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On Recursive Filters Inverse to Finite Sequences in the Minimum Mean Square Sense

LUDVÍK PROUZA

A finite sequence may be interpreted as a distorted impulse. A filter restoring the impulse from the sequence in the minimum mean square sense will be called "inverse".

In this contribution, some properties of the recursive "inversion" filter from [1], [4], [5] are investigated.

1. INTRODUCTION

Let a finite real sequence

$$\{b_0, \ldots, b_h\}, \quad h > 0, \quad b_0 \neq 0, \quad b_h \neq 0 \tag{1}$$

be given. To this sequence, the polynomial

$$\mathscr{B}_s(z) = b_0 z^h + \ldots + b_h \tag{2}$$

is formed. The roots z_i, $i = 1, \ldots, h$ of the equation $\mathscr{B}_s(z) = 0$ will be called the roots of $\mathscr{B}_s(z)$. It is clear that a nontrivial multiple of the sequence (vector) (1) possesses the same roots to (2).

To the sequence

$$\{b_h, \ldots, b_0\} \tag{3}$$

the polynomial

$$b_h z^h + \ldots + b_0 = \mathscr{B}_m(z) = z^h \mathscr{B}_s(z^{-1}) \tag{4}$$

is formed. For z_i being a root of (2), z_i^{-1} is a root of (4).

In what follows one will suppose that none of the roots of (2) lies precisely on the unit circle $\boldsymbol{C}_1$.

If the sequence (1) represents a discrete signal, then the sequence (3) represents the filter matched to (1).

2. SOME PROPERTIES OF A RECURSIVE FILTER "INVERSE" TO A FINITE SEQUENCE

We will be concerned with a recursive filter, "inverse" – in the sense of minimum sum of error squares [1] – to the sequence (1).

The weighting sequence of this filter will be denoted

$$\{a_j\}\,, \quad j = 0, 1, 2, \ldots \tag{5}$$

and the output sequence

$$\{c_j\}\,, \quad j = 0, 1, 2, \ldots \tag{6}$$

The maximum term of this sequence is c_T, where $T \geqq 0$ is given in advance. There is ([5], (2), (13))

$$c_T = b_0^2 \,.\, \frac{z_1 \ldots z_h}{\zeta_1 \ldots \zeta_h} \,.\, \sum_{k=0}^{T} p_k^2 \,. \tag{7}$$

In (7), z_j $(j = 1, \ldots, h)$ are the roots of (2), $\zeta_j = z_j$ for $|z_j| > 1$, $\zeta_j = z_j^{-1}$ for $|z_j| < 1$. The sequence $\{p_j\}$ $(j = 0, 1, \ldots)$ is sequence of coefficients of the expansion ([5], (10))

$$\frac{b_0 z^h + \ldots + b_h}{q_0 z^h + \ldots + q_h} = p_0 + p_1 z^{-1} + \ldots . \tag{8}$$

The denominator polynomial possesses the roots $\zeta_j = z_j$ for $|z_j| < 1$, $\zeta_j = z_j^{-1}$ for $|z_j| > 1$, and $q_0 = b_0 b_h$. Using these properties, one gets from (7):

$$c_T = b_0^2 \,.\, \prod |z_j|^2 \,.\, \sum_{k=0}^{T} p_k^2 \,, \tag{9}$$

where the product is formed from those roots of (2) with $|z_j| < 1$.

The left side of (8) can be expressed as follows

$$\frac{b_0 z^h + \ldots + b_h}{q_0 z^h + \ldots + q_h} = \frac{1}{b_h} \prod \frac{z - z_j}{z - \bar{z}_j^{-1}} = \frac{1}{b_h} \prod z_j \,.\, \prod \frac{z - z_j}{1 - \bar{z}_j \,.\, z} \,. \tag{10}$$

All products contain the roots of (2) for which $|z_j| > 1$. The last product on the right is a finite Blaschke product with absolute value 1 for $z = \exp(i\lambda)$. The preceding terms represent only a scale factor, thus for $z = \exp(i\lambda)$ the values of (10) lie on a circle with the center in the origin.

