KYBERNETIKA — VOLUME 15 (1979), NUMBER 3

Suboptimal Control on Finite Time Interval

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In the paper an algebraic approach to the numerical computation of suboptimal open-loop control on finite time interval is developed. The linear time-invariant continuous system with known transfer functions are considered.

INTRODUCTION

Some optimal control problems can be considered as finite time problems. In this paper only linear time-invariant continuous systems are considered and hence their $\mathscr{L}$ transfer functions are rational.

Our problem can be formulated as that of finding the best approximation $g(t)$ of the given impulse response $f(t)$ in the following sense:
the integral

$$\int_0^T (f(t) - g(t))^2 \, dt$$

is minimized subject to the conditions

$$\mathscr{L}(f(t)) = v(s)/p(s) ,$$
$$\mathscr{L}(g(t)) = x(s)/y(s) ,$$
$$\partial v < \partial p , \quad \partial x < \partial y ,$$

where v, p and x, y are polynomials and ∂v, ∂p and ∂x, ∂y are their respective degrees. For example an approximation of the high order $\mathscr{L}$ transfer function by a low order one may be needed. Assuming the impulse response $g(t)$ partially predetermined by the system with transfer function b/a, $\mathscr{L}(g(t)) = bx/(ay)$, the problem can be described more generally as that of open-loop control.

Consider the open-loop control system in Fig. 1, where

$\mathscr{S}$ is a realization of the transfer function $b(s)/a(s)$,
$u(t)$ – input signal with $\mathscr{L}$ transform $x(s)/y(s)$,
$f(t)$ – reference signal with $\mathscr{L}$ transform $v(s)/p(s)$,
$e(t)$ – error signal,

then for any given ∂y we can find an input $u(t)$ such that $\int_0^T e^2(t)\,dt$ is minimized.

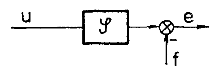

Fig. 1.

Problem formulation:

Let an integer K and polynomials b, a and v, p be given such that $\partial b \leqq \partial a$, $\partial v < \partial p$. Find the polynomials x, y such that $\partial y = K$, $\partial x < \partial y$ and the following integral

$$\int_0^T (f(t) - g(t))^2\, dt$$

is minimized for

$$f(t) = \mathscr{L}^{-1}\left(\frac{v(s)}{p(s)}\right), \quad g(t) = \mathscr{L}^{-1}\left(\frac{b(s)\,x(s)}{a(s)\,y(s)}\right).$$

The solution of this problem is complicated due to finite T.

To begin with we introduce same special notation and operations which are more precisely described in [1] and [2]. The mathematical background is the congruence of analytic functions modulo a polynomial.

Consider a polynomial $m(s) = m_0 + m_1 s + \ldots + m_{\partial m} s^{\partial m}$ with real coefficients and degree $\partial m > 0$. Then the set $\mathscr{M} = \{s : m(s) = 0\}$ is called the spectrum of the polynomial m.

Let functions f, g be analytic on $\mathscr{M}$. Then f, g are congruent modulo m, written $f = g \bmod m$, if there exists a function h analytic on $\mathscr{M}$ such that $f = g + hm$.

For any function f analytic on $\mathscr{M}$ only one polynomial r exists such that

$$(1) \qquad f = r \bmod m\,, \quad \partial r < \partial m\,.$$

The polynomial m is called the modulus. The operation which yields such a polynomial r is called the reduction of f modulo m and it is denoted as

$$(2) \qquad [f]_m = r\,.$$

Denote $\mathscr{F}_m$ the set of all functions analytic or having at worst removable singularities on $\mathscr{M}$.

Let a modulus m, $\partial m > 0$, and function f, $g \in \mathscr{F}_m$ be given. Then for $[f]_m = a$, $[g]_m = b$ and any complex number λ the next equations hold:

(3) $$[f + g]_m = [f]_m + [g]_m = a + b\,,$$

(4) $$[\lambda f]_m = \lambda[f]_m = \lambda a\,,$$

(5) $$[fg]_m = [[f]_m\,[g]_m]_m = [ab]_m\,,$$

(6)) if $f/g \in \mathscr{F}_m$ then

$$\left[\frac{f}{g}\right]_m = \left[\frac{[f]_m}{[g]_m}\right]_m = \left[\frac{a}{b}\right]_m.$$

If the function f is a polynomial then the reduction of f modulo m produces the remainder after dividing f by m. Procedures for the computation of $[f]_m$ for the functions $\ln(s)$, e^{ks}, $\sqrt{s}$, s^k with k real, and for b/a, $b\,.\,a$ with polynomials a, b are described in [1].

