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On a Certain Type of Discrete Two-Point Boundary Problem Arising in Discrete Optimal Control*)

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It is demonstrated that in difference to the continuous case the necessary optimality conditions used in discrete optimal control have the form of a discrete implicit two-point boundary-value problem. A possible application of the modified quasilinearization method to its solution is discussed.

1. INTRODUCTION

This communication deals with a certain discrete two-point boundary-value problem (TPBVP), which is encountered during the solution of discrete optimal control problems. More exactly, it is shown, that the application of necessary optimality conditions on a quite common nonlinear discrete optimal control problem results in a special type of the discrete TPBVP, which can be alternatively denoted as the implicit one.

For the numerical solution of this problem the so-called modified quasilinearization method is suggested. Finally, a brief comparison with the continuous case is performed, because the described phenomena has no counterpart when dealing with optimal control of continuous systems.

2. DISCRETE OPTIMAL CONTROL PROBLEM

Consider the following formulation of a discrete optimal control problem with system equations described by the recurrent equation

$$x_{k+1} = f_k(x_k, u_k)\,, \quad k = 0.1, \ldots, K - 1\,, \tag{1}$$

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 together with the given initial condition

$$x_0 = a \,, \tag{2}$$

where $x_k \in E^n$ is the state of the system and $u_k \in E^m$ is the control (input) applied to the system at the stage k, and $f_k : E^n \times E^m \to E^n$, $k = 0, 1, \ldots, K - 1$. The positive integer K denotes the prescribed number of stages. If not otherwise stated, all vectors are supposed to be column-vectors, while all gradients are treated as row-vectors.

The optimal control problem consists in finding a control sequence $\hat{u} = (\hat{u}_0, \hat{u}_1, \ldots \ldots, \hat{u}_{K-1})$ and a corresponding trajectory $\hat{x} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K)$, determined by (1), i.e., to find an admissible process $(\hat{x}, \hat{u})$, which minimize the objective functional

$$J = \gamma(x_K) + \sum_{k=0}^{K-1} h_k(x_k, u_k) \,, \tag{3}$$

where $\gamma : E^n \to E^1$ and $h_k : E^n \times E^m \to E^1$, $k = 0, 1, \ldots, K - 1$.

For the sake of simplicity, no further control and/or state constraints are being imposed. More involved formulation of this problem can be found in [1]–[4], where also the necessary optimality conditions or discrete maximum principle are derived. For our simple problem described by (1)–(3) these conditions can be stated in the following way.

Assume that functions γ and f_k, h_k, $k = 0, 1, \ldots, K - 1$, are continuously differentiable on E^n and $E^n \times E^m$, respectively. If $(\hat{x}, \hat{u})$ is an optimal process, then there exist multipliers (row-vectors) $\lambda_k \in E^n$, $k = 0, 1, \ldots K$ such that

$$\lambda_k = \frac{\partial}{\partial x} H_{k+1}(\hat{x}_k, \hat{u}_k) \,, \quad k = 0, 1, \ldots, K - 1 \,, \tag{4}$$

$$\lambda_K = - \frac{\partial}{\partial x} \gamma(\hat{x}_K) \,, \tag{5}$$

where

$$H_{k+1}(x, u) = -h_k(x, u) + \lambda_{k+1} f_k(x, u) \,, \quad k = 0, 1, \ldots, K - 1 \,, \tag{6}$$

and

$$\frac{\partial}{\partial u} H_{k+1}(\hat{x}_k, \hat{u}_k) = 0 \,, \quad k = 0, 1, \ldots, K - 1 \,. \tag{7}$$

Equations (4) and (5) define the so-called adjoint system and equations (7) represent the necessary optimality conditions.

To proceed further, let us additionally assume that for the studied class of optimal control problems it is possible to determine $\hat{u}_k$ as the explicit function of $\hat{x}_k$ and λ_{k+1} using (7) for all $k = 0, 1, \ldots, K - 1$. This means that (7) implies that (carets above x_k and u_k further omitted)

(8) $$u_k = g_k(x_k, \lambda_{k+1}), \quad k = 0, 1, \ldots, K - 1,$$

where $g_k : E^n \times E^n \to E^m$, $k = 0, 1, \ldots, K - 1$, are continuous functions on $E^n \times \times E^n$.

Although this assumption is restrictive from the theoretic point of view, it is not otherwise possible to treat the original discrete optimal control problem as further studied TPBVP, i.e., to solve the original optimal control problem by indirect methods. If this assumption does not hold, it is then necessary to use direct methods, e.g., of the gradient type.

