

Generalization of Sum Representation Functional Equations II Generalized Directed Divergence*)

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The measurable solution of a functional equation in three variables useful in the axiomatic characterization of generalized directed-divergence is given in this paper.

1. INTRODUCTION

Let

$$K =]0, 1[\times]0, 1[\times]0, 1[\cup \{(0, y, z)\} \cup \{(1, u, w)\}$$

with $y, z \in [0, 1[$ and $u, w \in]0, 1]$.

Let $F_{ij}, G_i, H_j : K \rightarrow R$ (reals) ($i = 1, 2; j = 1, 2, 3$) be functions which are measurable in each variable satisfying the functional equation

$$(1.1) \quad \sum_{i=1}^2 \sum_{j=1}^3 F_{ij}(x_i p_j, y_i q_j, z_i r_j) = \sum_{i=1}^2 G_i(x_i, y_i, z_i) + \sum_{j=1}^3 H_j(p_j, q_j, r_j)$$

where $P, Q, R \in \mathcal{A}_3$ and $X, Y, Z \in \mathcal{A}_2$ with

$$\mathcal{A}_n = \{P = (p_1, p_2, \dots, p_n) : p_i \geq 0, \sum_{i=1}^n p_i = 1\}$$

for all $n = 1, 2, \dots$, the set of all complete finite n -ary probability distributions.

The equation (1.1) in a particular case, is useful to characterize the following measure of information known as the generalized directed-divergence [6]

$$(1.2) \quad I_n(p_1, \dots, p_n : q_1, \dots, q_n : r_1, \dots, r_n) = \sum_{i=1}^n p_i \log (q_i / r_i),$$

where $P, Q, R \in \mathcal{A}_n$, with the convention that whenever a q_i or a r_i is zero then the

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corresponding p_i is also zero, and $0 \cdot \log 0 = 0$. The measure generalized directed divergence given by (1.2) satisfies many algebraic properties. It was shown by C.T. Ng that any measure I_n satisfying the properties of symmetry, expansibility and branching can be represented in the sum form $I_n(P||Q|R) = \sum_{i=1}^n F(p_i, q_i, r_i)$ where F is an arbitrary function satisfying $F(0, 0, 0) = 0$. In addition if I_n is additive, then F satisfies (2.27), that is (1.1) in which all functions appearing are the same. For details refer to [2].

The object of this paper is to find the measurable solutions of the equation (1.1).

2. THE MAIN THEOREM

The following theorem will be proved through a series of lemmas.

Theorem. Let $F_{ij}, G_i, H_j : K \rightarrow R$ ($i = 1, 2; j = 1, 2, 3$) be functions measurable in each variable satisfying the functional equation (1.1). Then the solutions of (1.1) are given by

$$\begin{aligned}
 (2.1) \quad & F_{11}(p, q, r) = A(p, q, r) + (d_1 - c_1)p + (d_2 - c_2)q + (d_3 - c_3)r + e_2, \\
 & F_{12}(p, q, r) = A(p, q, r) + (d'_1 - c_1)p + (d'_2 - c_2)q + (d'_3 - c_3)r + e_5, \\
 & F_{13}(p, q, r) = A(p, q, r) + (d''_1 - c_1)p + (d''_2 - c_2)q + (d''_3 - c_3)r + e_8, \\
 & F_{21}(p, q, r) = A(p, q, r) + (d_1 - c_4)p + (d_2 - c_5)q + (d_3 - c_6)r + e_3, \\
 & F_{22}(p, q, r) = A(p, q, r) + (d'_1 - c_4)p + (d'_2 - c_5)q + (d'_3 - c_6)r + e_6, \\
 & F_{23}(p, q, r) = A(p, q, r) + (d''_1 - c_4)p + (d''_2 - c_5)q + (d''_3 - c_6)r + e_9, \\
 & G_2(p, q, r) = g(p, q, r) \text{ which is arbitrary,} \\
 & G_1(p, q, r) = -g(p, q, r) - L_{p,q,r}(1, 1) + (c_4 - c_1)p + (c_5 - c_2)q + \\
 & \quad + (c_6 - c_3)r - c_4 - c_5 - c_6 - e_1 + e_2 + e_3 - e_4 + e_5 + \\
 & \quad + e_6 - e_7 + e_8 + e_9, \\
 & H_1(p, q, r) = A(p, q, r) + d_1p + d_2q + d_3r + e_1, \\
 & H_2(p, q, r) = A(p, q, r) + d'_1p + d'_2q + d'_3r + e_4, \\
 & H_3(p, q, r) = A(p, q, r) + d''_1p + d''_2q + d''_3r + e_7,
 \end{aligned}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, d_1, d_2, d_3, d'_1, d'_2, d'_3, d''_1, d''_2, d''_3, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ are arbitrary constants,

