has unproductive rules the system (1) need not have a unique solution.

Example. Let us consider the context-free grammar $G = (V_T, V_N, S, R)$ where $V_T = \{a\}$, $V_N = \{S\}$ and the set of rules is: $S \to a$, $S \to a^2S$, $S \to SS$. The third rule is unproductive. The associated equation is:

$$\mathscr{L} = a \cup a^2\mathscr{L} \cup \mathscr{L}\mathscr{L}\ .$$

One can prove that V_T^* is a solution and is not equal to the least solution.

3. THE SOLUTIONS OF THE ASSOCIATED SYSTEM WHEN G MAY HAVE UNPRODUCTIVE RULES

In this section we do not impose the restriction to have no unproductive rules on the grammar G, so that we are not sure that the solution is unique. The closure of the function F defined in (3) is a function

$$F^* : (2^{V^*_T})^{n+1} \to (2^{V^*_T})^{n+1}$$

defined in the following way:

$$F^*(\overline{X}) = \overline{X} \cup F(\overline{X}) \cup F^2(\overline{X}) \cup \ldots \cup F^k(\overline{X}) \cup \ldots,$$

where: $F^{k+1}(\overline{X}) = F(F^k(\overline{X}))$.

F is monotonically σ-distributive, so that F^* is also monotonically σ-distributive, and also monotonous [1]. Let us define the following subset of $(2^{V^*_T})^{n+1}$:

$$\mathscr{A} = \{\overline{X} \in (2^{V^*_T})^{n+1};\ \overline{X} \subseteq F(\overline{X})\}\ .$$

Proposition 4. $\mathscr{A}$ is a complete lattice.

Proof. $\mathscr{A}$ is an ordered set with respect to the inclusion i.e. $\overline{X} \subseteq \overline{Y}$ iff $X_i \subseteq Y_i$ for $i = 0, 1, \ldots, n$. The least upper bound (l.u.b.) $\bigvee \overline{X}_i$ $(i \in I)$ of a family $(\overline{X}_i)_{i \in I}$ is $\bigcup \overline{X}_i$ $(i \in I)$ because $\overline{X}_i \subseteq F(\overline{X}_i) \subseteq F(\bigcup_{i \in I} \overline{X}_i)$ which implies $\bigcup_{i \in I} \overline{X}_i \subseteq F(\bigcup_{i \in I} \overline{X}_i)$ and, hence, $\bigcup_{i\in I} \bar{X}_i \in \mathscr{A}$. Since $\mathscr{A}$ has a least element $\bar{\emptyset} = (\emptyset, \emptyset, \ldots, \emptyset)$, it is a complete lattice.

By $\bigwedge_{i\in I} \bar{X}_i$, we denote the greatest lower bound (g.l.b.) of the family $\bar{X}_i$ $(i \in I)$.

Note that $\mathscr{A}$ is not a sublattice of $(2^{V^*T})^{n+1}$ because it is not true that $\bar{X} \wedge \bar{Y} = \bar{X} \cap \bar{Y}$.

Example. Let us consider the function $F(\mathscr{L}) = a \cup b \cup b\mathscr{L} \cup \mathscr{L}a$, $F : 2^{V^*T} \to 2^{V^*T}$ for which if we take $X = \{a, ba\}$, $Y = \{b, ba\}$ we see that $X \cap Y \notin \mathscr{A}$ so that $X \cap Y \neq X \wedge Y$. In fact we have $X \wedge Y = \emptyset$.

Proposition 5. $\bar{L}$ is a solution of (2) iff there exists $\bar{X} \in \mathscr{A}$ such that $\bar{L} = F^*(\bar{X})$.

Proof. Since $\bar{X} \in \mathscr{A}$ we have $\bar{X} \subseteq F(\bar{X})$, using the property of F to be monotonous we obtain:

$$\bar{X} \subseteq F(\bar{X}) \subseteq F^2(\bar{X}) \subseteq \ldots$$

and, F being monotonically σ-distributive, we have:

$$F(F^*(\bar{X})) = F(\bigcup_{k=0}^{\infty} F^k(\bar{X})) = \bigcup_{k=1}^{\infty} F^k(\bar{X}) = F^*(\bar{X}) .$$

Thus we proved that if $\bar{L} = F^*(\bar{X})$, $\bar{X} \in \mathscr{A}$ then $\bar{L}$ is a solution of (2).

Conversely if $\bar{L}$ is such a solution then $F^*(\bar{L}) = \bar{L}$ and $\bar{L} \in \mathscr{A}$.

Consider now: $\mathscr{A}_1 = \{\bar{L}; F(\bar{L}) = \bar{L}\}$ i.e. the set of all solutions of (2).

