Kybernetika

Exercises in Stochastic Analysis

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ACADEMIA
PRAHA

0. INTRODUCTION

The present collection of exercises is intended to assist those who wish to deepen the knowledge of stochastic analysis by means of solving problems. It is assumed that the reader is acquainted with the basic facts about stochastic integrals and differentials, and has absorbed some information about stochastic differential equations. For the sake of precision the definitions of certain notions are restated in the introduction to each section or before the exercises. This serves also to determine the denotation. Exercises are often supplied with hints. Whenever the solution is an analytical expression or a formula, it is listed in the last section, which is followed by references to the sources in the literature.

The collection can be used as a manual to university courses on stochastic calculus. In fact, the authors were encouraged to publish this unlarge textbook by their awareness of the wide difference between the highly developed didactics of the classical analysis and the neglected pedagogical aspects of its stochastic counterpart.

Thanks are due to our colleagues from the Seminar on the Theory of Random Processes for their helpfull comments and criticism.

The material explained in the preceding supplement to Kybernetika *Elements of Stochastic Analysis* by P. Mandl is sufficient for solving the majority of the exercises, and the same system of denotation is employed there. But this collection is self-contained, and can be coupled with other texts on stochastic analysis as well.

The reader is asked to consider the subsequent survey of the denotations. A probability space $(\Omega, \mathscr{A}, P)$ supplied with σ-algebras $\mathscr{F}_t$, $t \geqq 0$, which perform the time-structuring of random events, is allways assumed to underlay the exercises, and to have the demanded properties, e.g. to carry a Wiener process with respect to $\mathscr{F}$.

General denotations:

$(\Omega, \mathscr{A}, P)$	basic probability structure
Ω	set of elementary events $(\omega \in \Omega)$
$\mathscr{A}$	σ-algebra of random events
P	probability measure on $\mathscr{A}$
$\mathscr{F} = \{\mathscr{F}_t, t \geqq 0\}$	increasing family of σ-algebras $\mathscr{F}_t \subset \mathscr{A}$
$\xi(\omega), \xi, \eta, \ldots$	random variables
$\xi = (\xi^1, \ldots, \xi^m)'$	m-dimensional random variable
$E\xi$	mathematical expectation of ξ
$E\{\xi \mid \mathscr{C}\}$	conditional expectation of ξ given σ-algebra $\mathscr{C} \subset \mathscr{A}$

a. s.	almost surely
l. i. m.	limit in mean square
p lim	limit in probability
$X = \{X_t, t \geqq 0\}$, $Y = \{Y_t, t \in [0, T]\}, \ldots$	random processes (random functions)
$\sigma a(\,)$	σ-algebra generated by random events or random processes in the brackets
$\mathscr{F}_t^X$	$= \sigma a(X_s, s \in [0, t])$
$W = \{W_t, t \geqq 0\}$	Wiener process with respect to $\mathscr{F}$ ($W_0 = 0$ a. s., $E(W_t - W_s)^2 = t - s$, $t \geqq s \geqq 0$, unless stated otherwise)
$\overline{X} = \{\overline{X}_t = ({}^1X_t, \ldots, {}^mX_t)', t \in [0, T]\}$	m-dimensional random process
$\overline{W} = \{\overline{W}_t = ({}^1W_t, \ldots, {}^rW_t)', t \geqq 0\}$	r-dimensional Wiener process (r mutually independent Wiener processes)
χ_A	indicator function of A
R^m	m-dimensional Euclidean space
$\lvert\bar{x}\rvert$	norm of $\bar{x} \in R^m$
const.	finite positive constants
$'$	transposition of vectors and matrices
*	exercise is solved in Elements of Stochastic Analysis
□	solution can be found in Section 6
Ex.	exercise

1. STOCHASTIC INTEGRAL

Random function $\Phi = \{\Phi_t, t \in [0, T]\}$ is called *nonanticipative*, if, for $t \in [0, T]$, $\{\Phi_s, s \in [0, t]\}$ is a measurable random process on $(\Omega, \mathscr{F}_t, P)$.