Now we introduce a new operation

(7) $$\left\langle\frac{s[f]_a}{a}\right\rangle_s = \lim_{s\to\infty}\frac{s[f]_a}{a} = \frac{c_{k-1}}{a_k}\,,$$

where $k = \partial a$

$$[f]_a = c_0 + c_1 s + \ldots + c_{k-1}s^{k-1}$$

and the subscript s denotes the variable with respect to which the operation $\langle . \rangle_s$ is performed.

The next Theorem is proved in [2].

Theorem 1. Let polynomials b, a, $\partial b < \partial a$, with real coefficients be given. Then the inverse Laplace transform of b/a denoted as $\mathscr{L}^{-1}(b/a)$, is the real-valued function $f(t)$ given by

(8) $$f(t) = \left\langle\frac{s}{a}\,[b\,e^{st}]_a\right\rangle_s$$

and furthermore

$$\frac{d^i}{dt^i}f(t) = \left\langle\frac{s}{a}\,[s^i b\,e^{st}]_a\right\rangle_s\,, \quad i = 1, 2, \ldots.$$

EVALUATION $\int_0^T f(t)\,g(t)\,dt$

The basic formula of this paper is in the following Theorem.

Theorem 2. Let polynomials b, a, v, p, $\partial b < \partial a$, $\partial v < \partial p$ be given. Denote $f(t) = \mathcal{L}^{-1}(b/a)$, $g(t) = \mathcal{L}^{-1}(v/p)$ then

$$(9) \qquad I = \int_0^T f(t)\, g(t)\, \mathrm{d}t = \left\langle \frac{s}{a} \left[b \, \frac{\bar{v} - \mathrm{e}^{sT} [\bar{v}\, \mathrm{e}^{-sT}]\, \bar{p}}{\bar{p}} \right]_a \right\rangle_s = \left\langle \frac{s}{p} \left[v \, \frac{\bar{b} - \mathrm{e}^{-sT} [\bar{b}\, \mathrm{e}^{-sT}]\, \bar{a}}{\bar{a}} \right]_p \right\rangle_s ,$$

where $\bar{v}(s) = v(-s)$, $\bar{b}(s) = b(-s)$.

Proof. Using Theorem 1 and the linearity of the operation $\langle . \rangle_s$ we can write

$$I = \left\langle \frac{s}{a} \left[b \int_0^T \mathrm{e}^{st}\, g(t)\, \mathrm{d}t \right]_a \right\rangle_s .$$

It is evident that

$$\int_0^T \mathrm{e}^{st}\, g(t)\, \mathrm{d}t = \int_0^\infty \mathrm{e}^{st}\, g(t)\, \mathrm{d}t - \int_T^\infty \mathrm{e}^{st}\, g(t)\, \mathrm{d}t$$

and $\int_0^\infty \mathrm{e}^{st}\, g(t)\, \mathrm{d}t = \bar{v}/\bar{p}$ by definition of $\mathcal{L}$ transform.
By Theorem 1

$$g(t) = \left\langle \frac{s}{p} \, [v\, \mathrm{e}^{st}]_p \right\rangle_s$$

and

$$g(T + \tau) = \left\langle \frac{s}{p} \, [v\, \mathrm{e}^{sT}\, \mathrm{e}^{s\tau}]_p \right\rangle_s = \left\langle \frac{s}{p} \, [[v\, \mathrm{e}^{sT}]_p \,.\, \mathrm{e}^{s\tau}]_p \right\rangle_s .$$

Hence

$$\int_T^\infty \mathrm{e}^{st}\, g(t)\, \mathrm{d}t = \mathrm{e}^{sT} \int_0^\infty \mathrm{e}^{s\tau}\, g(T + \tau)\, \mathrm{d}\tau = \mathrm{e}^{sT} \, \frac{[\bar{v}\, \mathrm{e}^{-sT}]_{\bar{p}}}{\bar{p}}$$

and the proof is complete. Considering $f(t) = \mathcal{L}^{-1}(v/p)$, $g(t) = \mathcal{L}^{-1}(b/a)$ the second formula follows.

