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# On Identical Response of Initially Excited and Relaxed Linear Discrete-Time System

VÁCLAV SOUKUP

The relation between the initially excited and relaxed linear discrete-time system is investigated. Given the system state-space description state initial conditions are transformed into the equivalent input signal respecting the input-output physical causality. The recursive as well as compact form of the transformation is given. Finally the problem simplification for stationary systems including $\mathscr{Z}$-transform approach is presented.

## FORMULATION OF THE PROBLEM

Let us consider a linear, single input-single output, discrete-time system described in the state-space form

$$\mathbf{x}(k+1) = \mathbf{A}(k)\,\mathbf{x}(k) + \mathbf{b}(k)\,u(k)\,, \qquad (1)$$
$$y(k) = \mathbf{c}(k)\;\mathbf{x}(k) + d(k)\,u(k)\,,$$

where $\mathbf{x}(k)$ is an $(s \times 1)$ state vector, $u(k)$ and $y(k)$ are input and output, respectively; $\mathbf{A}(k)$, $\mathbf{b}(k)$, $\mathbf{c}(k)$ and $d(k)$ are system parameters of the proper dimensions and time variable $k \in K$ ranges over the integers.

Before formulating our problem we introduce for the system (1)

a) the operator

$$L^i_M\,\mathbf{c}(k) = [L^{i-1}_M\,\mathbf{c}(k+1)]\,\mathbf{M}(k)\,; \quad \text{integer} \quad i \geqq 1 \qquad (2)$$

with

$$L^0_M\,\mathbf{c}(k) = \mathbf{c}(k)\,,$$

where $\mathbf{M}(k)$ is an arbitrary $(s \times s)$ matrix;

b) the scalar

$$l_i^{-1}(k) = [L^{i-1}_A\,\mathbf{c}(k+1)]\,\mathbf{b}(k), \quad i \geqq 1 \qquad (3)$$

 with

$$l_0^{-1}(k) = d(k)$$

c) the relative order $\varrho$, $0 \leqq \varrho \leqq s$, of the system (1) possessing the properties derived and proved in [2]:

$$l_{\varrho-j}^{-1}(k) = 0\,, \quad \varrho \geqq j \geqq 1\,, \tag{4}$$

$$l_\varrho^{-1}(k) \neq 0\,, \quad k \in K\,.$$

We can distinguish two separate cases of the system operation.

At first, the system (1) operates under the given non-zero initial conditions, i.e., previously excited.

Secondly, the system is initially relaxed, i.e., it operates under zero initial conditions.

Roughly speaking, the problem of interest is to find such an (equivalent) input which if applied to the initially relaxed system yields the response identical with the response of the previously excited system for all $k \geqq k_0$, $k \in K$.

Obviously an additional external input of the linear system (1) does not affect the solution of the problem and therefore it can be neglected in further considerations.

To formulate and solve the above problem more exactly with respect to input-output physical causality we assume the state initial conditions of the system (1) to be $\mathbf{x}(k_0 - \varrho)$; $k_0 - \varrho \in K$.

Then the output of the system can be expressed as [1]

$$y_1(k) = \mathbf{c}(k)\,\boldsymbol{\Phi}(k, k_0 - \varrho)\,\mathbf{x}(k_0 - \varrho)\,, \quad k \geqq k_0 - \varrho\,, \tag{5}$$

where the system transition matrix

$$\boldsymbol{\Phi}(k, m) = \mathbf{A}(k)\,\mathbf{A}(k-1)\ldots\mathbf{A}(m)\,; \quad k > m$$

with

$$\boldsymbol{\Phi}(k, k) = \mathbf{I}\,.$$

On the other hand the relaxed system has the response

$$y_2(k) = \mathbf{c}(k)\sum_{m=k_0-\varrho}^{k-1}\boldsymbol{\Phi}(k, m+1)\,\mathbf{b}(m)\,u(m) + d(k)\,u(k) \tag{7}$$

if the input sequence

$$u(m)\,; \quad m = k_0 - \varrho\,, \quad k_0 - \varrho + 1, \ldots, k$$

is applied to it.