Random function $\Phi = \{\Phi_t, t \in [0, T]\}$, $T < \infty$, is a *simple function*, if it is nonanticipative, and if there exists a division $t_0 = 0 < t_1 < t_2 < \ldots < t_n = T$ of interval $[0, T]$ together with random variables $\varphi_0, \ldots, \varphi_{n-1}$ such that

$$\Phi_t = \varphi_j\,, \quad t_j \leqq t < t_{j+1}\,, \quad j = 0, \ldots, n-1\,.$$

For $t \in [0, T]$ its *stochastic integral* is defined as

$$\int_0^t \Phi_s \, dW_s = \sum_{j=0}^{k-1} \varphi_j(W_{t_{j+1}} - W_{t_j}) + \varphi_k(W_t - W_{t_k}) \quad \text{if} \quad t_k \leqq t \leqq t_{k+1}\,.$$

Let $\Phi = \{\Phi_t, t \in [0, T]\}$, $T < \infty$, be nonanticipative, $\int_0^T \Phi_t^2 \, dt < \infty$ a. s. Let $\{{}^n\Phi, n = 1, 2, \ldots\}$ be a sequence of simple functions such that

$$\operatorname*{p\,lim}_{n \to \infty} \int_0^T ({}^n\Phi_t - \Phi_t)^2 \, dt = 0\,.$$

Then

$$\operatorname*{p\,lim}_{n \to \infty} \sup_{t \in [0,T]} \left| \int_0^t {}^n\Phi_s \, dW_s - \int_0^t \Phi_s \, dW_s \right| = 0\,.$$

1. Prove that

$$\frac{1}{\sqrt{(2\pi)}} W_{2\pi}\,, \quad \frac{1}{\sqrt{\pi}} \int_0^{2\pi} \sin nt \, dW_t\,, \quad \frac{1}{\sqrt{\pi}} \int_0^{2\pi} \cos nt \, dW_t\,, \quad n = 1, 2, 3, \ldots,$$

are mutually independent random variables having normal distribution $N(0, 1)$.

2. Find

$$E\left(\int_0^T W_t^n \, dW_t\right)^2, \quad n = 0, 1, \ldots, \quad E\left(\int_0^T e^{W_t} \, dW_t\right)^2, \quad T < \infty\,. \qquad \square$$

3. Let

$$\Phi = \{\Phi_t, t \in [0, T]\}\,, \quad T < \infty\,, \quad E\Phi_t^2 < \infty\,, \quad t \in [0, T]\,,$$

be nonanticipative and continuous in quadratic mean. Consider divisions $0 =$

$= t_0 < t_1 < \ldots < t_k = T$ of $[0, T]$, and denote $\Delta = \max_{j=0,\ldots,k-1} (t_{j+1} - t_j)$. Prove

$$\int_0^T \Phi_t \, dW_t = \operatorname*{l.i.m.}_{\Delta \to 0} \sum_{j=0}^{k-1} \Phi_{t_j}(W_{t_{j+1}} - W_{t_j}) .$$

4. Let $\Phi = \{\Phi_t, t \in [0, T]\}$ be as in Ex. 3. Let $0 \leqq t_0^m < t_1^m < \ldots t_{k_m}^m = T$, $m = 1, 2, \ldots$, be a sequence of divisions of $[0, T]$ such that

$$E(\Phi_t - \Phi_{t_j^m})^2 \leqq 2^{-m} , \quad t_j^m \leqq t \leqq t_{j+1}^m , \quad j = 0, \ldots, k_m - 1 , \quad m = 1, 2, \ldots .$$

Prove

$$\int_0^T \Phi_t \, dW_t = \lim_{m \to \infty} \sum_{j=0}^{k_m - 1} \Phi_{t_j^m}(W_{t^m_{j+1}} - W_{t_j^m}) \quad \text{a. s.}$$

5. Compute $\int_0^T W_t \, dW_t$, $T < \infty$, and prove

$$E\int_0^T W_t \, dW_t \left(W_T^2 - \int_0^T W_t \, dW_t \right) = \tfrac{1}{2}T^2 .$$

Hint. Use Ex. 3 and

$$\operatorname*{l.i.m.}_{\Delta \to 0} \sum_{j=0}^{k-1} (W_{t_{j+1}} - W_{t_j})^2 = T .$$

6. Let $0 = t_0 < t_1 < \ldots < t_k = T < \infty$ and Δ be as in Ex. 3. Prove

$$\operatorname*{l.i.m.}_{\Delta \to 0} \sum_{j=0}^{k-1} W_{t_{j+1}}(W_{t_{j+1}} - W_{t_j}) = \int_0^T W_t \, dW_t + T .$$

7. Let $0 = t_0 < t_1 < \ldots < t_k = T < \infty$ and Δ be as in Ex. 3. Prove

$$\lim_{\Delta \to 0} EW_T^2 \sum_{j=1}^{k-1} W_{t_j}(W_{t_{j+1}} - W_{t_j}) = T^2 .$$