3. DISCRETE IMPLICIT TWO-POINT BOUNDARY-VALUE PROBLEM

After inserting (8) into (1) and (4) one obtains

(9) $$x_{k+1} = f_k(x_k, g_k(x_k, \lambda_{k+1})) = \tilde{f}_k(x_k, \lambda_{k+1}), \; k = 0, 1, \ldots, K - 1,$$

(10) $$\lambda_k = \frac{\partial}{\partial x} H_{k+1}(x_k, g_k(x_k, \lambda_{k+1})) = \tilde{g}_k(x_k, \lambda_{k+1}), \quad k = 0, 1, \ldots, K - 1,$$

with $\tilde{f}_k : E^n \times E^n \to E^n$, $\tilde{g}_k : E^n \times E^n \to E^n$, $k = 0, 1, \ldots, K - 1$, being continuous on $E^n \times E^n$. Relations (9) and (10) together with the boundary conditions (2) and (5) form a discrete TPBVP. However, this problem has a rather special structure, because it is not solved with respect to the $(k + 1)$-st or k-th stage. To do this, it is necessary to exclude λ_{k+1} from (9) using (10) or to exclude x_k from (10) using (9).

For example, the first possibility would lead to the equations

(11) $$x_{k+1} = \tilde{\tilde{f}}_k(x_k, \lambda_k) = \tilde{f}_k(x_k, \tilde{\tilde{g}}_k(x_k, \lambda_k)), \quad k = 0, 1, \ldots, K - 1,$$

(12) $$\lambda_{k+1} = \tilde{\tilde{g}}_k(x_k, \lambda_k), \quad k = 0, 1, \ldots, K - 1,$$

which describe together with (2) and (5) a discrete TPBVP of the more familiar type, considered in [4].

However, this representation is not usually possible due to the general form of (9) and (10). Therefore, it seems reasonable to treat the original TPBVP separately.

It is easy to see that such problem, described by (9) and (10) with boundary conditions (2) and (5), is a special case of the following discrete implicit TPBVP given by the implicit recurrent equation

(13) $$F_k(y_k, y_{k+1}) = 0, \quad k = 0, 1, \ldots, K - 1,$$

and boundary conditions

(14) $$\varphi(y_0) = 0\,, \quad \psi(y_K) = 0\,, \quad \omega(y_0, y_K) = 0\,,$$

where $y_k \in E^N$, and $F_k : E^N \times E^N \to E^N$, $k = 0, 1, \ldots, K - 1$, $\varphi : E^N \to E^p$, $\psi : E^N \to E^q$ and $\omega : E^N \times E^N \to E^r$. Clearly, $N = p + q + r$ must hold. This formulation of a discrete boundary-value problem covers rather broad class of discrete optimal problems. For example, the problems with various constraints on initial and/or final state or problems with feedback can be handled in this way.

4. MODIFIED QUASILINEARIZATION ALGORITHM

To solve the implicit TPBVP given by (13) and (14), an iterative methods must be applied. A quasilinearization version of the Newton-Raphson method is described in [4] for discrete TPBVP of the explicit type, i.e., the equation (13) has the form

(15) $$y_{k+1} = G_k(y_k)\,, \quad k = 0, 1, \ldots, K - 1\,.$$

However, only the case of linear initial and final conditions is considered there. Further presented quasilinearization algorithm for discrete TPBVP includes the additional modification (stepsize control) as suggested in [5] for continuous TPBVP.

This modification is based on the consideration of the auxiliary performance index P, which measures the cumulative error in the recurrent equations and the boundary conditions. This algorithm is generated by the requirement that the first variation of the performance index δP be negative. It differs from the ordinary quasilinearization algorithm, because of the inclusion of the stepsize α in the system of variations. The main property of the modified quasilinearization algorithm is the descent property. Namely, if the stepsize α is sufficiently small, the reduction in P is quaranteed. Convergence to the desired solution is achieved when the inequality $P \leqq \varepsilon$ is met, where ε is a small, preselected number.

For the explicite formulation of TPBVP, given by (15), the implementation of the modified quasilinearization method was studied in [6]. Now let us briefly describe the generalization of [6] to the implicit case (13) and (14). It is assumed that the all functions in these relations are continuously differentiable and that a solution of this discrete implicit TPBVP exists.

The derivation of the algorithm is rather straightforward if we combine the results of [4] and [5]. The details of this procedure are discussed in [6]. The resulting numerical algorithm can be summarized as follows (superscript i denotes iteration number).

Step 0. Select $\varepsilon > 0$ and nominal estimate $y^0 = (y_0^0, y_1^0, \ldots, y_K^0)$.

Step 1. Set $i = 0$.

Step. 2. Compute $F_k(y_k^i, y_{k+1}^i)$, $k = 0, 1, \ldots, K - 1$, and $\varphi(y_0^i)$, $\psi(y_K^i)$, $\omega(y_0^i, y_K^i)$, and evaluate the performance index (here $\mathrm{N}(c) = c^T c = \|c\|^2$ for a vector c)

$$(16) \qquad P(y^i) = \mathrm{N}(\varphi(y_0^i)) + \mathrm{N}(\psi(y_K^i)) + \mathrm{N}(\omega(y_0^i, y_K^i)) + \sum_{k=0}^{K-1} \mathrm{N}(F_k(y_k^i, y_{k+1}^i)) .$$

Step 3. If $P \leqq \varepsilon$, stop; else go to Step 4.