The sequence $\{p_j\}$ may be computed by a recurrent system of linear equations ([5], (9)) or, what is the same to say, by a linear difference equation. Moreover, using the middle expression in (10), the order of this difference equation can be reduced knowing the roots of (2) with $|z_j| > 1$.

Or, alternatively, $\{p_j\}$ can be computed from the right side of (10) as Fourier coefficients

$$p_k = \frac{1}{b_h} \prod z_j \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(\alpha(\lambda) + k\lambda)} \, . \, d\lambda \,, \tag{11}$$

where

$$\alpha(\lambda) = \sum_{|z_j| > 1} \arg \frac{e^{i\lambda} - z_j}{1 - \bar{z}_j e^{i\lambda}} \,. \tag{12}$$

Thus, the behavior of the "phase function" (12), determined by the roots of (2) lying outside of C_1, is substantial for the behavior of c_T for growing T.

3. CHOOSING A RECURSIVE "INVERSION" FILTER TO A GIVEN FINITE SEQUENCE AND COMPARING TWO SEQUENCES

There may be shown easily

$$\frac{c_T^2}{\sum\limits_{j \neq T} c_j^2} = \frac{c_T}{1 - c_T} \,. \tag{13}$$

Since $c_T \to 1$ for $T \to \infty$ ([5], (14)), the expression (13) can be made arbitrarily large letting T to be sufficiently large.

Since the sequence (1) is real, one can replace in what follows $\bar{z}_j^{-1}$ by z_j^{-1} in the respective products and sums.

Let $D(z)$ be a polynomial with those roots of (2) for which $|z_j| > 1$,

$$D(z) = z^m + d_1 z^{m-1} + \ldots + d_m \,. \tag{14}$$

There is $m < h$ for practically useful sequences. $D(z)$ is the numerator of the product in the middle of (10). The denominator will be denoted $F(z)$, it possesses roots reciprocal to $D(z)$, thus

$$F(z) = z^m + \frac{d_{m-1}}{d_m} z^{m-1} + \ldots + \frac{d_1}{d_m} z + \frac{1}{d_m} \,. \tag{15}$$

Denoting

$$p_0 b_h = r_0 \,, \quad p_1 b_h = r_1 \,, \ldots \tag{16}$$

one gets from the middle expression in (10) the system of equations

(17)
$$
\begin{aligned}
r_0 &= 1 , \\
r_1 &= d_1 - (d_{m-1}/d_m)\, r_0 , \\
r_2 &= d_2 - (d_{m-2}/d_m)\, r_0 - (d_{m-1}/d_m)\, r_1 , \\
&\vdots
\end{aligned}
$$

Instead of (9), one can write

(18)
$$c_T = b_0^2 \cdot \frac{1}{\prod_{|z_j|>1} |z_j|^2} \cdot \sum_{k=0}^{T} r_k^2 .$$

For a given sequence (1), knowing the roots of (2), one can compute c_T from (17), (18), for growing T.

Since c_T is used practically for detection and T is a time delay, T cannot be too large (e.g. $h \leqq T \leqq 2h$), the complexity of the inversion filter being also important. The rapidity of convergence of the series in (18) for growing T is dependent on the original sequence (1). Given two sequences, the one with more rapidly converging $\{c_T\}$ $(T = 0, 1, \ldots)$ is better from this standpoint.

Moreover, for a detection inversion filter, the signal/noise ratio must also be considered. For the white noise,

(19)
$$N_{\text{out}} = N_{\text{in}} \cdot \sum_{k=0}^{\infty} a_k^2 .$$

For a given T, the sequence $\{a_i\}$ is to be computed. For a rough guess, one can be content with the result for $T \to \infty$.

