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On the Core of an Incomplete n-Person Game

ANTONÍN OTÁHAL

Analogously as Bondareva's theorem (cf [1]) we state that the core of a game is nonempty if and only if the game is balanced but general incomplete games are considered, i.e. some coalitions may be unfeasible.

0. INTRODUCTION

Our method of studying incomplete games is that of an approximation of any incomplete game by some complete one.

Parts 1, 2 introduce basic concepts, the 3rd part contains the main results of the paper.

1. PREREQUISITIES

We suppose that I is a finite nonempty set and K is some system of subsets of I. $\mathbf{R}$ denotes the set of all real numbers.

A set function $v : K \to \mathbf{R}$ is superadditive if, whenever S is a system of disjoint sets from K such that $\bigcup S \in K$, then

$$\sum_{s \in S} v(s) \leqq v(\bigcup S) .$$

For $K \subset \exp I$, $s \subset I$ we denote

$$\mathbf{r}(K, s) = \{R : R \subset K , \bigcup R = s, \text{ the elements of } R \text{ are disjoint sets}\} .$$

We write $\mathbf{r}(K)$ instead of $\mathbf{r}(K, I)$.

Let $\mathbf{r}(K) \neq \emptyset$. Then a *game* $v = (v, K, I)$ is any superadditive set function $v : K \to \mathbf{R}$. The game v is *complete* if $K = \exp I$ and *incomplete* otherwise.

We make no difference between vectors $\mathbf{x} \in \mathbf{R}^I$ and additive set functions $\mathbf{x}$: : $\exp I \to \mathbf{R}$, $x(s) = \sum_{i \in s} x_i$.

The *core* of the game $v = (v, K, I)$ is the set of all $\mathbf{x} \in \mathbf{R}^I$ for which

(a) $\mathbf{x}(s) \geqq v(s)$ for all $s \in K$,

(b) there exists $R \in \mathbf{r}(K)$ such that $\mathbf{x}(t) = v(t)$ for all $t \in R$.

We denote the core of the game v through $C(v)$ (or $C(v, K, I)$ if needed).

If $f : K \to \mathbf{R}$ is any set function and if $R \subset K$, we denote

$$\langle f, R \rangle = \sum_{s \in R} f(s) .$$

To every game $v = (v, K, I)$ we assigne the number

$$m(v) = \max \{\langle v, R \rangle : R \in \mathbf{r}(K)\} .$$

1.1. Lemma. Let $\mathbf{x} \in \mathbf{R}^I$. Then $\mathbf{x} \in C(v, K, I)$ if and only if

(a) $\mathbf{x}(s) \geqq v(s)$ for all $s \in K$,

(b′) $x(I) = m(v)$.

Proof. Under holding (a), any of (b), (b′) is clearly equivalent to the condition

$$\mathbf{x}(t) = v(t) \quad \text{for all} \quad t \in R ,$$

where R is that element of $\mathbf{r}(K)$ for which $m(v) = \langle v, R \rangle$.

2. BALANCED GAMES

Let S be a nonempty subsystem of K such that $\emptyset \notin S$, $\mathbf{c} \in \mathbf{R}^S$ be a vector whose coordinates are strictly positive real numbers. S is said to be a *balanced K-cover* with the *weight vector* $\mathbf{c}$ if for every $i \in I$

$$\sum_{s \in S_i} c_s = 1$$

holds, where $S_i = \{s : s \in S, i \in s\}$.

A game $v = (v, K, I)$ is *balanced* if, whenever S is a balanced K-cover with the weight vector $\mathbf{c}$, we have

$$\sum_{s \in S} c_s\, v(s) \leqq m(v) .$$

It is natural to say that S is a balanced K-cover (write $S \in \mathscr{B}_K$) if there exists $\mathbf{c} \in \mathbf{R}^S$ such that S is the balanced K-cover with the weight vector $\mathbf{c}$.

Moreover, S is said to be the balanced cover (write $S \in \mathscr{B}$) instead of saying S is the balanced $\exp I$-cover.

A balanced K-cover T is the *minimal balanced K-cover* if from $S \subset T$ and $S \in \mathscr{B}_K$ the equality $S = T$ follows. In this case we write $T \in \mathscr{M}_K$. That is, $\mathscr{M}_K$ is the set of all in inclusion sense minimal elements of $\mathscr{B}_K$. As previously omitting K means $K = \exp I$.