Step 4. Compute

$$(17) \qquad \frac{\partial}{\partial y_k} F_k(y_k^i, y_{k+1}^i) , \quad \frac{\partial}{\partial y_{k+1}} F_k(y_k^i, y_{k+1}^i) , \quad k = 0, 1, \ldots, K - 1 ,$$

and

$$(18) \qquad \frac{\partial}{\partial y_0} \varphi(y_0^i) , \quad \frac{\partial}{\partial y_K} \psi(y_K^i) , \quad \frac{\partial}{\partial y_0} \omega(y_0^i, y_K^i) , \quad \frac{\partial}{\partial y_K} \omega(y_0^i, y_K^i) ,$$

and solve the following discrete implicit linear TPBVP for $z_0^i, z_1^i, \ldots, z_K^i$:

$$(19) \qquad \frac{\partial}{\partial y_{k+1}} F_k(y_k^i, y_{k+1}^i)\, z_{k+1}^i + \frac{\partial}{\partial y_k} F_k(y_k^i, y_{k+1}^i)\, z_k^i + F_k(y_k^i, y_{k+1}^i) = 0 ,$$

$$k = 0, 1, \ldots, K - 1 ,$$

$$(20) \qquad \frac{\partial}{\partial y_0} \varphi(y_0^i)\, z_0^i + \varphi(y_0^i) = 0 ,$$

$$\frac{\partial}{\partial y_K} \psi(y_K^i)\, z_K^i + \psi(y_K^i) = 0 ,$$

$$\frac{\partial}{\partial y_0} \omega(y_0^i; y_K^i)\, z_0^i + \frac{\partial}{\partial y_K} \omega(y_0^i, y_0^i)\, z_K^i + \omega(y_0^i, y_K^i) = 0 .$$

Step 5. Consider one-parameter family of solutions $\tilde{y}^i(\alpha^i)$:

$$(21) \qquad \tilde{y}_k^i = y_k^i + \alpha^i z_k^i , \quad k = 0, 1, \ldots, K ,$$

and perform a one-dimensional search on function $\tilde{P}(\alpha^i) = P(\tilde{y}^i)$ for the minimizing α^{i*}; specifically, perform a bisection process on α^i (starting from $\alpha^i = 1$), and continue the process until the inequality

$$(22) \qquad \tilde{P}(\alpha^{i*}) < \tilde{P}(0) = P(y^i)$$

is satisfied. Then compute

$$(23) \qquad y_k^{i+1} = y_k^i + \alpha^{i*} z_k^i , \quad k = 0, 1, \ldots, K ,$$

set $i = i + 1$ and go to Step 2.

Remark 1. It is clear, that the performance index P given by (16) is zero if and only if the corresponding sequence y solves (13) and (14). For any other sequence y, P is always positive. It can be shown that the function $\tilde{P}(\alpha^i)$ exhibits relative minimum with respect to α^i, i.e., there exists a point α^{i*} such that

$$\frac{\partial}{\partial \alpha} \tilde{P}(\alpha^{i*}) = 0 , \tag{24}$$

because

$$\frac{\partial}{\partial \alpha} \tilde{P}(0) = -2\tilde{P}(0) . \tag{25}$$

As the exact determination of such α^{i*} takes usually excessive computer time, it is better to perform this search in a noniterative fashion, e.g., using the indicated bisection process starting with $\alpha^i = 1$ and terminating if (22) is met. The existence of such α^{i*} is guaranteed by (25).

Remark 2. The linear TPBVP given by (19) and (20) can be solved, for example, using the discrete version of the method of adjoints [7], as described in [4] and [6] more in detail. To apply this scheme, it is sufficient to assume that the $(N \times N)$-matrices

$$\frac{\partial}{\partial y_{k+1}} F_k(y_k, y_{k+1}) , \quad k = 0, 1, \ldots, K - 1 ,$$

are nonsingular. Clearly this assumption also implies that (13) can be locally resolved with respect to y_{k+1}. The numerical experience is reported in [8].

5. CONCLUSIONS

The studied problem connected with discrete optimal control systems has no counterpart in the continuous case. Namely, for continuous systems, the necessary optimality conditions lead always to the explicit TPBVP for a system of first-order ordinary differential equations, which are resolved with respect to the derivative, i.e. they have form analogical to (11) and (12). Of course, the implicit formulation of continuous TPBVP is also possible [9], however, it is not necessary when solving continuous optimal control problems.

On the other hand, the discrete implicit TPBVP of the type (13) cannot be generally avoided during the solution of discrete optimal control problems by indirect methods, as demonstrated by (9) and (10). It is further clear, that the somewhat special structure of (9) and (10) can be sometimes exploited during the solution of a problem in question. It was the aim of this communication to point out and discuss this interesting problem arising in discrete optimal control theory.

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