$$\begin{aligned}
 (2.2) \quad & A(p, q, r) = p[a \log p + b \log q + c \log r] + dq \log q + er \log r, \\
 & ((p, q, r) \in K)
 \end{aligned}$$

$$(2.3) \quad L_{x,y,z}(p, q, r) = p[a I(x, x) + b I(x, y) + c I(x, z)] + dq I(y, y) + er I(z, z),$$

$$((p, q, r), (x, y, z) \in K)$$

with

$$(2.4) \quad I(x, y) = -x \log y - (1 - x) \log (1 - y), \quad x, y \in [0, 1].$$

The theorem will be established with the help of four lemmas which will be proved one by one in what follows:

Lemma 1. Let $F : K \rightarrow R$ be measurable in each variable and satisfy

$$(2.5) \quad F(p, q, r) - F(px, qy, rz) - F((1 - x)p, (1 - y)q, (1 - z)r) =$$

$$= A_{x,y,z}(p, q, r),$$

where $A_{x,y,z} : K \rightarrow R$, for fixed $(x, y, z) \in K$, satisfies the equation

$$(2.6) \quad A_{x,y,z}(p_1 + p_2, q_1 + q_2, r_1 + r_2) = A_{x,y,z}(p_1, q_1, r_1) +$$

$$+ A_{x,y,z}(p_2, q_2, r_2).$$

Then the solutions of (2.5) are given by

$$(2.7) \quad F(p, q, r) = A(p, q, r) + d_1 p + d_2 q + d_3 r,$$

$$A_{x,y,z}(p, q, r) = L_{x,y,z}(p, q, r),$$

where the functions A and L are given by (2.2) and (2.3) respectively and d_1, d_2, d_3 are arbitrary constants.

Proof of Lemma 1. In order to prove Lemma 1, first we derive the measurable solution of (2.6).

For $p_1 = 0 = p_2$, (2.6) reduces to

$$A_{x,y,z}(0, q_1 + q_2, r_1 + r_2) = A_{x,y,z}(0, q_1, r_1) + A_{x,y,z}(0, q_2, r_2)$$

which is the same as discussed in [3]. Hence

$$(2.8) \quad A_{x,y,z}(0, q, r) = B_2(x, y, z) q + B_3(x, y, z) r$$

where B_2 and B_3 are arbitrary functions of x, y, z .

For $p_1 = 0$, (2.6) gives

$$A_{x,y,z}(p_2, q_1 + q_2, r_1 + r_2) = A_{x,y,z}(0, q_1, r_1) + A_{x,y,z}(p_2, q_2, r_2) =$$

$$= B_2(x, y, z) q_1 + B_3(x, y, z) r_1 + A_{x,y,z}(p_2, q_2, r_2)$$

$$(\text{by using (2.8)})$$

$$= B_2(x, y, z) q_2 + B_3(x, y, z) r_2 + A_{x,y,z}(p_2, q_1, r_1)$$

$$(\text{by interchanging } q_1 \text{ and } q_2 \text{ and } r_1 \text{ and } r_2 \text{ respectively}).$$

This implies that

$$\begin{aligned}
 & A_{x,y,z}(p_2, q_2, r_2) - B_2(x, y, z) q_2 - B_3(x, y, z) r_2 = \\
 & = A_{x,y,z}(p_2, q_1, r_1) - B_2(x, y, z) q_1 - B_3(x, y, z) r_1 = \\
 & = \text{Independent of } q_i \text{ and } r_i \ (i = 1, 2) = \\
 & = \alpha_{x,y,z}(p_2), \text{ say.}
 \end{aligned}$$

Then

$$(2.9) \quad A_{x,y,z}(p_2, q_2, r_2) = \alpha_{x,y,z}(p_2) + B_2(x, y, z) q_2 + B_3(x, y, z) r_2.$$

Hence (2.6) and (2.9) yield

$$(2.10) \quad \alpha_{x,y,z}(p_1 + p_2) = \alpha_{x,y,z}(p_1) + \alpha_{x,y,z}(p_2), \quad p_1, p_2, p_1 + p_2 \in [0, 1].$$

Clearly, the measurable solution of (2.10) is given by

$$(2.11) \quad \alpha_{x,y,z}(p) = B_1(x, y, z) p, \quad p \in [0, 1],$$

where B_1 is an arbitrary function of x, y, z .