Remark. The computation of the formula (9) can be rearranged in the following way. Denote

$$(10) \qquad [\bar{v}\, \mathrm{e}^{-sT}]_{\bar{p}a} = x ,$$

then $\partial x < \partial p + \partial a$. Use division algorithm for $x/\bar{p}$, then

$$(11) \qquad x = \bar{p} q + r , \quad \partial r < \partial \bar{p}$$

and

$$I = \int_0^T f(t)\, g(t)\, \mathrm{d}t = \left\langle \frac{s}{a} \left[b\, \mathrm{e}^{sT} \, \frac{x - [x]_{\bar{p}}}{\bar{p}} \right]_a \right\rangle_s = \left\langle \frac{s}{a} \, [b\, \mathrm{e}^{sT} q]_a \right\rangle_s .$$

Denote

$$g(t) = \mathscr{L}^{-1}\left(\frac{bx}{ay}\right), \quad f(t) = \mathscr{L}^{-1}\left(\frac{v}{p}\right),$$

and

$$I = \int_0^T (f(t) - g(t))^2 \, dt .$$

Assuming the polynomials a, y, p in normalized form, $a_{\partial a} = 1$, $y_{\partial y} = 1$, $p_{\partial p} = 1$, we can write

$$f(t) = \left\langle \frac{s}{s^{\partial p}} \left[e^{st} v\right]_p \right\rangle_s ,$$

$$g(t) = \left\langle \frac{s}{s^{\partial a + \partial y}} \left[e^{st} bx\right]_{ay} \right\rangle_s .$$

From the problem formulation the next formula follows.

(12) $$I(x, y) = \int_0^T (f(t) - g(t))^2 \, dt .$$

Using the variation approach the condition for minimizing (12) is given by

(13) $$\delta I = 2 \int_0^T (f(t) - g(t)) \, \delta g(t) \, dt = 0$$

for any variation of $g(t)$ which is given as

$$\delta \, g(t) = \mathscr{L}^{-1}\left(\frac{b \delta x}{ay} - \frac{bx}{ay^2} \delta y\right).$$

Hence using Theorem 2

(14) $$\delta I = \left\langle \frac{1}{s^{\partial a + \partial y}} \left[b \delta x (\bar{F} - \bar{G})\right]_{ny} \right\rangle_s - \left\langle \frac{1}{s^{\partial a + 2\partial y}} \left[bx \delta y (\bar{F} - \bar{G})\right]_{ny^2} \right\rangle_s = 0 ,$$

for all δx, δy, where

$$\bar{F} = \frac{\bar{v} - e^{+sT}\left[e^{-sT} \bar{v}\right]_{\bar{p}}}{\bar{p}} ,$$

$$\bar{G} = \frac{\bar{b}\bar{x} - e^{sT}\left[e^{-sT} \bar{b}\bar{x}\right]_{\bar{a}\bar{y}}}{\bar{a}\bar{y}} ,$$

$$\bar{a}(s) = a(-s) .$$

Since

$$\delta x = \delta x_0 + s\delta x_1 + \ldots + s^{\partial y - 1}\delta x_{\partial y - 1} ,$$
$$\delta y = \delta y_0 + s\delta y_1 + \ldots + s^{\partial y - 1}\delta y_{\partial y - 1} ,$$
$$y_{\partial y} = 1$$

the equation (14) may be written as a set of conditions

(15)
$$\frac{\partial I}{\partial x_i} = I_1^{(i)} = \left\langle \frac{s}{s^{\partial a + \partial y}} \left[s^i b(\bar{F} - \bar{G})\right]_{ay} \right\rangle_s = 0 ,$$
$$\frac{\partial I}{\partial y_i} = I_2^{(i)} = \left\langle \frac{s}{s^{\partial a + 2\partial y}} \left[s^i bx(\bar{F} - \bar{G})\right]_{ay^2} \right\rangle_s = 0 ,$$
$$i = 0, 1, \ldots, \partial y - 1 .$$

To solve these nonlinear equations we use Newton's iteration method in the form

(16)
$$I_1^{(i)} + \delta I_1^{(i)} = 0 , \quad \text{for} \quad i = 0, 1, \ldots, \partial y - 1,$$
$$I_2^{(i)} + \delta I_2^{(i)} = 0 ,$$

where $\delta I_1^{(i)}$, $\delta I_2^{(i)}$ denote variations of $I_1^{(i)}$, $I_2^{(i)}$ with respect to δx, δy.