Proposition 6. $\mathscr{A}_1$ is a complete lattice.

Proof. $\mathscr{A}_1$ is an ordered set with the order induced by $\subseteq$. Consider a family $(\bar{L}_i)_{i\in I}$ with $\bar{L}_i \in \mathscr{A}_1$ for $i \in I$. We shall prove that $F^*(\bigvee \bar{L}_i)$, $(i \in I)$ and $F^*(\bigwedge \bar{L}_i)$, $(i \in I)$ are respectively the l.u.b. and g.l.b. of the previous family.

Clearly $F^*(\bigvee \bar{L}_i)$, $(i \in I)$ and $F^*(\bigwedge \bar{L}_i)$, $(i \in I)$ are in $\mathscr{A}_1$, because $\bigvee \bar{L}_i$, $(i \in I)$ and $\bigwedge \bar{L}_i$, $(i \in I)$ are in $\mathscr{A}$.

To prove that $F^*(\bigvee \bar{L}_i)$, $(i \in I)$ is the l.u.b. of the family $(\bar{L}_i)_{i\in I}$ in $\mathscr{A}_1$, consider $\bar{L} \in \mathscr{A}_1$ such that $\bar{L}_i \subseteq \bar{L}$ for $i \in I$. Thus $\bigvee \bar{L}_i \subseteq \bar{L}$. This implies $F^*(\bigvee \bar{L}_i) \subseteq F^*(\bar{L}) = \bar{L}$ and also $\bar{L}_i \subseteq \bigvee \bar{L}_i \subseteq F^*(\bigvee \bar{L}_i)$ for every $i \in I$.

To prove that $F^*(\bigwedge \bar{L}_i)$, $(i \in I)$ is the g.l.b. let us observe that we have: $\bigwedge \bar{L}_i \subseteq \bar{L}_i$ thus $F^*(\bigwedge \bar{L}_i) \subseteq F^*(\bar{L}_i)$. Moreover if $\bar{L} \subseteq \bar{L}_i$ for any $i \in I$, then, because $\bar{L}$ is in $\mathscr{A}_1$, we have $\bar{L} \subseteq \bigwedge \bar{L}_i$. This implies $\bar{L} = F^*(\bar{L}) \subseteq F^*(\bigwedge \bar{L}_i)$.

Proposition 7. If $\bar{X}$, $\bar{Y} \in \mathscr{A}$ and $\bar{X} \subseteq \bar{Y} \subseteq F^*(\bar{X})$ then $F^*(\bar{X}) = F^*(\bar{Y})$.

Proof. From $\bar{X} \subseteq \bar{Y}$ we get $F^*(\bar{X}) \subseteq F^*(\bar{Y})$. Since $F^*(F^*(\bar{X})) = F^*(\bar{X})$ and because $\bar{Y} \subseteq F^*(\bar{X})$ we have $F^*(\bar{Y}) \subseteq F^*(\bar{X})$. Thus $F^*(\bar{X}) = F^*(\bar{Y})$.

Although very simple this proposition is very useful if we want to find the solutions of a system of equations. By Proposition 5 any solution is of the form $\bar{L} = F^*(\bar{X})$ with $\bar{X} \in \mathscr{A}$; thus theoretically we must check all $\bar{X} \in \mathscr{A}$. In this way a set $\bar{X} \in \mathscr{A}$ once inspected we can eliminate all the sets $\bar{Y} \in \mathscr{A}$ such that $\bar{X} \subseteq \bar{Y} \subseteq F^*(\bar{X})$.

Example. Let us consider the function: $F(\mathscr{L}) = \mathscr{L}\mathscr{L} \cup \{a\}$ associated with the context-free grammar: $G = \{V_T, V_N, S, R\}$ where $V_T = \{a\}$, $V_N = \{S\}$, $R = \{S \to a, S \to SS\}$.

Consider now $X = \emptyset$ and $Y = \{\varepsilon\}$. One can prove that $X, Y \in \mathscr{A}$ and $F^*(X) = \{a, a^2, \dots\}$, $F^*(Y) = \{\varepsilon, a, a^2, \dots\}$. Clearly for every $Z \in \mathscr{A}$ we have at least one of the following situations: $X \subseteq Z \subseteq F^*(X)$; $Y \subseteq Z \subseteq F^*(Y)$. Thus, $F^*(X)$ and $F^*(Y)$ are the only solutions of the equation $\mathscr{L} = F(\mathscr{L})$.

Proposition 8. If the function F is associated with a context-free grammar without unproductive rules, then for any $\bar{X} \in \mathscr{A}$ we have $\bar{X} \subseteq F^*(\bar{\emptyset})$, where $\bar{\emptyset} = (\emptyset, \emptyset, \dots, \emptyset)$.