Thus (2.9) and (2.11) give

$$\begin{aligned}
 (2.12) \quad & A_{x,y,z}(p, q, r) = B_1(x, y, z) p + B_2(x, y, z) q + B_3(x, y, z) r, \\
 & (p, q, r), (x, y, z) \in K.
 \end{aligned}$$

Hence (2.5), on using (2.12), becomes

$$\begin{aligned}
 (2.13) \quad & F(p, q, r) - F(px, qy, rz) - F(p(1-x), q(1-y), r(1-z)) = \\
 & = B_1(x, y, z) p + B_2(x, y, z) q + B_3(x, y, z) r.
 \end{aligned}$$

For fixed r and z , (2.13) takes the following form

$$(2.14) \quad g_1(p, q) - g_2(px, qy) - g_3(p(1-x), q(1-y)) = A'_{x,y}(p, q) + B(x, y),$$

where $A'_{x,y}(p, q) = B_1(x, y, z) p + B_2(x, y, z) q$ is additive in both p and q . The measurable solution of (2.14) are given by [4].

$$\begin{aligned}
 (2.15) \quad & g_1(p, q) = ap \log p + bp \log q + cq \log q + dp + eq + d_1, \\
 & g_2(p, q) = ap \log p + bp \log q + cq \log q + (d - c_1) p + \\
 & \quad + (e - c_2) q + d_2, \\
 & g_3(p, q) = ap \log p + bp \log q + cq \log q + (d - c_3) p + \\
 & \quad + (e - c_4) q + d_3,
 \end{aligned}$$

$$A'_{x,y}(p, q) = p[a I(x, x) + b I(x, y)] + q I(y, y) + (c_1 - c_3) px + \\ + (c_2 - c_4) qy + c_3 p + c_4 q ,$$

$$B(x, y) = d_1 - d_2 - d_3 ,$$

where $a, b, c, d, e, d_1, d_2, d_3, c_1, c_2, c_3, c_4$ are arbitrary constants (here functions of r and z). Thus

$$(2.16) \quad F(p, q, r) = a(r, z) p \log p + b(r, z) p \log q + c(r, z) q \log q + \\ + d(r, z) p + e(r, z) q + d_1(r, z) , \quad \text{etc;} ,$$

that is,

$$(2.17) \quad F(p, q, r) = a(r) p \log p + b(r) p \log q + c(r) q \log q + \\ + d(r) p + e(r) q + d_1(r) .$$

Now (2.17) in (2.13) yields

$$(2.18) \quad \begin{aligned} a(r) &= \text{constant} = a , \\ b(r) &= \text{constant} = b , \\ c(r) &= \text{constant} = d . \end{aligned}$$

Hence (2.17) and (2.18) give

$$(2.19) \quad F(p, q, r) = ap \log p + bp \log q + dq \log q + d(r) p + e(r) q + d_1(r) .$$

Substituting (2.19) in (2.13) and equating similar terms on both sides, we get

$$(2.20) \quad B_1(x, y, z) = a I(x, x) + b I(x, y) + d(r) - x d(rz) - (1 - x) d(r(1 - z)) ,$$

$$(2.21) \quad B_2(x, y, z) = c I(y, y) + e(r) - y e(rz) - (1 - y) e(r(1 - z))$$

and

$$(2.22) \quad B_3(x, y, z) r = d_1(r) - d_1(rz) - d_1(r(1 - z)) .$$

Equation (2.21) for $y = 0, z = 1$ yields

$$(2.23) \quad e(r) = \text{constant} = d_2 , \quad \text{say} .$$

The measurable solution of equation (2.22) is given by ([2], [5])

$$(2.24) \quad d_1(r) = er \log r + d_3 r ,$$

where e and d_3 are arbitrary constants.