For $S \in \mathscr{B}_K$ we denote

$$\mathbf{V}(S) = \{\mathbf{c} : \mathbf{c} \text{ is the weight vector of } S\} .$$

2.1. Lemma. Let S be the balanced K-cover. Then

(i) $\mathbf{V}(S)$ is a convex set in $\mathbf{R}^S$,

(ii) $\mathbf{V}(S)$ contains exactly one point if and only if S is minimal,

(iii) $\bigcup\{\mathbf{V}(T) : T \subset S, T \in \mathscr{M}_K\}$ is the set of all extremal points of $\operatorname{cl} V(S)$ ($\operatorname{cl} X$ denotes the topological closure of a set X).

Proof is given in [2].

2.2. Theorem. The game $v = (v, K, I)$ is balanced if and only if

$$(1) \qquad \sum_{t \in T} c_t\, v(t) \leqq m(v)$$

for every T being a minimal balanced K-cover with the (unique) weight vector $\mathbf{c}$.

Proof. The "only if" is obvious.

According to the previous lemma, if $S \in \mathscr{B}_K$ and $\mathbf{c} \in \mathbf{V}(S)$ then there exist $T^1, \ldots$ $\ldots, T^m \in \mathscr{M}_K$ with weight vectors $\mathbf{c}^1, \ldots, \mathbf{c}^m$ (respectively) such that $\mathbf{c}$ is some convex linear combination of $\mathbf{c}^1, \ldots, \mathbf{c}^m$. ($\mathbf{c}^i$ taken as an element of $\mathbf{R}^{T^i} \times \{0\}^{S-T^i}$).

As

$$(1^i) \qquad \sum_{t \in T^i} c_t^i\, v(t) \leqq m(v)$$

holds for every $i = 1, \ldots, m$ the same convex combination applied on $(1^1), \ldots, (1^m)$ implies

$$\sum_{s \in S} c_s\, v(s) \leqq m(v) .$$

2.3. Remark. Let v be a complete game. Then

1) v is superadditive if and only if for every disjoint pair s, t of subsets of I

$$v(s) + v(t) \leqq v(s \cup t) ,$$

2) $m(v) = v(I)$.

3.1. Theorem. Let v be a complete game. Then v is balanced if and only if the core of v is nonempty.

Proof cf. [1], [3].

We shall generalize this result now.

$\mathbf{R}^+$ denotes the set of all positive real numbers.

For the brevity we shall use this terminology: Let $P(h)$ be a statement depending on $h \in \mathbf{R}^+$. Then "$P(h)$ holds for h large" means the same as "there exists $h_0 \in \mathbf{R}^+$ such that for every $h \geqq h_0$ the statement $P(h)$ holds".

Let us denote $\bar{K} = K \cup \{\{i\} : i \in I\}$ and for every $h \in \mathbf{R}^+$ define set functions

$$\bar{v}_h : \bar{K} \to \mathbf{R}, \quad \bar{v}_h(s) = \begin{cases} v(s) & \text{for} \quad s \in K \\ -h & \text{for} \quad s \in \bar{K} - K \end{cases}$$

$$v_h : \exp I \to \mathbf{R}, \quad v_h(s) = \max \{\langle \bar{v}_h, R \rangle : R \in \mathbf{r}(K, s)\} \quad \text{for} \quad s \neq \emptyset,$$
$$v_h(\emptyset) = 0.$$

Obviously, for every $h \in \mathbf{R}^+$, the set function v_h is superadditive and so v_h is a complete game.

We denote $\mathbf{d}(K)$ the set of all $s \subset I$ for which $\mathbf{r}(K, s) \neq \emptyset$.

3.2. Lemma. Let $v = (v, K, I)$ be a game, $s \subset I$.

(i) if $s \in \mathbf{d}(K)$ then

$$v_h(s) = \max \{\langle v, R \rangle : R \in \mathbf{r}(K, s)\}$$

holds for h large,

(ii) if $s \notin \mathbf{d}(K)$ then $\lim_{h \to \infty} v_h(s) = -\infty$.

Proof. Let $s \subset I$, $R \in \mathbf{r}(\bar{K}, s) - \mathbf{r}(K, s)$, $R \neq \emptyset$. Then clearly $\langle v_h, R \rangle \to -\infty$, $h \to \infty$.

Both (i) and (ii) follow immediately.

3.3. Lemma. Let $v = (v, K, I)$. Then for h large

(i) $v_h/K = v$,

(ii) $m(v_h) = m(v)$

hold.

Proof. Let $s \in K$. Then $\{\{s\}\} \in \mathbf{r}(K, s)$, i.e. $v(s) = \max \{\langle v, R \rangle : R \in \mathbf{r}(K, s)\}$ (v is superadditive). So 3.2. (i) and the finiteness of K imply (i). (ii) follows from 3.2. (i) as $\mathbf{r}(K) \neq \emptyset$.