In this case, the inversion filter is the formal two-sided inversion filter with

(20)
$$|A(z)|^2 = \frac{1}{|b_0 + b_1 z^{-1} + \;..\; + b_h z^{-h}|^2} .$$

Using the Parseval identity, one gets

(21)
$$\lim_{T\to\infty} \sum_{k=0}^{\infty} a_k^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|\mathscr{B}_s(e^{i\lambda})|^2}\, d\lambda .$$

For the matched filter

(22)
$$(S/N)_{\text{out}} = \frac{\sum_0^h b_i^2}{\sigma^2}$$

and for the inversion filter

(23) $$(S/N)_{\text{out}} = \frac{c_T^2}{\sigma^2 \, . \sum_0^\infty a_i^2} \, .$$

Thus, the ratio

(24) $$\nu = \frac{c_T^2}{\sum_0^h b_i^2 \, . \sum_0^\infty a_i^2}$$

gives the diminution of S/N of the inversion filter compared with the matched one. For $T \to \infty$, $c_T \to 1$, thus

(25) $$\nu = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\mathscr{B}_s(e^{i\lambda})|^2 \, d\lambda \, . \, \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|\mathscr{B}_s(e^{i\lambda})|^2} \, d\lambda \right)^{-1} .$$

The second integral can be computed practically e.g. by numerical integration. From (25) there is clear that for the recursive inversion filter, (2) with roots lying on C_1 is not admissible ($\nu = 0$ in this case).

There is

(26) $$|\mathscr{B}_s(e^{i\lambda})|^2 = \mu_0 + 2\mu_1 \cos \lambda + \ldots + 2\mu_h \cos h\lambda$$

with

(27) $$\mu_i = b_0 b_i + \ldots + b_{h-i} b_h \, , \quad i = 0, \ldots, h \, .$$

4. A PROPERTY OF FINITE SEQUENCES

From [5] one knows that the transfer function of the recursive inversion filter to (1) is

(28) $$A(z) = z^{h-T} \, . \, \frac{p_0 + p_1 z + \ldots + p_T z^T}{(1 - \zeta_1 z) \ldots (1 - \zeta_h z)} \, ,$$

where $\zeta_i = z_i$ for $|z_i| > 1$, $\zeta_i = z_i^{-1}$ for $|z_i| < 1$, z_i being the roots of (2). Substituting $\zeta = z^{-1}$, one gets

(29) $$\mathscr{A}(\zeta) = a_0 + a_1 \zeta + \ldots = \frac{p_0 \zeta^T + \ldots + P_T}{(\zeta - \zeta_1) \ldots (\zeta - \zeta_h)} =$$
$$= \frac{p_0 \zeta^T + \ldots + p_T}{\zeta^h + \gamma_{h-1} \zeta^{h-1} + \ldots + \gamma_0} \, .$$

The polynomial in the denominator is the "minimum phase" polynomial with the same power spectrum as (2) (neglecting a constant scale factor).

From (29), one gets the equations system

$$\begin{aligned} &\gamma_0 a_T + \gamma_1 a_{T-1} + \dots + \qquad a_{T-h} &= p_0\,, \\ &\qquad \gamma_0 a_{T-1} + \dots + \gamma_{h-1} a_{T-h} + a_{T-h-1} &= p_1\,, \\ &\qquad\qquad \vdots \\ &\qquad\qquad \gamma_0 a_{T-h} + \dots + a_{T-2h} &= p_h\,. \end{aligned} \tag{30}$$

Let now

$$|\gamma_0| \gg |\gamma_i| \, (i > 0\,,\ \gamma_h = 1)\,. \tag{31}$$

Then, approximately

$$a_{T-i} \doteq p_i/\gamma_0 \quad (i = 0, \dots, h)\,. \tag{32}$$

Further, from (8)

$$\frac{1}{b_h} z^h + \frac{b_1}{b_0 b_h} z^{h-1} + \dots + \frac{1}{b_0} = \tag{33}$$

$$= \left(z^h + \delta_1 z^{h-1} + \dots + \delta_h\right) . \left(p_0 + p_1 z^{-1} + \dots\right).$$

Since the roots of the polynomial at the right are reciprocal to the roots of the polynomial in the denominator in (29), this polynomial is one with "maximum phase" and (neglecting a constant scale factor) with the same power spectrum as (2).