The equation (2.20) can be rewritten as follows:

$$\begin{aligned}
 d(r) - d(r(1 - z)) &= B_1(x, y, z) - a I(x, x) - b I(x, y) + \\
 &\quad + x[d(rz) - d(r(1 - z))] \\
 &= \text{independent of } r \text{ (by letting } x = 0) \\
 &= h(z), \text{ say.}
 \end{aligned}$$

This is a Pexider equation with measurable solution given by

$$(2.25) \quad d(r) = c \log r + d_1,$$

where c and d_1 are arbitrary constants.

Thus (2.19), (2.23), (2.24) and (2.25) yield

$$\begin{aligned}
 (2.26) \quad F(p, q, r) &= ap \log p + bp \log q + dq \log q + (c \log r + d_1) p + \\
 &\quad + d_2 q + er \log r + d_3 r
 \end{aligned}$$

which is the same as (2.7), proving Lemma 1.

Remark 1. The Lemma 1 leads to the measurable solution of the functional equation

$$(2.27) \quad \sum_{i=1}^2 \sum_{j=1}^3 F(x_i p_j, y_i q_j, z_i r_j) = \sum_{i=1}^2 F(x_i, y_i, z_i) + \sum_{j=1}^3 F(p_j, q_j, r_j)$$

with $\sum x_i = 1 = \sum y_i = \sum z_i = \sum p_j = \sum q_j = \sum r_j$, as

$$(2.28) \quad F(p, q, r) = \Delta(p, q, r) + d_1 p + d_2 q + d_3 r + d_1 + d_2 + d_3,$$

connected with the generalized directed divergence (1.2), mentioned in Section 1 [2].

Lemma 2. Let $F : K \rightarrow R$ be measurable in each variable and satisfy

$$\begin{aligned}
 (2.29) \quad F(p, q, r) - F(px, qy, rz) - F(p(1 - x), q(1 - y), r(1 - z)) &= \\
 &= A_{x,y,z}(p, q, r) + B(x, y, z),
 \end{aligned}$$

where $A_{x,y,z}$ satisfies (2.6) and $B(\cdot, \cdot, \cdot)$ is a function of x, y, z . Then the solutions of (2.29) are given by

$$\begin{aligned}
 (2.30) \quad F(p, q, r) &= \Delta(p, q, r) + d_1 p + d_2 q + d_3 r - l, \\
 B(x, y, z) &= l, \\
 A_{x,y,z}(p, q, r) &= L_{x,y,z}(p, q, r).
 \end{aligned}$$

34 Proof of Lemma 2. For $p = q = r = 0$, (2.29) yields

$$(2.31) \quad B(x, y, z) = -F(0, 0, 0) = l, \quad \text{say}.$$

Then $G(p, q, r) = F(p, q, r) + l$ satisfies the functional equation (2.5). Hence (2.7) and (2.31) give (2.30), proving the Lemma 2.

Lemma 3. Let the functions $F_i : K \rightarrow R$ ($i = 1, 2, 3$) be measurable, in each variable and satisfy

$$(2.32) \quad \begin{aligned} F_1(p, q, r) - F_2(px, qy, rz) - F_3(p(1-x), q(1-y), r(1-z)) = \\ = A_{x,y,z}(p, q, r) + B(x, y, z), \end{aligned}$$

where $A_{x,y,z}$ satisfies (2.6). Then the solutions of (2.32) are given by

$$(2.33) \quad \begin{aligned} F_1(p, q, r) &= \Delta(p, q, r) + d_1p + d_2q + d_3r + a_1, \\ F_2(p, q, r) &= \Delta(p, q, r) + (d_1 - c_1)p + (d_2 - c_2)q + (d_3 - c_3)r + a_2, \\ F_3(p, q, r) &= \Delta(p, q, r) + (d_1 - c_4)p + (d_2 - c_5)q + (d_3 - c_6)r + a_3, \\ A_{x,y,z}(p, q, r) &= L_{x,y,z}(p, q, r) + (c_1 - c_4)px + (c_2 - c_5)qy + (c_3 - c_6)rz + \\ &\quad + c_4p + c_5q + c_6r, \\ B(x, y, z) &= a_1 - a_2 - a_3. \end{aligned}$$

Proof of Lemma 3. The equation (2.32) with $p = q = r = 0$ yields

$$(2.34) \quad B(x, y, z) = a_1 - a_2 - a_3,$$

where $a_i = F_i(0, 0, 0)$ ($i = 1, 2, 3$).