Note that the application of Theorem 2 to equations (15) gives

$$I_1^{(i)} = \int_0^T (f(t) - g(t))\, h^{(i)}(t)\, \mathrm{d}t = 0 ,$$

where $h^{(i)} = \mathscr{L}^{-1}(s^i H)$, $H = b/ay$
and similarly

$$I_2^{(i)} = \int_0^T (f(t) - g(t))\, c^{(i)}(t)\, \mathrm{d}t = 0 ,$$

where

$$c^{(i)}(t) = \mathscr{L}^{-1}(s^i C) , \quad C = \frac{bx}{ay^2} .$$

Hence

$$\delta I_1^{(i)} = \int_0^T ((f(t) - g(t))\, \delta\, h^{(i)}(t) - h^{(i)}(t)\, \delta\, g(t))\, \mathrm{d}t ,$$

$$\delta I_2^{(i)} = \int_0^T ((f(t) - g(t))\, \delta c^{(i)}(t) - c^{(i)}(t)\, \delta\, g(t))\, \mathrm{d}t ,$$

where

$$\delta h^{(i)}(t) = \mathscr{L}^{-1}\left(-\frac{s^i b}{ay^2}\delta y\right),$$

$$\delta c^{(i)}(t) = \mathscr{L}^{-1}\left(\frac{s^i b\delta x}{ay^2} - 2\frac{s^i bx\delta y}{ay^3}\right),$$

$$\delta g(t) = \mathscr{L}^{-1}\left(\frac{b\delta x}{ay} - \frac{bx\delta y}{ay^2}\right).$$

Define

$$\bar{H}^{(i)} = \frac{\bar{b}(-s)^i - e^{sT}[e^{-sT}(-s)^i\,\bar{b}]_{\bar{a}\bar{y}}}{\bar{a}\bar{y}},$$

$$\bar{C}^{(i)} = \frac{\bar{b}\bar{x}(-s)^i - e^{sT}[e^{-sT}\bar{b}\bar{x}(-s)^i]_{\bar{n}\bar{y}\bar{p}^2}}{\bar{a}\bar{y}^2}$$

and the elements of the matrices A, B, C, D of the dimension $\partial y \times \partial y$ in the following way.

(17) $$A_{i+1,j+1} = \left\langle\frac{s}{s^{\partial a+\partial y}}[bs^j\bar{H}^{(i)}]_{ay}\right\rangle_s,$$

$$B_{i+1,j+1} = \left\langle\frac{s}{s^{\partial a+2\partial y}}[bxs^j\bar{H}^{(i)}]_{ay^2}\right\rangle_s - \left\langle\frac{s}{s^{\partial a+2\partial y}}[bs^{i+j}(\bar{F} - \bar{G})]_{ay^2}\right\rangle_s,$$

$$C_{i+1,j+1} = B_{j+1,i+1},$$

$$D_{i+1,j+1} = \left\langle\frac{s}{s^{\partial a+3\partial y}}[2bs^{i+j}(F - G)]_{ay^3}\right\rangle_s - \left\langle\frac{s}{s^{\partial a+2\partial y}}[bxs^j\bar{C}^{(i)}]_{ay^2}\right\rangle_s,$$

$$i = 0, 1, \ldots, \partial y - 1, \quad j = 0, 1, \ldots, \partial y - 1.$$

Using Theorem 2 it is simple to prove that A, D are symmetric matrices.
Using the matrix notation

$$J_1(X, Y) = \begin{bmatrix} I_1^{(0)} \\ I_1^{(1)} \\ \vdots \\ I_1^{(\partial y-1)} \end{bmatrix}, \quad J_2(X, Y) = \begin{bmatrix} I_2^{(0)} \\ I_2^{(1)} \\ \vdots \\ I_2^{(\partial y-1)} \end{bmatrix},$$

$$X = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\partial y-1} \end{bmatrix}, \quad Y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{\partial y-1} \end{bmatrix}, \quad y_{\partial y} = 1,$$

then

(18) $$\delta J_1 = -A\delta X + B\delta Y,$$
$$\delta J_2 = B'\delta X + D\delta Y.$$