By the same reasoning

$$\delta_i = \gamma_i/\gamma_0 \quad (i = 1, \dots, h) \tag{34}$$

and from (33) one gets

$$\begin{aligned} &p_0 &= 1/b_h\,, \\ &\frac{\gamma_1}{\gamma_0} p_0 + p_1 &= b_1/b_0 b_h\,, \\ &\frac{\gamma_2}{\gamma_0} p_0 + \frac{\gamma_1}{\gamma_0} p_1 + p_2 &= b_2/b_0 b_h\,. \\ &\qquad\qquad \vdots \end{aligned} \tag{35}$$

And with the condition (31), there is approximately

$$p_i \doteq b_i/b_0 b_h \quad (i = 0, \dots, h) \tag{36}$$

and from (32), (36)

$$a_{T-i} \doteq b_i/b_0 b_h \gamma_0 \quad (i = 0, \dots, h)\,. \tag{37}$$

Conversely, there can be deduced from (30) and (35) that (37) cannot be valid, not even approximately, without fulfilling (31). Thus, one will ask what is the meaning of (31). Remembering what is said about the denominator on the right of (29), one sees that a sequence (1) fulfilling (31) is "approximately white". Since always not only $\gamma_0 \neq 0$, but also $\gamma_h = 1 \neq 0$, there follows heuristically that the best what can be done to meet (31) is to choose for (1) a Huffman sequence.

In [6], [7], the property (37) is postulated in a somewhat less general situation for a binary sequence to be "good" in the correlation sense. Moreover, in [2], there has been shown that to the Barker sequences "near" Huffman sequences can be found. From what has been shown the connection of (37) with the simpler condition (31) and with the Huffman sequences is clearly seen.

There may be of interest to show "precise" formulas for Huffman sequences.

Let ϱ $(0 < \varrho < 1)$ be the radius of the circle inside $\boldsymbol{C}_1$ containing the roots of a Huffman polynomial. Then from (33) with (16)

$$z^h + \frac{b_1}{b_0} z^{h-1} + \ldots + \frac{b_h}{b_0} = (z^h \mp \varrho^h)(r_0 + r_1 z^{-1} + \ldots). \tag{38}$$

From (38) one gets (the upper and the lower signs being always corresponding in (38) and the following formulas)

$$\begin{aligned} p_i &= b_i/b_0 b_h \qquad (i = 0, \ldots, h-1), \\ p_h &= 1/b_0 \pm \varrho^h/b_h, \\ p_{h+j} &= \pm \varrho^h p_j \qquad (j = 1, 2, \ldots). \end{aligned} \tag{39}$$

Further, from (29)

$$p_0 \zeta^T + \ldots + p_T = \left(\zeta^h \mp \left(\frac{1}{\varrho}\right)^h\right)(a_0 + a_1 \zeta + \ldots). \tag{40}$$

And from (40)

$$\begin{aligned} a_i &= \mp p_{T-i} \varrho^h \qquad (i = 0, \ldots, h-1), \\ a_h &= \mp p_{T-h} \varrho^h \pm a_0 \varrho^h, \\ &\vdots \\ a_T &= \mp p_0 \varrho^h \pm a_{T-h} \varrho^h. \end{aligned} \tag{41}$$

From (41)

$$\begin{aligned} a_{T-h} &= \mp p_h \varrho^h \qquad\qquad \text{for} \quad h < T < 2h, \\ a_{T-h} &= \mp p_h \varrho^h - p_{2h} \varrho^{2h} \quad \text{for} \quad 2h \leqq T < 3h. \\ &\vdots \end{aligned} \tag{42}$$

For sequences not "too differing" from the Huffman ones, one can expect – see (39) – an "abrupt" diminution of $|p_j|$ going from p_h to p_{h+1}. Furthermore, one expects from (41) that at least

$$\text{sign}\, a_{T-j} = \text{sign}\, b_j \quad (j = 0, \ldots, h) \tag{43}$$

will hold in this case.