**Theorem.** Let the response of the system (1) of the relative order $\varrho$, $0 \leqq \varrho \leqq s$, operating under only initial conditions $\mathbf{x}(k_0 - \varrho) \neq \mathbf{0}$ be $y_1(k)$, $k_0 - \varrho \leqq k \in K$. Then the relaxed system (1) with $\mathbf{x}(k_0 - \varrho) = \mathbf{0}$ yields the response

$$y_2(k) = 0 \quad \text{for} \quad k < k_0 \,,$$
$$y_2(k) = y_1(k) \quad \text{for} \quad k \geqq k_0$$

when excited by the equivalent input given either
a) in the recursive form

$$(8) \qquad u(k_0 - \varrho + i) = l_\varrho(k_0 - \varrho + i) \{[L_A^{\varrho+i} \mathbf{c}(k_0 - \varrho)] \mathbf{x}(k_0 - \varrho) - \\ - \sum_{j=0}^{i-1} l_{\varrho+i-j}^{-1}(k_0 - \varrho + j) \, u(k_0 - \varrho + j)\}$$

gradually for $i = 0, 1, 2, \ldots$ or
b) in the compact form

$$(9) \qquad u(k_0 - \varrho + i) = l_\varrho(k_0 - \varrho + i) [L_{H_\varrho}^i \mathbf{c}(k_0)] \mathbf{A}(k_0 - 1) \ldots \\ \ldots \mathbf{A}(k_0 - \varrho) \mathbf{x}(k_0 - \varrho) \,, \quad i \geqq 0 \,,$$

where the $(s \times s)$ matrix

$$(10) \quad \mathbf{H}_\varrho(k) = \mathbf{A}(k) - l_\varrho(k - \varrho) \mathbf{A}(k) \mathbf{A}(k-1) \ldots \mathbf{A}(k - \varrho + 1) \mathbf{b}(k - \varrho) \mathbf{c}(k) \,.$$

Proof. a) Using the denotations (2) and (3) with the property (4) and putting $k = k_0 + i$, the output (5) of the previously excited system can be expressed as

$$(11) \qquad y_1(k_0 + i) = [L_A^{\varrho+i} \mathbf{c}(k_0 - \varrho)] \mathbf{x}(k_0 - \varrho) \,, \quad i = -\varrho, 1 - \varrho, \ldots$$

On the other hand the response (6) of the initially relaxed system can be written in the form

$$(12) \qquad y_2(k_0 + i) = \sum_{m=k_0-\varrho}^{k_0+i} l_{k_0+i-m}^{-1}(m) \, u(m) = \sum_{m=k_0-\varrho}^{k_0+i-\varrho} l_{k_0+i-m}^{-1}(m) \, u(m) \,.$$

Substituting $m = k_0 - \varrho + j$ we obtain

$$(13) \qquad y_2(k_0 + i) = \sum_{j=0}^{i} l_{\varrho+i-j}^{-1}(k_0 - \varrho + j) \, u(k_0 - \varrho + j) \,, \quad i \geqq 0$$

and

$$y_2(k_0 + i) = 0 \,, \quad i < 0$$

because of $u(k_0 - \varrho + j) = 0; j < 0$ for the system relaxed until $k_0 - \varrho$.

The relation (13) rewritten in the form

$$(14)\qquad y_2(k_0 + i) = l_\varrho^{-1}(k_0 - \varrho + i)\, u(k_0 - \varrho + i) + \\ + \sum_{j=0}^{i-1} l_{\varrho+i-j}^{-1}(k_0 - \varrho + j)\, u(k_0 - \varrho + j)$$

can be compared with (11); the comparison of the both outputs for $i \geqq 0$ simply yields the recursive formula (8), q.e.d.

b) Replacing the input values $u(k_0 - \varrho + j)$, $j = 0, 1, \ldots, i$ in the recursive formula (8) by the assumed ones given by the compact form (9) we obtain the equation

$$(15)\qquad l_\varrho(k_0 - \varrho + i)\,[L_{H_\varrho}^i\, \mathbf{c}(k_0)]\, \mathbf{A}(k_0 - 1) \ldots \mathbf{A}(k_0 - \varrho)\, \mathbf{x}(k_0 - \varrho) = \\ = l_\varrho(k_0 - \varrho + i)\, \{[L_A^{\varrho+i}\, \mathbf{c}(k_0 - \varrho)]\, \mathbf{x}(k_0 - \varrho) - \\ - \sum_{j=0}^{i-1} l_{\varrho+i-j}^{-1}(k_0 - \varrho + j)\, l_\varrho(k_0 - \varrho + j)\, . \\ . [(L_{H_\varrho}^j\, \mathbf{c}(k_0)]\, \mathbf{A}(k_0 - 1) \ldots \mathbf{A}(k_0 - \varrho)\, \mathbf{x}(k_0 - \varrho)\}\, .$$

Obviously the compact form (9) is correct if the both sides of the equation (15) will be found to be identical.