Substituting $x = y = z = 1$ and $x = y = z = 0$ respectively in (2.32), and using (2.34), we get

$$(2.35) \quad F_2(p, q, r) = F_1(p, q, r) - A_{1,1,1}(p, q, r) + a_2 - a_1,$$

and

$$(2.36) \quad F_3(p, q, r) = F_1(p, q, r) - A_{0,0,0}(p, q, r) + a_3 - a_1.$$

Hence (2.32) with the help of (2.34), (2.35) and (2.36) gives

$$(2.37) \quad \begin{aligned} F_1(p, q, r) - F_1(px, qy, rz) - F_1(p(1-x), q(1-y), r(1-z)) = \\ = A_{x,y,z}^1(p, q, r) - a_1, \end{aligned}$$

where

$$(2.38) \quad \begin{aligned} A_{x,y,z}^1(p, q, r) &= A_{x,y,z}(p, q, r) - A_{1,1,1}(px, qy, rz) - \\ &\quad - A_{0,0,0}(p(1-x), q(1-y), r(1-z)). \end{aligned}$$

It is easy to see that $A_{x,y,z}^1(p, q, r)$ also satisfies (2.6). Hence, applying the Lemma 2 to (2.37), we have

$$(2.39) \quad F_1(p, q, r) = A(p, q, r) + d_1p + d_2q + d_3r + a_1,$$

$$(2.40) \quad A_{x,y,z}^1(p, q, r) = L_{x,y,z}(p, q, r).$$

Also from (2.12) it follows that

$$(2.41) \quad A_{1,1,1}(p, q, r) = c_1p + c_2q + c_3r$$

and

$$(2.42) \quad A_{0,0,0}(p, q, r) = c_4p + c_5q + c_6r,$$

where c_1, c_2, c_3, c_4, c_5 and c_6 are arbitrary constants.

Hence (2.35), (2.39), (2.41); (2.36), (2.39), (2.42); (2.38), (2.40), (2.41), (2.42) respectively yield

$$(2.43) \quad F_2(p, q, r) = A(p, q, r) + (d_1 - c_1)p + (d_2 - c_2)q + (d_3 - c_3)r + a_2,$$

$$(2.44) \quad F_3(p, q, r) = A(p, q, r) + (d_1 - c_4)p + (d_2 - c_5)q + (d_3 - c_6)r + a_3$$

and

$$(2.45) \quad \begin{aligned} A_{x,y,z}(p, q, r) &= L_{x,y,z}(p, q, r) + (c_1 - c_4)px + (c_2 - c_5)qy + \\ &\quad + (c_3 - c_6)rz + c_4p + c_5q + c_6r. \end{aligned}$$

Thus (2.39), (2.43), (2.44), (2.45) and (2.34) prove Lemma 3.

Lemma 4. Let $F_i : K \rightarrow R$ ($i = 1, 2, \dots, 6$) be measurable in each variable and satisfy

$$(2.46) \quad \begin{aligned} &[F_1(p_1, q_1, r_1) - F_2(p_1x, q_1y, r_1z) - F_3(p_1(1-x), q_1(1-y), r_1(1-z))] + \\ &+ [F_4(p_2, q, r_2) - F_5(p_2x, q_2y, r_2z) - F_6(p_2(1-x), q_2(1-y), r_2(1-z))] = \\ &= A_{x,y,z}(p_1 + p_2, q_1 + q_2, r_1 + r_2). \end{aligned}$$