5. THE BEHAVIOR OF INVERSION IN THE z AND FREQUENCY DOMAIN

If (28) were the transfer function of the formal inversion filter, there would be

$$z^{T-h}\, \mathscr{B}_s(z)\, A(z) = 1\,. \tag{44}$$

For the recursive inversion filter, there is with (8), (10) and (27)

$$\begin{aligned} z^{T-h}\, \mathscr{B}_s(z)\, A(z) &= \\ = (-1)^h \frac{b_0}{b_h} \Big(\prod_{|z_i|<1} z_i \Big/ \prod_{|z_j|>1} z_j \Big) &\prod_{|z_j|>1} \frac{z - z_j}{z - z_j^{-1}} = \\ = (-1)^h \frac{b_0}{b_h} \Big(\prod_{|z_i|<1} z_i \Big/ \prod_{|z_j|>1} z_j \Big) &(r_0 + \ldots + r_T z^T)(r_0 + r_1 z^{-1} + \ldots)\,. \end{aligned} \tag{45}$$

The structure of (45) is clear. The roots of $\mathscr{B}_s(z)$ inside $\boldsymbol{C}_1$ are compensated by the same roots of the denominator of $A(z)$. The roots of $\mathscr{B}_s(z)$ lying outside of $\boldsymbol{C}_1$ cannot be compensated by the poles of $A(z)$, $A(z)$ being stable. The "noncomplete" compensation results in a phase delay expressed by the last term on the right. This delay is corrected by the last but one term on the right, at the expense of a time delay T.

There is

$$\begin{aligned} &(r_0 + r_1 z + \ldots)(r_0 + r_1 z^{-1} + \ldots)_{z=e^{i\lambda}} = \\ &= |r_0 + r_1 z^{-1} + \ldots|^2_{z=e^{i\lambda}} = \\ &= \sum_{j=0}^{\infty} r_j^2 + 2\cos\lambda \sum_{j=0}^{\infty} r_j r_{j+1} + 2\cos 2\lambda \sum_{j=0}^{\infty} r_j r_{j+2} + \ldots \end{aligned} \tag{46}$$

This is a non-constant function of λ. Furthermore ([1] and [4], (35), (37))

$$\begin{aligned} &\min \frac{1}{2\pi i} \int_{C_1} |z^{T-h}\, \mathscr{B}_s(z)\, A(z) - 1|^2 \frac{dz}{z} = \\ &= \min\,(c_0^2 + \ldots + c_{T-1}^2 + (1 - c_T)^2 + c_{T+1}^2 + \ldots) = 1 - c_T\,. \end{aligned} \tag{47}$$

6. EXAMPLES

Some simple examples will be shown here for illustration.

Example 1. Let the sequence

$$1{\cdot}000\,,\quad -1{\cdot}000\,,\quad 1{\cdot}618\,,\quad -2{\cdot}618\,,\quad -2{\cdot}618 \tag{48}$$

be given. It is a Huffman sequence, its correlation sequence is

$$-2{\cdot}618\,,\quad 0{\cdot}000\,,\quad 0{\cdot}000\,,\quad 0{\cdot}000\,,\quad 18{\cdot}326\,,\quad \ldots \tag{49}$$

Thus $h = 4$, $b_0 = 1$, $b_4 = -2{\cdot}618$. There is

$$\begin{array}{llllll} r_0 = & 1{\cdot}000 & r_3 = & -2{\cdot}618 & r_6 = & 0{\cdot}236 \\ r_1 = & -1{\cdot}000 & r_4 = & -2{\cdot}472 & r_7 = & -0{\cdot}382 \\ r_2 = & 1{\cdot}618 & r_5 = & -0{\cdot}146 & & \vdots \end{array} \tag{50}$$

One may easily verify the formulas (39), remembering that $\varrho = 0{\cdot}618$ and since b_4 is negative, the plus sign is valid in (39). Also from (18), $c_T = 0{\cdot}980$ for $T = 4$, and from (24), $v = 0{\cdot}96$.