Hint. Write

$$W_T^2 = [(W_T - W_{t_{j+1}}) + (W_{t_{j+1}} - W_{t_j}) + W_{t_j}]^2 .$$

8. Let $\Phi = \{\Phi_t, t \in [0, T]\}$, $T < \infty$, be nonanticipative and let, with probability 1, its trajectory have continuous derivative $\Phi' = \{\Phi'_t, t \in [0, T]\}$. Prove

$$(1) \qquad \int_0^T \Phi_t \, dW_t = \Phi_0 W_T + \int_0^T \Phi'_t(W_T - W_t) \, dt .$$

Hint. Use partial summation in

$$\int_0^T \Phi_t \, dW_t = \operatorname*{p\,lim}_{\Delta \to 0} \sum_{j=0}^{k-1} \Phi_{t_j}(W_{t_{j+1}} - W_{t_j}) .$$

9. Consider the right-hand side of (1) as definition of $\int_0^T \Phi_t \, \mathrm{d}W_t$, and establish the properties of the integral.

10. Construct a sequence of real functions

$$\{f_n(t),\ t \in [0, 1],\ n = 1, 2, \ldots\} \quad \text{such that} \quad \lim_{n\to\infty} \int_0^1 f_n(t)^2 \, \mathrm{d}t = 0 \,,$$

and

$$\liminf_{n\to\infty} \int_0^1 f_n(t) \, \mathrm{d}W_t = -\infty \,, \quad \limsup_{n\to\infty} \int_0^1 f_n(t) \, \mathrm{d}W_t = +\infty \,, \quad \text{a. s.}$$

Hint. Take divisions

$$\frac{m-1}{m} = t_0^m < t_1^m < \ldots < t_{k_m}^m = \frac{m}{m+1} \tag{2}$$

of intervals $[(m-1)/m,\ m/(m+1)]$ for which

$$\lim_{m\to\infty} \sum_{j=0}^{k_m} \left| W_{t^m_{j+1}} - W_{t_j^m} \right| = +\infty \quad \text{a. s.}$$

For

$$1 + 2^{k_1} + \ldots + 2^{k_{m-1}} \leqq n < 1 + 2^{k_1} + \ldots + 2^{k_m}$$

let f_n run through all functions equal 1 or -1 on the intervals of division (2), and equal 0 outside $[(m-1)/m,\ m/(m+1)]$.

11. Let $h(s, t)$ be a Borel measurable function on $[0, T] \times [0, T]$,

$$T < \infty \,, \quad \int_0^T h(s, t)^2 \, \mathrm{d}s \, \mathrm{d}t < \infty \,.$$

Prove

$$\int_0^T \left\{ \int_0^T h(s, t) \, \mathrm{d}W_s \right\} \mathrm{d}t = \int_0^T \left\{ \int_0^T h(s, t) \, \mathrm{d}t \right\} \mathrm{d}W_s \,. \tag{3}$$

Hint. The class of functions satisfying (3) contains the indicators of rectangles, and is closed with respect to linear combinations and limits in $L^2([0, T] \times [0, T])$.

2. THE ITÔ FORMULA

Random process $X = \{X_t, t \in [0, T]\}$ has *stochastic differential*

$$\mathrm{d}X_t = A_t \, \mathrm{d}t + \sum_{i=1}^{r} {}^iB_t \, \mathrm{d}\, {}^iW_t \,, \quad t \in [0, T] \,,$$

if

$$X_t = X_0 + \int_0^t A_s\,\mathrm{d}s + \sum_{l=1}^{r} \int_0^t {}^l B_s\,\mathrm{d}\,{}^l W_s\,, \quad t \in [0, T]\,,$$

where X_0 is $\mathscr{F}_0$-measurable, and $A = \{A_t, t \in [0, T]\}$, ${}^l B = \{{}^l B_t, t \in [0, T]\}$, $l = 1, \ldots, r$, are nonanticipative processes satisfying

$$\int_0^T |A_t|\,\mathrm{d}t < \infty \quad \text{a. s.}\,, \quad \int_0^T {}^l B_t^2\,\mathrm{d}t < \infty \quad \text{a. s.}\,, \quad l = 1, \ldots, r\,.$$

Let random processes ${}^i X = \{{}^i X_t, t \in [0, T]\}$, $i = 1, \ldots, m$, have stochastic differentials

$$\mathrm{d}\,{}^i X_t = {}^i A_t\,\mathrm{d}t + \sum_{l=1}^{r} {}^{il} B_t\,\mathrm{d}\,{}^l W_t\,, \quad t \in [0, T]\,, \quad i = 1, \ldots, m\,.$$