Then the solutions of (2.46) are given by

$$(2.47) \quad F_1(p, q, r) = A(p, q, r) + d_1p + d_2q + d_3r + e_1,$$

$$F_2(p, q, r) = A(p, q, r) + (d_1 - c_1)p + (d_2 - c_2)q + (d_3 - c_3)r + e_2,$$

$$\begin{aligned}
F_3(p, q, r) &= \Delta(p, q, r) + (d_1 - c_4)p + (d_2 - c_5)q + (d_3 - c_6)r + e_3, \\
F_4(p, q, r) &= \Delta(p, q, r) + d'_1p + d'_2q + d'_3r + e_4, \\
F_5(p, q, r) &= \Delta(p, q, r) + (d'_1 - c_1)p + (d'_2 - c_2)q + (d'_3 - c_3)r + e_5, \\
F_6(p, q, r) &= \Delta(p, q, r) + (d'_1 - c_4)p + (d'_2 - c_5)q + (d'_3 - c_6)r + e_6, \\
A_{x,y,z}(p, q, r) &= L_{x,y,z}(p, q, r) + (c_1 - c_4)px + (c_2 - c_5)qy + (c_3 - c_6)rz + \\
&\quad + c_4p + c_5q + c_6r + e_1 - e_2 - e_3 + e_4 - e_5 - e_6,
\end{aligned}$$

where c_i ($i = 1, \dots, 6$), $d_1, d_2, d_3, d'_1, d'_2, d'_3$, are constants.

Proof of Lemma 4. The equation (2.46) for $p_2 = q_2 = r_2 = 0$ and $p_1 = q_1 = r_1 = 0$ respectively gives

$$\begin{aligned}
(2.48) \quad F_1(p_1, q_1, r_1) - F_2(p_1x, q_1y, r_1z) - F_3(p_1(1-x), q_1(1-y), r_1(1-z)) &= \\
&= A_{x,y,z}(p_1, q_1, r_1) + e_5 + e_6 - e_4,
\end{aligned}$$

and

$$\begin{aligned}
(2.49) \quad F_4(p_2, q_2, r_2) - F_5(p_2x, q_2y, r_2z) - F_6(p_2(1-x), q_2(1-y), r_2(1-z)) &= \\
&= A_{x,y,z}(p_2, q_2, r_2) + e_2 + e_3 - e_1,
\end{aligned}$$

where $e_i = F_i(0, 0, 0)$ ($i = 1, 2, \dots, 6$).

Adding (2.48) and (2.49) and using (2.46), we get

$$\begin{aligned}
(2.50) \quad A_{x,y,z}(p_1 + p_2, q_1 + q_2, r_1 + r_2) &= A_{x,y,z}(p_1, q_1, r_1) + A_{x,y,z}(p_2, q_2, r_2) + \\
&\quad + e_2 + e_3 + e_5 + e_6 - e_1 - e_4.
\end{aligned}$$

Let

$$(2.51) \quad A_{x,y,z}^2(p, q, r) = A_{x,y,z}(p, q, r) - e_1 + e_2 + e_3 - e_4 + e_5 + e_6.$$

Then (2.50) becomes

$$(2.52) \quad A_{x,y,z}^2(p_1 + p_2, q_1 + q_2, r_1 + r_2) = A_{x,y,z}^2(p_1, q_1, r_1) + A_{x,y,z}^2(p_2, q_2, r_2).$$

Hence $A_{x,y,z}^2(p, q, r)$ satisfies (2.6).

The equations (2.48) and (2.49) due to (2.51) take the following forms

$$\begin{aligned}
(2.53) \quad F_1(p_1, q_1, r_1) - F_2(p_1x, q_1y, r_1z) - F_3(p_1(1-x), q_1(1-y), r_1(1-z)) &= \\
&= A_{x,y,z}^2(p_1, q_1, r_1) + e_1 - e_2 - e_3
\end{aligned}$$

and

$$\begin{aligned}
(2.54) \quad F_4(p_2, q_2, r_2) - F_5(p_2x, q_2y, r_2z) - F_6(p_2(1-x), q_2(1-y), r_2(1-z)) &= \\
&= A_{x,y,z}^2(p_2, q_2, r_2) + e_4 - e_5 - e_6.
\end{aligned}$$