As

$$[L_A^{\varrho+i}\, \mathbf{c}(k_0 - \varrho)]\, \mathbf{x}(k_0 - \varrho) = \\ = \mathbf{c}(k_0 + i)\, \mathbf{A}(k_0 + i - 1) \ldots \mathbf{A}(k_0)\, \mathbf{A}(k_0 - 1) \ldots \mathbf{A}(k_0 - \varrho)\, \mathbf{x}(k_0 - \varrho) = \\ = [L_A^i\, \mathbf{c}(k_0)]\, \mathbf{A}(k_0 - 1) \ldots \mathbf{A}(k_0 - \varrho)\, \mathbf{x}(k_0 - \varrho)$$

the comparison given by the equation (15) can be simply transformed into the form

$$(16)\qquad L_{H_\varrho}^i\, \mathbf{c}(k_0) = L_A^i\, \mathbf{c}(k_0) - \sum_{j=0}^{i-1} l_{\varrho+i-j}^{-1}(k_0 - \varrho + j)\, l_\varrho(k_0 - \varrho + j)\, L_{H_\varrho}^j\, \mathbf{c}(k_0)\, .$$

Using the relations (2), (3) and (10) the equation (16) may be rewritten in the form

$$(17)\qquad L_{H_\varrho}^i\, \mathbf{c}(k_0) = L_A^i\, \mathbf{c}(k_0) + \sum_{j=0}^{i-1} [L_A^{i-j-1}\, \mathbf{c}(k_0 + j + 1)]\, \mathbf{H}_\varrho(k_0 + j) \ldots \mathbf{H}_\varrho(k_0) - \\ - \sum_{j=0}^{i-1} [L_A^{i-j-1}\, \mathbf{c}(k_0 + j + 1)]\, \mathbf{A}(k_0 + j)\, \mathbf{H}_\varrho(k_0 + j - 1) \ldots \mathbf{H}_\varrho(k_0) = \\ = L_A^i\, \mathbf{c}(k_0) + \sum_{j=0}^{i-1} [L_A^{i-j-1}\, \mathbf{c}(k_0 + j + 1)]\, \mathbf{H}_\varrho(k_0 + j) \ldots \mathbf{H}_\varrho(k_0) - \\ - \sum_{j=-1}^{i-2} [L_A^{i-j-1}\, \mathbf{c}(k_0 + j + 1)]\, \mathbf{H}_\varrho(k_0 + j) \ldots \mathbf{H}_\varrho(k_0)\, .$$

Then the common identical terms of the both sums (for $j = 0, 1, \ldots, i - 2$) in the equation (17) are mutually compensated. Hence 

$$L_{H_\varrho}^{i}\, \mathbf{c}(k_0) = L_A^i\, \mathbf{c}(k_0) + \mathbf{c}(k_0 + i)\, \mathbf{H}_\varrho(k_0 + i - 1) \ldots \mathbf{H}_\varrho(k_0) - L_A^i\, \mathbf{c}(k_0) = = L_{H_\varrho}^{i}\, \mathbf{c}(k_0)\,, \tag{18}$$

q.e.d.

## STATIONARY SYSTEM

The solution of the above problem becomes more simple for a stationary system. The system parameters $\boldsymbol{A}$, $\boldsymbol{b}$, $\mathbf{c}$ and $d$ are time-invariant in this case and the relations (2), (3) and (10) can be simplified into

$$L_M^i \mathbf{c} = c\boldsymbol{M}^i\,, \quad i = 0, 1, \ldots \tag{2a}$$

$$l_i^{-1} = [L_A^{i-1}\mathbf{c}]\, \boldsymbol{b} = \mathbf{c}\boldsymbol{A}^{i-1}\boldsymbol{b}\,, \quad i = 1, 2, \ldots \tag{3a}$$

with

$$l_0^{-1} = d$$

and

$$\boldsymbol{H}_\varrho = \boldsymbol{A} - l_\varrho \boldsymbol{A}^\varrho \boldsymbol{b}\mathbf{c}\,. \tag{10a}$$