Let $f(t, x^1, \ldots, x^m)$ be a function on $[0, T] \times R^m$ with continuous derivatives

$$\dot{f} = \frac{\partial}{\partial t} f\,, \quad f_i = \frac{\partial}{\partial x^i} f\,, \quad i = 1, \ldots, m\,, \quad f_{ij} = \frac{\partial^2}{\partial x^i\,\partial x^j} f\,, \quad i, j = 1, \ldots, m\,,$$

and let $Y_t = f(t, {}^1 X_t, \ldots, {}^m X_t)$, $t \in [0, T]$. Then Y has stochastic differential

$$\mathrm{d}Y_t = \dot{f}\,\mathrm{d}t + \sum_{i=1}^{m} f_i\,\mathrm{d}\,{}^i X_t + \tfrac{1}{2} \sum_{i,j=1}^{m} f_{ij}\,\mathrm{d}\,{}^i X_t\,\mathrm{d}\,{}^j X_t\,, \quad t \in [0, T]\,,$$

where $\dot{f} = \dot{f}(t, {}^1 X_t, \ldots, {}^m X_t)$, ..., and

$$\mathrm{d}\,{}^i X_t\,\mathrm{d}\,{}^j X_t = \sum_{l=1}^{r} {}^{il} B_t\,{}^{jl} B_t\,\mathrm{d}t\,, \quad t \in [0, T]\,, \quad i, j = 1, \ldots, m\,.$$

***12.** Let $X = \{X_t, t \geqq 0\}$, $X_0 \neq 0$, $Y = \{Y_t, t \geqq 0\}$ have stochastic differentials

$$\mathrm{d}X_t = -\sin \Phi_t\,\mathrm{d}W_t\,, \quad \mathrm{d}Y_t = \cos \Phi_t\,\mathrm{d}W_t\,, \quad \Phi_t = \operatorname{arctg}(Y_t/X_t)\,, \quad t \geqq 0\,.$$

Prove

$$X_t^2 + Y_t^2 = X_0^2 + Y_0^2 + t\,, \quad t \geqq 0\,.$$

Hint. Compute $\mathrm{d}(X_t^2 + Y_t^2)$.

13. Prove that stochastic differential equation

$$\begin{aligned} \mathrm{d}\,{}^1 X_t &= -\tfrac{1}{2}\,{}^1 X_t\,\mathrm{d}t + {}^2 X_t\,\mathrm{d}W_t\,, \\ \mathrm{d}\,{}^2 X_t &= -\tfrac{1}{2}\,{}^2 X_t\,\mathrm{d}t - {}^1 X_t\,\mathrm{d}W_t\,, \quad t \geqq 0\,, \end{aligned}$$

with initial condition ${}^1 X_0 = 0$, ${}^2 X_0 = 1$ a. s. has solution

$${}^1 X_t = \sin W_t\,, \quad {}^2 X_t = \cos W_t\,, \quad t \geqq 0\,.$$

14. Prove that stochastic differential equation

$$dX_t = (-\sin^2 X_t + \sin^3 X_t \cos X_t)\,dt - \sin^2 X_t\,dW_t, \quad t \geqq 0, \tag{4}$$

with initial condition $X_0 = x_0$ a. s. has solution

$$X_t = \operatorname{Arcctg}(t + W_t + \operatorname{ctg} x_0), \quad t \geqq 0.$$

Consider the case $x_0 = 0$ and examine the behaviour of X for x_0 close to 0.

15. Let $f(x)$ be a function on $(-\infty, \infty)$, twice continuously differentiable. Set $\bar{X} = \{\bar{X}_t = (W_t, f(W_t))', t \geqq 0\}$.

Find

$$\bar{a}(\bar{x}) = (a^1(\bar{x}), a^2(\bar{x}))', \quad \bar{b}(\bar{x}) = (b^1(\bar{x}), b^2(\bar{x}))', \quad \bar{x} \in R^2,$$

such that

$$d\bar{X}_t = \bar{a}(\bar{X}_t)\,dt + \bar{b}(\bar{X}_t)\,dW_t, \quad t \geqq 0. \qquad \square$$

***16.** Let $f(x)$ be a function on $(-\infty, \infty)$ with continuous derivative $f'(x)$. Denote by F the integral of f, i.e. $F'(x) = f(x)$. Prove

$$\int_0^t f(W_s)\,dW_s = F(W_t) - F(W_0) - \frac{1}{2}\int_0^t f'(W_s)\,ds, \quad t \geqq 0.$$

17. Let f be as in Ex. 16. Introduce divisions $0 = t_0 < t_1 < \ldots < t_k = T < \infty$, and set $\Delta = \max_{j=0,\ldots,k-1} (t_{j+1} - t_j)$.