Proof. Consider the following property P(k) depending on k.

P(k): For every i such that $0 \leqq i \leqq n$ and for every $x \in X_i$, $|x| \leqq k$ we have $x \in (F^*(\bar{\emptyset}))_i$ where by $(F^*(\bar{\emptyset}))_i$ is understood the i-th component of the vector $F^*(\bar{\emptyset})$.

One can prove that this property is true for any nonnegative integer k, following the method used in the proof of Proposition 3. But this implies Proposition 8.

Corollary. If F is associated with a context-free grammar without unproductive rules, then the solution of (2) is unique.

Proof. By Proposition 5 any solution is of the form $\bar{L} = F^*(\bar{X})$ with $\bar{X} \in \mathscr{A}$. By Proposition 7 and Proposition 8 we have $\bar{L} = F^*(\bar{\emptyset})$.

Definition. A vector $\bar{X} = (X_0, X_1, \dots, X_n) \in \mathscr{A}$ is said to be finite if X_i is finite for $i = 0, 1, \dots, n$. A solution $\bar{L}$ of (2) is said to be finitely generated iff there exists a finite $\bar{X} \in \mathscr{A}$ such that $\bar{L} = F^*(\bar{X})$.

Proposition 9. If $\bar{X}, \bar{Y} \in \mathscr{A}$, $\bar{X}$ finite then $F^*(\bar{X}) \subseteq F^*(\bar{Y})$ iff there exists an integer k such that $\bar{X} \subseteq F^k(\bar{Y})$.

Proof. From $F^*(\bar{X}) \subseteq F^*(\bar{Y})$ we get $\bar{X} \subseteq F^*(\bar{Y})$. Since $\bar{X}$ is finite and $\bar{Y}, F(\bar{Y}), \dots, F^k(\bar{Y}), \dots$ is an increasing sequence, we get $\bar{X} \subseteq F^k(\bar{Y})$ for some integer k.

Conversely if $\bar{X} \subseteq F^k(\bar{Y})$ then $F^*(\bar{X}) \subseteq F^k(\bar{Y}) \cup F^{k+1}(\bar{Y}) \cup \dots = F^*(\bar{Y})$.

Corollary. If $\bar{X}, \bar{Y} \in \mathscr{A}$, $\bar{X}, \bar{Y}$ finite then $F^*(\bar{X}) = F^*(\bar{Y})$ iff $\bar{X} \subseteq F^k(\bar{Y})$ and $\bar{Y} \subseteq F^k(\bar{X})$ for some integer k.

Proof. We apply two times Proposition 9.

Proposition 10. If $\bar{L}$ is a finitely generated solution, $\bar{L} = F^*(\bar{X})$, $\bar{X} \in \mathscr{A}$, $\bar{X}$ finite then $\bar{L}$ is the minimal solution of the system associated with the context-free grammar G' obtained from G by adding the rules $A_i \to x$ for all $x \in X_i$ and for all $0 \leqq i \leqq n$.

Proof. Let us denote by F' the function associated with G'. Clearly $F'(\bar{Y}) = \bar{X} \cup F(\bar{Y})$ so that $F'(\bar{\emptyset}) = \bar{X} \cup F(\bar{\emptyset}) \subseteq \bar{X} \cup F(\bar{X}) = F(\bar{X})$. Suppose that $F'^k(\bar{\emptyset}) \subseteq F^k(\bar{X})$. Then $F'^{(k+1)}(\bar{\emptyset}) = F'(F'^k(\bar{\emptyset})) \subseteq F'(F^k(\bar{X})) = \bar{X} \cup F^{(k+1)}(\bar{X}) = F^{(k+1)}(\bar{X})$. This implies $F'^*(\bar{\emptyset}) \subseteq F^*(\bar{X})$.

Conversely $\bar{X} \subseteq F'(\bar{\emptyset})$. If we assume that $F^k(\bar{X}) \subseteq F'^{(k+1)}(\bar{\emptyset})$ then $F^{(k+1)}(\bar{X}) \subseteq F(F'^{(k+1)}(\bar{\emptyset})) \subseteq F'^{(k+2)}(\bar{\emptyset})$. Thus $F'^*(\bar{\emptyset}) = F^*(\bar{X})$. But $F^*(\bar{X}) = \bar{L}$ and $F'^*(\bar{\emptyset})$ is the minimal solution for associated system of G'.

Definition. A set $P \subseteq V_N$ is said to be a parasitic set of nonterminals iff for every $A \in P$ there exist a rule $A \to \alpha$ with $\alpha \in P^*$.

Proposition 11. The system (1) associated with a context-free grammar has a unique solution if and only if the nonterminal vocabulary does not contain a non-void parasitic set of nonterminals.