Hence applying Lemma 3 to (2.53) and (2.54) respectively, we have

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$$\begin{aligned}
 (2.55) \quad F_1(p, q, r) &= \Delta(p, q, r) + d_1 p + d_2 q + d_3 r + e_1, \\
 F_2(p, q, r) &= \Delta(p, q, r) + (d_1 - c_1) p + (d_2 - c_2) q + (d_3 - c_3) r + e_2, \\
 F_3(p, q, r) &= \Delta(p, q, r) + (d_1 - c_4) p + (d_2 - c_5) q + (d_3 - c_6) r + e_3, \\
 A_{x,y,z}(p, q, r) &= L_{x,y,z}(p, q, r) + (c_1 - c_4) px + (c_2 - c_5) qy + (c_3 - c_6) rz + \\
 &\quad + c_4 p + c_5 q + c_6 r, \\
 (2.56) \quad F_4(p, q, r) &= \Delta'(p, q, r) + d'_1 p + d'_2 q + d'_3 r + e_4, \\
 F_5(p, q, r) &= \Delta'(p, q, r) + (d'_1 - c'_1) p + (d'_2 - c'_2) q + (d'_3 - c'_3) r + e_5, \\
 F_6(p, q, r) &= \Delta'(p, q, r) + (d'_1 - c'_4) p + (d'_2 - c'_5) q + (d'_3 - c'_6) r + e_6, \\
 A^2_{x,y,z}(p, q, r) &= L'_{x,y,z}(p, q, r) + (c'_1 - c'_4) px + (c'_2 - c'_5) qy + (c'_3 - c'_6) rz + \\
 &\quad + c'_4 p + c'_5 q + c'_6 r,
 \end{aligned}$$

where Δ' and L' are given by (2.2) and (2.3) with a, b, c, d, e changed to a', b', c', d', e' respectively.

Comparing the expressions for $A^2_{x,y,z}(p, q, r)$ in (2.55) and (2.56), we get

$$\begin{aligned}
 a &= a', \quad b = b', \quad c = c', \quad d = d', \quad e = e', \quad c_4 = c'_4, \quad c_5 = c'_5, \quad c_6 = c'_6, \\
 c_1 &= c'_1, \quad c_2 = c'_2, \quad c_3 = c'_3.
 \end{aligned}$$

This proves Lemma 4.

Now we are in a position to prove the main theorem.

Proof of the theorem. Substituting $x_1 = x, x_2 = 1 - x, y_1 = y, y_2 = 1 - y, z_1 = z, z_2 = 1 - z$ in (1.1), we have

$$\begin{aligned}
 (2.57) \quad & [H_1(p_1, q_1, r_1) - F_{11}(p_1 x, q_1 y, r_1 z) - F_{21}(p_1(1 - x), q_1(1 - y), r_1(1 - z))] + \\
 & + [H_2(p_2, q_2, r_2) - F_{12}(p_2 x, q_2 y, r_2 z) - F_{22}(p_2(1 - x), q_2(1 - y), r_2(1 - z))] = \\
 & = F_{13}(p_3 x, q_3 y, r_3 z) + F_{23}(p_3(1 - x), q_3(1 - y), r_3(1 - z)) - H_3(p_3, q_3, r_3) - \\
 & \quad - G_1(x, y, z) - G_2(1 - x, 1 - y, 1 - z) = \\
 & \quad = A^3_{x,y,z}(1 - p_3, 1 - q_3, 1 - r_3), \quad \text{say} \\
 & \quad = A^3_{x,y,z}(p_1 + p_2, q_1 + q_2, r_1 + r_2).
 \end{aligned}$$

38 Hence applying Lemma 4 to (2.57), we get

$$\begin{aligned}
 (2.58) \quad H_1(p, q, r) &= A(p, q, r) + d_1 p + d_2 q + d_3 r + e_1, \\
 F_{11}(p, q, r) &= A(p, q, r) + (d_1 - c_1) p + (d_2 - c_2) q + (d_3 - c_3) r + e_2, \\
 F_{21}(p, q, r) &= A(p, q, r) + (d_1 - c_4) p + (d_2 - c_5) q + (d_3 - c_6) r + e_3, \\
 H_2(p, q, r) &= A(p, q, r) + d'_1 p + d'_2 q + d'_3 r + e_4, \\
 F_{12}(p, q, r) &= A(p, q, r) + (d'_1 - c_1) p + (d'_2 - c_2) q + (d'_3 - c_3) r + e_5, \\
 F_{22}(p, q, r) &= A(p, q, r) + (d'_1 - c_4) p + (d'_2 - c_5) q + (d'_3 - c_6) r + e_6, \\
 A^3_{x,y,z}(p, q, r) &= L_{x,y,z}(p, q, r) + (c_1 - c_4) p x + (c_2 - c_5) q y + (c_3 - c_6) r z + \\
 &\quad + c_4 p + c_5 q + c_6 r + e_1 - e_2 - e_3 + e_4 - e_5 - e_6.
 \end{aligned}$$