Prove

$$\operatorname{p\,lim}_{\Delta \to 0} \sum_{j=0}^{k-1} f(W_{t_{j+1}})(W_{t_{j+1}} - W_{t_j}) = F(W_T) - F(W_0) + \frac{1}{2}\int_0^T f'(W_t)\,dt.$$

***18.** Let ${}^iX = \{{}^iX_t, t \in [0, T]\}$, $i = 1, 2$, have stochastic differentials

$$d\,{}^iX_t = {}^iA_t\,dt + {}^iB_t\,dW_t, \quad t \in [0, T].$$

Prove

$$d({}^1X_t\,{}^2X_t) = {}^1X_t\,d\,{}^2X_t + {}^2X_t\,d\,{}^1X_t + {}^1B_t\,{}^2B_t\,dt, \quad t \in [0, T].$$

***19.** Let $\Phi = \{\Phi_t, t \in [0, T]\}$ be nonanticipative, $\int_0^T \Phi^2\,dt < \infty$ a. s. Prove that

$$Z_t = \exp\left\{\int_0^t \Phi_s\,dW_s - \frac{1}{2}\int_0^t \Phi_s^2\,ds\right\}, \quad t \in [0, T],$$

is the unique solution of the equation

$$dZ_t = \Phi_t Z_t\,dW_t, \quad t \in [0, T],$$

with initial condition $Z_0 = 1$ a. s.

Hint. To establish the unicity differentiate the ratio of two solutions.

20. The Hermite polynomials $H_n(t, x)$, $n = 0, 1, \ldots$, satisfy

$$\sum_{n=0}^{\infty} z^n H_n(t, x) = \exp\{zx - \tfrac{1}{2}z^2 t\}\,, \quad z \in (-\infty, \infty)\,.$$

Define the iterated integrals

$${}^0I_t = 1\,, \quad t \geqq 0\,, \quad {}^{n+1}I_t = \int_0^t {}^nI_s \, \mathrm{d}W_s\,, \quad t \geqq 0\,, \quad n = 0, 1, 2, \ldots\,.$$

Prove

$${}^nI_t = H_n(t, W_t)\,, \quad t \geqq 0\,, \quad n = 0, 1, 2, \ldots\,.$$

Hint. Use $\sum_{n=0}^{\infty} z^n \, {}^nI_t$ and Ex. 19.

***21.** Let $a(t, x)$, $b(t, x)$ be functions on $[0, T] \times (-\infty, \infty)$, b continuous together with $(\partial/\partial x)\, b$. Let function $u(t, y)$, $(t, y) \in [0, T] \times (-\infty, \infty)$, have continuous derivatives $(\partial/\partial t)\, u$, $(\partial/\partial y)\, u$, and $(\partial^2/\partial y^2)\, u$, and let

$$\frac{\partial}{\partial t} u(t, y) = a(t, u) - \frac{1}{2} b \frac{\partial}{\partial x} b(t, u)\,, \tag{5}$$

$$\frac{\partial}{\partial y} u(t, y) = b(t, u)\,, \quad (t, y) \in [0, T] \times (-\infty, \infty)\,, \quad u(0, 0) = x_0\,. \tag{6}$$

Prove that $X_t = u(t, W_t)$, $t \in [0, T]$, satisfies

$$\mathrm{d}X_t = a(t, X_t)\, \mathrm{d}t + b(t, X_t)\, \mathrm{d}W_t\,, \quad t \in [0, T]\,, \quad X_0 = x_0 \quad \text{a.s.}$$

22. Under additional differentiability assumptions about a and b find a necessary condition for the solvability of (5), (6). □

Hint. $(\partial^2/\partial t\, \partial y)\, u = (\partial^2/\partial y\, \partial t)\, u$.

23. Solve

$$\mathrm{d}X_t = \tfrac{1}{4}\, \mathrm{d}t + \sqrt{(X_t)}\, \mathrm{d}W_t\,, \quad t \geqq 0\,, \quad X_0 = x_0 \geqq 0\,. \tag{7}$$ □

Hint. Ex. 21.

24. Solve

$$\mathrm{d}X_t = (1 + X_t)(1 + X_t^2)\, \mathrm{d}t + (1 + X_t^2)\, \mathrm{d}W_t\,, \quad t \geqq 0\,, \quad X_0 = x_0\,, \tag{8}$$

and notice that X reaches ∞ or $-\infty$ in finite time. □

Hint. Ex. 21.