Proof. To prove the necessity, let us consider that there exists a non-void parasitic set of nonterminals P. By Proposition 5 a solution of (1) is of the form $F^*(\bar{Y})$ with $\bar{Y} \in \mathscr{A}$. If we define $\bar{Y}$ in the following way:

$$Y_i = \begin{cases} \emptyset & \text{if } A_i \notin P\,, \\ \{\varepsilon\} & \text{if } A_i \in P\,, \end{cases}$$

obviously $Y_i \subseteq f_i(Y_0, Y_1, \ldots, Y_n)$ so that $\bar{Y} \in \mathscr{A}$. Moreover at least one component of $F^*(\bar{Y})$ contains the null string so that $F^*(\bar{Y}) \neq F^*(\bar{\emptyset})$. Thus we proved that the system (1) has at least two solutions.

Conversely suppose that the nonterminal vocabulary does not contain a non-void set of parasitic nonterminals. By Proposition 7 it remains to prove that for any $\bar{Y} \in \mathscr{A}$ we have $\bar{Y} \subseteq F^*(\bar{\emptyset})$.

Consider the following property P(k) depending on k:

P(k): For every i such that $0 \leqq i \leqq n$ and for any $x \in Y_i$ such that $|x| \leqq k$ we have $x \in (F^*(\bar{\emptyset}))_i$. To show that P(0) is true we shall prove that $\varepsilon \notin Y_i$ for $0 \leqq i \leqq n$. Since $\bar{Y} \in \mathscr{A}$ we have $\bar{Y} \subseteq F(\bar{Y})$. If $\varepsilon \in Y_i$ for some $0 \leqq i \leqq n$ then necessarily there exists in $f_i(Y_0, Y_1, \ldots, Y_n)$ a term of the form $Y_{i_1} Y_{i_2} \ldots Y_{i_p}$ corresponding to a rule $A_i \to A_{i_1} \ldots A_{i_p}$ such that $\varepsilon \in Y_{i_1} Y_{i_2} \ldots Y_{i_p}$. But this implies $\varepsilon \in Y_{i_q}$ for $q = 1, 2, \ldots, p$. If we repeat this procedure for each of these Y's and if we put together the nonterminals appearing in the above procedure we get a non-void set of parasitic nonterminals. Thus P(0) is true.

Assume now that P(k) is true and let us prove that $P(k + 1)$ is also true. If $x \in Y_i$ for some $0 \leqq i \leqq n$ and $|x| = k + 1$, we have to prove that $x \in (F^*(\bar{\emptyset}))_i$.

Since $Y_i \subseteq f_i(Y_0, Y_1, \ldots, Y_n)$ two cases are to be considered:

a) x belongs to a term of the form $\alpha_1 Y_{i_1} \alpha_2 Y_{i_2} \ldots \alpha_p Y_{i_p} \alpha_{p+1}$ where either $p \geqq 2$ or one of α is not null. This implies $x = \alpha_1 x_{i_1} \alpha_2 x_{i_2} \ldots x_{i_p} \alpha_{p+1}$ and $|x_{i_q}| < k$ for $q = 1, 2, \ldots, p$. From the induction hypothesis we get $x_{i_q} \in (F^*(\emptyset))_{i_q}$ for $q = 1, 2, \ldots, p$. We find that the rule $A_i \to \alpha_1 A_{i_1} \alpha_2 A_{i_2} \ldots \alpha_p A_{i_p} \alpha_{p+1}$ is in R. Thus we have $x \in \in (F(F^*(\emptyset)))_i = (F^*(\emptyset))_i$.

b) x belongs to term of the form Y_{i_1}. Thus we have the rule $A_i \to A_{i_1}$. If we repeat this procedure we are led to the case a) because if we rest in this case we get a non-void set of parasitic nonterminals $\{A_i, A_{i_1}, A_{i_2}, \ldots\}$.

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REFERENCES

[1] A. Blikle: Equations in a space of languages. CC PAS Reports 43, Computation Center, Polish Academy of Sciences, Warsaw 1971.

[2] A. Blikle: Equations in nets-computer oriented lattices. CC PAS Reports 99, Computation Center, Polish Academy of Sciences, Warsaw 1973.

[3] Gh. Borcan: Sisteme de ecuatii intr-un spatiu de limbaje cu aplicatii la gramatici independente de context. Studii și cercetări matematice *26* (1974).

[4] M. Gross, A. Lentin: Introduction to Formal Grammars. Springer Verlag, Berlin—Heidelberg 1970.

[5] S. Marcus: Algebraic Linguistics, Analytical Models. Academic Press, New York—London 1967.

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