Therefore the recursive form (8) generating the equivalent input of the stationary system is

$$u(k_0 - \varrho + i) = (\mathbf{c}\boldsymbol{A}^{\varrho-1}\boldsymbol{b})^{-1}\, [\mathbf{c}\boldsymbol{A}^{\varrho+i}\, \boldsymbol{x}(k_0 - \varrho) - - \mathbf{c}\sum_{j=0}^{i-1} \boldsymbol{A}^{\varrho+i-1-j}\boldsymbol{b}\, u(k_0 - \varrho + j)]\,, \quad i = 0, 1, 2, \ldots \tag{8a}$$

and the compact relation (9) may be written in the form

$$u(k_0 - \varrho + i) = l_\varrho \mathbf{c}\boldsymbol{H}_\varrho^i \boldsymbol{A}^\varrho\, \boldsymbol{x}(k_0 - \varrho)\,; \quad i \geqq 0\,. \tag{9a}$$

The solution can be expressed in $z$-domain too. Applying the $\mathscr{Z}$ transform to the signals started at the time $k_0 - \varrho$ we have

$$Y(z) = \sum_{i=0}^{\infty} y(k_0 - \varrho + i)\, z^{-i} \tag{19}$$

and

$$U(z) = \sum_{i=0}^{\infty} u(k_0 - \varrho + i)\, z^{-i}\,. \tag{20}$$

Then the $\mathscr{Z}$ transform of the relation (9a) gives

$$(21)\quad U(z) = u(k_0 - \varrho) + u(k_0 - \varrho + 1)\, z^{-1} + u(k_0 - \varrho + 2)\, z^{-2} + \ldots =$$
$$= l_\varrho \mathbf{c}[\mathbf{I} + \mathbf{H}_\varrho z^{-1} + \mathbf{H}_\varrho^2 z^{-2} + \ldots]\, \mathbf{A}^\varrho\, \mathbf{x}(k_0 - \varrho) =$$
$$= l_\varrho \mathbf{c}[\mathbf{I} - z^{-1}\mathbf{H}_\varrho]^{-1}\, \mathbf{A}^\varrho\, \mathbf{x}(k_0 - \varrho)$$

seeing that [1]

$$\mathscr{Z}[\mathbf{H}_\varrho^i] = [\mathbf{I} - z^{-1}\mathbf{H}_\varrho]^{-1}\,.$$

EXAMPLE

Assuming the system described by

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)\,,$$
$$y(k) = [-1 \quad 1]\, \mathbf{x}(k)$$

we determine the input equivalent to the state initial conditions

$$\mathbf{x}(k_0 - \varrho) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We have

$$l_0^{-1} = 0\,, \quad l_1^{-1} = \mathbf{cb} = 1 \quad \text{and} \quad \varrho = 1 \quad \text{at first}\,.$$

a) Using the recursive form (8a) we can write gradually

$$u(k_0 - 1) = -2\,, \qquad u(k_0 + 1) = -3\,,$$
$$u(k_0) \quad = -3\,, \qquad \vdots$$

b) According to (10a) the matrix

$$\mathbf{H}_1 = \mathbf{A} - l_1 \mathbf{Abc} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$$

and the compact form (9a) of the equivalent input yields

$$u(k_0 - \varrho + i) = l_1 \mathbf{c} \mathbf{H}_1^i \mathbf{A}\, \mathbf{x}(k_0 - 1) = [-1 \;\; 1] \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}^i \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad i \geqq 0\,.$$

Using the $\mathscr{Z}$ transform relation (21) we have

$$U(z) = -\frac{2 + z^{-1}}{1 - z^{-1}} = -2 - 3z^{-1} - 3z^{-2} - \ldots$$

To verify the results we compare the both outputs

$$Y_1(z) = \mathbf{c}[\mathbf{I} - z^{-1}\mathbf{A}]^{-1}\,\mathbf{x}(k_0 - 1) = \frac{1 - z^{-1}}{1 + z^{-1} + z^{-2}} =$$

$$= 1 - 2z^{-1} + z^{-2} + z^{-3} - 2z^{-4} + \ldots$$

and

$$Y_2(z) = \mathbf{c}\,z^{-1}[\mathbf{I} - z^{-1}\mathbf{A}]^{-1}\,\mathbf{b}U(z) = \frac{z^{-1} - z^{-2}}{1 + z^{-1} + z^{-2}} \cdot \frac{-2 - z^{-1}}{1 - z^{-1}} =$$

$$= -2z^{-1} + z^{-2} + z^{-3} - 2z^{-4} + \ldots$$

The responses are identical for $k \geqq k_0$.

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*Ing. Václav Soukup, CSc., katedra řídicí techniky elektrotechnické fakulty ČVUT (Czech Technical University — Department of Automatic Control), Karlovo nám. 13, 121 35 Praha 2. Czechoslovakia.*