25. Solve

$$\mathrm{d}X_t = \tfrac{1}{4}\sin 2X_t\,\mathrm{d}t + \sin X_t\,\mathrm{d}W_t\,, \quad X_0 = x_0 \in (-\pi, \pi)\,. \tag{9}$$ □

Hint. Ex. 21.

26. Solve

$$\mathrm{d}X_t = \left(\frac{2}{1+t} X_t - a(1+t)^2\right)\mathrm{d}t + a(1+t)^2\,\mathrm{d}W_t\,, \quad t \geqq 0\,, \quad X_0 = x_0\,,$$

where a is a positive number. □

Hint. Ex. 21.

27. Generalize the statement of Ex. 21, and prove that $X_t = u(t, W_t, \int_0^t g(s)\,\mathrm{d}W_s)$, $t \in [0, T]$, satisfies

$$\mathrm{d}X_t = a(t, X_t)\,\mathrm{d}t + b(t, X_t)\,\mathrm{d}W_t\,, \quad t \in [0, T]\,,$$

if

$$\frac{\partial}{\partial t} u(t, y, z) = a(t, u) - \frac{1}{2} b \frac{\partial}{\partial x} b(t, u)\,, \tag{10}$$

$$\frac{\partial}{\partial y} u(t, y, z) = b(t, u) - \frac{g(t)}{g'(t)} h(t, u)\,, \tag{11}$$

$$\frac{\partial}{\partial z} u(t, y, z) = \frac{1}{g'(t)} h(t, u)\,, \quad (t, y, z) \in [0, T] \times R^2\,, \tag{12}$$

where

$$g' = \frac{d}{dt} g\,, \quad h(t, x) = \frac{\partial b}{\partial t} + a \frac{\partial b}{\partial x} - b \frac{\partial a}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 b}{\partial x^2}\,.$$

28. Assume a, b, g of Ex. 27 sufficiently smooth, and prove that the following conditions are necessary for solvability of (10), (11), (12). There exists a function $c(t)$ such that

1. $$c(t) = \frac{g''(t)}{g'(t)} - \frac{c'(t)}{c(t)}\,.$$

2. $$\frac{\partial b}{\partial t} = -\frac{1}{2} b^2 \frac{\partial^2 b}{\partial x^2} - a \frac{\partial b}{\partial x} + b\left(\frac{\partial a}{\partial x} + c(t)\right).$$

Hint. $(\partial^2/\partial y\,\partial z)\,u = (\partial^2/\partial z\,\partial y)\,u\,, \quad (\partial^2/\partial t\,\partial z)\,u = (\partial^2/\partial z\,\partial t)\,u$.

29. Let $X = \{X_t, t \in [0, T]\}$ have stochastic differential

$$\mathrm{d}X_t = a(t, X_t)\,\mathrm{d}t + b(t, X_t)\,\mathrm{d}W_t\,, \quad t \in [0, T]\,,$$

where function b is continuously differentiable, $b > 0$. Find function $f(t, x)$, $(t, x) \in [0, T] \times (-\infty, \infty)$, so that $Y_t = f(t, X_t)$, $t \in [0, T]$, has stochastic differential

$$dY_t = a^*(t, Y_t)\,dt + dW_t\,, \quad t \in [0, T]\,,$$

for appropriate $a^*(t, y)$, $(t, y) \in [0, T] \times (-\infty, \infty)$. □

30. Let $X = \{X_t, t \geqq 0\}$ have stochastic differential

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t\,, \quad t \geqq 0\,,$$

where a, b are sufficiently smooth functions on $(-\infty, \infty)$, $b > 0$. Find function $f(x)$, $x \in (-\infty, \infty)$, such that $Y_t = f(X_t)$, $t \geqq 0$, has stochastic differential

$$dY_t = b^*(Y_t)\,dW_t\,, \quad t \geqq 0\,,$$

for appropriate $b^*(y)$, $y \in (-\infty, \infty)$. □

31. Let $X = \{X_t, t \geqq 0\}$ be as in Ex. 30. Find a necessary and sufficient condition for the existence of a function $f(t, x)$, $(t, x) \in [0, \infty) \times (-\infty, \infty)$, such that $Y_t = f(t, X_t)$, $t \geqq 0$, has stochastic differential

$$dY_t = a^*(t)\,dt + b^*(t)\,dW_t\,, \quad t \geqq 0\,,$$

for appropriate $a^*(t)$, $b^*(t)$, $t \geqq 0$, and determine $f(t, x)$. □

32. Let $\Phi = \{\Phi_t, t \in [0, T]\}$, $T < \infty$, be nonanticipative and such that $\int_0^T E\Phi_t^{2m}\,dt < \infty$, where m is a positive integer. Prove

$$E\left(\int_0^t \Phi_s\,dW_s\right)^{2m} \leqq [m(2m - 1)]^m t^{m-1} \int_0^t E\Phi_s^{2m}\,ds\,, \quad t \in [0, T]\,.$$

Hint. Assume first Φ bounded. Integrate the differential of $(\int_0^t \Phi_s\,dW_s)^{2m}$, and apply Hölder's inequality to the mathematical expectation of the integral.