From (2.57) and (2.58), we get

$$\begin{aligned}
 (2.59) \quad & H_3(p_3, q_3, r_3) - F_{13}(p_3 x, q_3 y, r_3 z) - F_{23}(p_3(1-x), q_3(1-y), r_3(1-z)) = \\
 & = -A^3_{x,y,z}(1-p_3, 1-q_3, 1-r_3) - G_1(x, y, z) - G_2(1-x, 1-y, 1-z) = \\
 & = [p_3\{a I(x, x) + b I(x, y) + c I(x, z)\} + q_3 d I(y, y) + r_3 e I(z, z) - \\
 & \quad - (c_4 - c_1) p_3 x - (c_5 - c_2) q_3 y - (c_6 - c_3) r_3 z + c_4 p_3 + c_5 q_3 + c_6 r_3] + \\
 & \quad + [-\{a I(x, x) + b I(x, y) + c I(x, z)\} - d I(y, y) - e I(z, z) + \\
 & \quad + (c_4 - c_1) x + (c_5 - c_2) y + (c_6 - c_3) z - c_4 - c_5 - c_6 - e_1 + e_2 + e_3 - \\
 & \quad - e_4 + e_5 + e_6 - G_1(x, y, z) - G_2(1-x, 1-y, 1-z)] = \\
 & = A_{x,y,z}(p_3, q_3, r_3) + B^*(x, y, z), \text{ say}
 \end{aligned}$$

where A is the same as given in (2.32) and $B^*(x, y, z)$ is the remaining expression on the right hand side of (2.59).

Applying Lemma 3 to (2.59), we get

$$\begin{aligned}
 (2.60) \quad & H_3(p, q, r) = A(p, q, r) + d''_1 p + d''_2 q + d''_3 r + e_7, \\
 F_{13}(p, q, r) &= A(p, q, r) + (d''_1 - c_1) p + (d''_2 - c_2) q + (d''_3 - c_3) r + e_8, \\
 F_{23}(p, q, r) &= A(p, q, r) + (d''_1 - c_4) p + (d''_2 - c_5) q + (d''_3 - c_6) r + e_9,
 \end{aligned}$$

$$B^*(x, y, z) = e_7 - e_8 - e_9,$$

where $e_7 = H_3(0, 0, 0)$, $e_8 = F_{13}(0, 0, 0)$, $e_9 = F_{23}(0, 0, 0)$.

The last equation of (2.60) yields

$$(2.61) \quad G_1(x, y, z) = -G_2(1-x, 1-y, 1-z) - L_{x,y,z}(1, 1, 1) + (c_4 - c_1)x + \\ + (c_5 - c_2)y + (c_6 - c_3)z - c_4 - c_5 - c_6 - e_1 + e_2 + e_3 - e_4 + \\ + e_5 + e_6 - e_7 + e_8 + e_9.$$

The equation (2.61) expresses $G_1(x, y, z)$ in terms of $G_2(1-x, 1-y, 1-z)$ which is arbitrary.

Thus (2.58), (2.60) and (2.61) prove the theorem. It is easy to verify that the eleven functions thus obtained do satisfy (1.1).

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REFERENCES

- [1] Z. Daróczy: On the measurable solution of a functional equation. *Acta Math. Acad. Sci. Hung.* 22 (1971), 11–24.
- [2] Pl. Kannappan: On a functional equation connected with generalized directed divergence. *Aeq. Math.* 11 (1974), 51–56.
- [3] Pl. Kannappan, P. N. Rathie: On the measurable solution of a functional equation in two variables (1978).
- [4] Pl. Kannappan, V. Sathyabhama: Generalization of sum representation functional equation – I. (1978).
- [5] C. T. Ng: On the measurable solutions of the functional equation $\sum_{i=1}^2 \sum_{j=1}^3 F_{i,j}(p, q_j) = \sum_{i=1}^2 G_i(p_i) + \sum_{j=1}^3 H_j(q_j)$. *Acta Math. Acad. Sci. Hung.* 25 (1974), 249–254.
- [6] H. Theil: *Economics and Information Theory*. North-Holland, Amsterdam 1967.

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