Example 2. Let the sequences be

$$\begin{array}{ccccccc} +, & +, & +, & -, & -, & +, & - \\ +, & +, & +, & +, & -, & +, & - \end{array} \tag{51}$$

The first one is the known Barker-7 sequence, the second one results changing the sign of the middle term.

Here, $c_T = 0{\cdot}98$ for $T = 11$ in the first case, and for $T = 10$ in the second case. Thus, from this viewpoint, both sequences are almost equally "good". But $v = 0{\cdot}72$ for Barker sequence, and $v = 0{\cdot}37$ for the second one, so that this sequence is definitely worse.

Moreover,

(52)

	B-7	2.
r_0	1·000	1·000
r_1	1·000	1·000
r_2	1·240	0·318
r_3	−0·760	0·318

r_4	$-0{\cdot}503$	$-1{\cdot}148$
r_5	$1{\cdot}016$	$0{\cdot}852$
r_6	$-0{\cdot}714$	$0{\cdot}021$

Thus, there exists a Huffman sequence near to the Barker-7 one and none to the second one.

Example 3. Let the sequences be

$$+,\ +,\ +,\ -,\ -,\ -,\ +,\ -,\ -,\ +,\ - \tag{53}$$

$$+,\ +,\ +,\ +,\ +,\ -,\ -,\ +,\ -,\ +,\ -$$

The first one is the known Barker-11 sequence, the second one may be called Golay-Schroeder sequence. Here, $c_T = 0{\cdot}98$ for $T = 15$, and $v = 0{\cdot}71$ in the first case, $c_T = 0{\cdot}98$ for $T = 10$, and $v = 0{\cdot}77$ in the second case. Also, the signs in the $\{r_i\}$ sequences are the same as the sings of the terms of the sequences in both cases, so that near Huffman sequences exist in both cases.

7. CONCLUDING REMARKS

The property of a sequence to possess an "inversion" filter the signs of the weighting coefficients $a_T, \ldots, a_{T-h}$ thereof are the same as the signs of the weighting coefficients of the matched filter, introduces a semiordering into a set of corresponding sequences (e.g. binary of the same "length").

Consider e.g. binary sequences with 5 terms. Considering the sequences with all signs reversed or the sequences with reversed order of terms as the same and excluding further those with none or 5 sign changes, one obtains 8 sequences

$+ + + + -$		$+ + + - +$	$+ + - + -$	$+ + - - +$
$\uparrow$	$\nearrow$	$\nwarrow$	$\uparrow$	
$+ - - - +$	$+ + + - -$	$+ - + + -$	$+ + - + +$	

Thus, e.g. in the second column, the signs of $r_0, \ldots, r_4$ of the sequences in the second row are the same as the signs $r_0, \ldots, r_4$ of the sequence in the first row, this sequence being the Barker-5 sequence. With the arrows, the semiordering is illustrated.

Thus, precisely for the sequences in the first row, near Huffman sequences exist. Applying further the Varakin sieve [3], (that is, retaining only the sequences with $k = (h + 2)/2 = 3$ "groups" of $+$ and $-$ signs, only the second and the last sequence in the first row remain, the second one being the Barker-5 sequence and the last one is formed thereof changing the sign of the middle term. Both sequences result also applying Golay's "skew-symmetry" sieve [8]. For the Barker sequence $c_T = 0{\cdot}98$ for $T = 6$, and $v = 0{\cdot}88$, for the second one $c_T = 0{\cdot}98$ for $T = 7$, $v = 0{\cdot}36$.

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Dr. Ludvík Prouza, DrSc., Tesla - Ústav pro výzkum radiotechniky (Institute of Radioengineering), Opočínek, 533 31 p. Lány na Důlku. Czechoslovakia.