33. Let $g(s, t)$ be a Borel measurable function on $[0, T] \times [0, T]$, $T < \infty$. Let for $s \in [0, T]$, g be absolutely continuous in t, and let

$$\int_0^T g(s, t)^2\,ds < \infty\,, \quad t \in [0, T]\,, \quad \int_0^T g(t, t)^2\,dt < \infty\,,$$

$$\int_0^T \int_0^T \left(\frac{\partial}{\partial t} g(s, t)\right)^2 ds\,dt < \infty\,.$$

Prove that

$$X_t = \int_0^t g(s, t)\,dW_s\,, \quad t \in [0, T]\,, \tag{13}$$

has stochastic differential

$$(14)\qquad \mathrm{d}X_t = \left(\int_0^t \frac{\partial}{\partial t} g(s, t)\,\mathrm{d}W_s\right)\mathrm{d}t + g(t, t)\,\mathrm{d}W_t\,, \quad t \in [0, T]\,.$$

(A suitable version of the integral on the right-hand side of (13) is to be considered.)

34. Let

$$X_t = x + \int_0^t g(t - s)\,\mathrm{d}W_s\,, \quad t \geqq 0\,,$$

where $g(t) = \mathrm{e}^{ct}(a \cos t + b \sin t)$, $t \geqq 0$, and x, a, b, c are numbers, $a^2 + b^2 > 0$. Construct $Y = \{Y_t, t \geqq 0\}$ so that $(X, Y) = \{(X_t, Y_t)', t \geqq 0\}$ is Markovian. □

Hint. Use Ex. 33. Choose Y so that the first term on the right in (14) is a linear combination of X_t and Y_t.

35. Let $X = \{X_t, t \geqq -\Delta\}$, $\Delta > 0$, have stochastic differential

$$\mathrm{d}X_t = A_t\,\mathrm{d}t + B_t\,\mathrm{d}W_t\,, \quad t \geqq -\Delta\,.$$

Let $f(u, x, y)$ be a function on $[-\Delta, 0] \times R^2$, continuous together with its derivatives

$$\dot{f} = \frac{\partial}{\partial u} f\,, \quad f' = \frac{\partial}{\partial y} f\,, \quad f'' = \frac{\partial^2}{\partial y^2} f\,.$$

Prove that

$$Z_t = \int_{t-\Delta}^t f(s - t, X_s, X_t)\,\mathrm{d}s\,, \quad t \geqq 0\,,$$

has stochastic differential

$$\mathrm{d}Z_t = \left[f(0, X_t, X_t) - f(-\Delta, X_{t-\Delta}, X_t) - \int_{t-\Delta}^t \dot{f}(s - t, X_s, X_t)\,\mathrm{d}s\right]\mathrm{d}t +$$
$$+ \int_{t-\Delta}^t f'(s - t, X_s, X_t)\,\mathrm{d}s\,\mathrm{d}X_t + \frac{1}{2} B_t^2 \int_{t-\Delta}^t f''(s - t, X_s, X_t)\,\mathrm{d}s\,\mathrm{d}t\,, \quad t \geqq 0\,.$$

Let $\bar{X} = \{\bar{X}_t = ({}^1X_t, \ldots, {}^mX_t)', t \in [0, T]\}$, $T < \infty$, be nonanticipative, and let $b(t, \bar{x})$ be a Borel measurable function on $[0, T] \times R^m$. The *Stratonovich integral* of $B = \{B_t = b(t, \bar{X}_t), t \in [0, T]\}$ is defined as follows. With divisions $0 = t_0 < t_1 < \ldots < t_k = T$ of $[0, T]$ are associated integral sums

$$S_t^{t_0, t_1, \ldots} = \sum_{j=0}^{i-1} b(t_j, \tfrac{1}{2}(\bar{X}_{t_{j+1}} + \bar{X}_{t_j}))(W_{t_{j+1}} - W_{t_j}) +$$
$$+ b(t_i, \tfrac{1}{2}(\bar{X}_{t_{i+1}} + \bar{X}_{t_i}))(W_t - W_{t_i})\,, \quad t_i \leqq t \leqq t_{i+1}\,, \quad i = 0, \ldots, k - 1\,.$$

B is said to have the Stratonovich integral with respect to W on $[0, T]$, if there exists a random process $I = \{I_t, t \in [0, T]\}$ such that

$$\operatorname{p\,lim}_{\Delta \to 0} \sup_{t \in [0,T]} \left| S_t^{t_0, t_1, \dots} - I_t \right| = 0 ,$$

where $\Delta = \max_{j=0,\dots,k-1} (t_{j+1} - t_j)$.