3.4. Lemma. A game v is balanced if and only if v_h is balanced for h large.

Proof. The "if" is obvious with regard to 3.3. For the proof of the "only if" it is sufficient to prove (cf. 2.2, 3.3. (ii)): whenever $T \in \mathcal{M}$ and $\{\mathbf{c}\} = \mathbf{V}(T)$ then

$$\sum_{t \in T} c_t\, v_h(t) \leqq m(v) \tag{1}$$

holds for h large. So let $T \in \mathcal{M}$, $\{\mathbf{c}\} = \mathbf{V}(T)$. There are two possibilities.

a) $T \subset \mathbf{d}(K)$. Then we define

$$S = \bigcup_{t \in T} R_t\,,$$

$$b_s = \sum_{\{t : s \in R_t\}} c_t \quad \text{for} \quad s \in S\,,$$

where $R_t (t \in T)$ are defined by the relation

$$\langle v, R_t \rangle = \max \{\langle v, R \rangle : R \in \mathbf{r}(K, t)\}\,,$$

Obviously $S \in \mathcal{B}_K$, $\boldsymbol{b} \in \mathbf{V}(S)$. According to 3.2. (i)

$$\sum_{t \in T} c_t\, v_h(t) = \sum_{s \in S} b_s\, v(s)$$

holds for h large.

Now (1) follows from v being balanced.

b) $T - \mathbf{d}(K) \neq \emptyset$. Then

$$\lim_{k \to \infty} \sum_{t \in T} c_t\, v_h(t) = -\infty$$

(from 3.2. (ii)). That is, (1) holds for h large.

3.5. Lemma. Let $h, k \in \mathbf{R}^+$, $h \leqq k$, v_h, v_k be defined as previously. Then for h large

(i) if v_h is balanced then v_k is also balanced,

(ii) $C(v_h) \subset C(v_k)$.

Proof. If h is large enough then $v_h(I) = v_k(I) = m(v)$ and $v_h/K = v_k/K = v$. Both (i), (ii) follow from obvious relation

$$v_h(s) \geqq v_k(s) \quad \text{for all} \quad s \subset I$$

now.

3.6. Theorem. Let $v = (v, K, I)$ be a game (complete or incomplete). Then the core of v is nonempty if and only if v is the balanced game.

Proof. 1) Let $\mathbf{x} \in C(v)$. With regard to 3.3 there exists $h \in \mathbf{R}^+$ such that

(2) $$\mathbf{x}(I) = m(v_h) = v_h(I),$$

(3) $$v_h/K = v$$

and

(4) $$h \geqq -x_i \quad \text{for all} \quad i \in I$$

hold. Let $s \subset I$. It is $v_h(s) = \langle \bar{v}_h, R \rangle$ where $R \in \mathbf{r}(\bar{K}, s)$. If $t \in R$ then $t \in K$ or $t \in \bar{K} - K$, i.e. $\bar{v}_h(t) = v(t) \leqq \mathbf{x}(t)$ or $\bar{v}_h(t) = -h \leqq \mathbf{x}(t)$ (respectively). Consequently

(5) $$v_h(s) \leqq \mathbf{x}(s) \quad \text{for all} \quad s \subset I$$

is valid.

(3), (5) mean $\mathbf{x} \in C(v_h)$. According to 3.1 the complete game v_h is balanced, (2), (3) imply that v is a balanced game, too.

2) Let v be a balanced game. From 3.3, 3.4 it follows the existence of h for which (3) and

(6) $$m(v) = v_h(I)$$

hold and, moreover, v_h is also balanced. So there exists $\mathbf{x} \in C(v_h)$. We define

(7) $$k = \max\,(h, \max_{i \in I}\,(-x_i)).$$

According to 3.5 it is $\mathbf{x} \in C(v_k)$ and with regard to (7) we obtain (analogously as in 1))

(8) $$v(s) \leqq \mathbf{x}(s) \quad \text{for all} \quad s \in K.$$

As $\mathbf{x} \in C(v_k)$, the relation (6) is the same as

(9) $$\mathbf{x}(I) = m(v).$$

(8) and (9) establish $\mathbf{x} \in C(v)$.

3.7. Remark. Let $v = (v, K, I)$ be a game, v_h be as above. We denote $C_h = C(v_h)$. Then

$$\lim_{h \to \infty} C_h = C$$

is true in the following sense:

(i) $(C_h)_{h \geqq h_0}$ is a monotone increasing system of sets and

(ii) $\bigcup_{h \geqq h_0} C_h = C$

hold for h_0 large.

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Antonín Otáhal, matematicko-fyzikální fakulta University Karlovy (Mathematical and Physical Faculty — Charles University), Sokolovská 83, 186 00 Praha 8. Czechoslovakia.