Let us point out that $I_1^{(i)}$ can be computed as

$$I_1^{(i)} = \left\langle \frac{s}{s^{\partial a + \partial y}} [s^i b\bar{F}]_{ay} \right\rangle_s - \left\langle \frac{s}{s^{\partial a + \partial y}} [bx\bar{H}^{(i)}]_{ay} \right\rangle_s ,$$
$$I_1^{(i)} = \left\langle \frac{s}{s^{\partial a + \partial y}} [s^i b\bar{F}]_{ay} \right\rangle_s - \sum_{j=0}^{\partial y - 1} A_{i+1,j+1} x_j$$

and $I_2^{(i)}$ corresponds to the second term in the formula for $B_{i+1,1}$. Hence in our iteration process, we can define for given Y the optimal X by

(19) $$X = A^{-1} J_1(0, Y).$$

Substituting (19) into (18) and using Newton's formula

$$- A\delta X \times B\delta Y = 0 ,$$
$$J_2 + B'\delta X + D\delta Y = 0$$

we obtain

$$\delta Y = -(D + B'A^{-1}B)^{-1} J_2 .$$

Now we summarize the iteration algorithm.

(1) Start from initial condition $y = s^{\partial y}$.

(2) Compute optimal X given by $X = A^{-1}J_1$.

(3) Compute the value $Y = Y + \delta Y$, where

$$\delta Y = -(D + B'A^{-1}B)^{-1} J_2 .$$

(4) If the n-th iteration and the (n-1)-th iteration are not sufficiently near go to (2).

(5) Print results.

This algorithm was programed and tested on IBM 370/135 computer using PL/I language.

Example 1. Consider the reference signal $f(t) = \mathscr{L}^{-1}(v/p)$, where

$$v = 0{\cdot}03387 - 0{\cdot}06093s - 1{\cdot}1634s^2 - 0{\cdot}43164s^3 + 8{\cdot}6775s^4 - $$
$$- 9{\cdot}525s^5 + 0{\cdot}9898s^6 + 1{\cdot}5379s^7 ,$$

$$p = 0{\cdot}01129 - 0{\cdot}03329s - 1{\cdot}1056s^2 + 5{\cdot}82213s^3 - 1{\cdot}77933s^4 - \\ - 6{\cdot}36952s^5 + 1{\cdot}97685s^6 + s^7 + 0{\cdot}5s^8\,,$$

and the system transfer function b/a, where

$$b = 1\,,$$
$$a = 1 + 0{\cdot}2s + 0{\cdot}04s^2\,.$$

Find polynomials x, y such that $\partial y = 3$ and the integral

$$\int_0^5 (f(t) - g(t))^2\,\mathrm{d}t\,, \quad \text{where} \quad g(t) = \mathscr{L}^{-1}\left(\frac{bx}{ay}\right),$$

is minimized.

The result is obtained after five iterations in the form

$$x = 0{\cdot}8023 - 1{\cdot}225s + 0{\cdot}1849s^2\,,$$
$$y = -0{\cdot}01124 - 0{\cdot}007296s - 0{\cdot}000608s^2 + s^3\,.$$

The functions $f(t)$, $g(t)$ are plotted in Fig. 2.

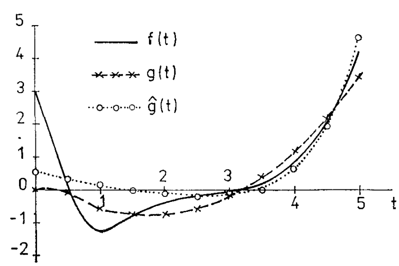

Fig. 2.

Example 2 Consider the transfer function v/p as in Example 1. Find a 3-order approximation of this transfer function such that the integral

$$\int_0^5 (f(t) - \hat{g}(t))^2\,\mathrm{d}t\,, \quad \text{where} \quad \hat{g}(t) = \mathscr{L}^{-1}(x/y)$$

is minimized. The solution is similar as in the previous Example, but $b = 1$, $a = 1$.

After five iterations we obtain the result

$$x = 0{\cdot}69452 - 1{\cdot}2376s + 0{\cdot}5877s^2 ,$$

$$y = 0{\cdot}6626 - 0{\cdot}6487s - 1{\cdot}011s^2 + s^3 .$$

The function $\vartheta(t)$ is plotted in Fig. 2.

(Received December 6, 1978.)

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