We set

$$I_t = \int_0^t B_s \, \mathrm{d}^* W_s , \quad t \in [0, T] .$$

Let $\bar{X}$ and b be as above. Further, let $a(t, \bar{x})$ be a Borel measurable function on $[0, T] \times R^m$. $Y = \{Y_t, t \in [0, T]\}$ has *Stratonovich differential*

$$\mathrm{d}^* Y_t = a(t, \bar{X}_t) \, \mathrm{d}t + b(t, \bar{X}_t) \, \mathrm{d}^* W_t , \quad t \in [0, T] ,$$

if

$$Y_t = Y_0 + \int_0^t a(s, \bar{X}_s) \, \mathrm{d}s + \int_0^t b(s, \bar{X}_s) \, \mathrm{d}^* W_s , \quad t \in [0, T] ,$$

where Y_0 is $\mathscr{F}_0$-measurable, $\int_0^T |a(t, \bar{X}_t)| \, \mathrm{d}t < \infty$ a. s.

36. Let $f(t, x)$ be a function on $[0, T] \times (-\infty, \infty)$ having continuous derivatives $(\partial/\partial t) f$, $(\partial/\partial y) f$, $(\partial^2/\partial y^2) f$. Prove

$$\mathrm{d}^* f(t, W_t) = \frac{\partial}{\partial t} f(t, W_t) \, \mathrm{d}t + \frac{\partial}{\partial y} f(t, W_t) \, \mathrm{d}^* W_t , \quad t \in [0, T] .$$

37. Let $a(t, x)$, $b(t, x)$ be functions on $[0, T] \times (-\infty, \infty)$, b continuous together with $(\partial/\partial x)\, b$. Let ξ be an $\mathscr{F}_0$-measurable random variable. Prove that the equation in Stratonovich differentials

$$\mathrm{d}^* X_t = a(t, X_t) \, \mathrm{d}t + b(t, X_t) \, \mathrm{d}^* W_t , \quad t \in [0, T] , \quad X_0 = \xi ,$$

and the Itô equation

$$\mathrm{d}X_t = \left(a(t, X_t) + \frac{1}{2} b \frac{\partial}{\partial x} b(t, X_t) \right) \mathrm{d}t + b(t, X_t) \, \mathrm{d}W_t , \quad t \in [0, T] , \quad X_0 = \xi ,$$

are equivalent.

38. Write equations (4), (7), (8), (9) in Stratonovich differentials. □

39. Let a, b be as in Ex. 37. Let function $u(t, y)$, $(t, y) \in [0, T] \times (-\infty, \infty)$ have continuous derivatives $(\partial/\partial t)\, u$, $(\partial/\partial y)\, u$ and $(\partial^2/\partial y^2)\, u$, and let

$$\frac{\partial}{\partial t} u(t, y) = a(t, u), \quad \frac{\partial}{\partial y} u(t, y) = b(t, u),$$
$$(t, y) \in [0, T] \times (-\infty, \infty), \quad u(0, 0) = x_0 .$$

Prove that $X_t = u(t, W_t)$, $t \in [0, T]$, satisfies

$$d^* X_t = a(t, X_t)\, dt + b(t, X_t)\, d^* W_t, \quad t \in [0, T], \quad X_0 = x_0 \text{ a. s.}$$

3. $d\bar{X} = (\bar{a}(t) + A(t)\bar{X})\, dt + B(t)\, d\bar{W}$

Let

$$\bar{a}(t) = (a^1(t), \ldots, a^m(t))', \quad A(t) = \|a_{ij}(t)\|_{i,j=1}^m ,$$
$$B(t) = \|b_{ij}(t)\|_{i=1,j=1}^{m\ \ r} ,$$

be Borel measurable functions on $[0, T]$ such that

$$\int_0^T |a^i(t)|\, dt < \infty , \quad \int_0^T |a_{ij}(t)|\, dt < \infty , \quad \int_0^T b_{ik}(t)^2\, dt < \infty ,$$
$$i, j = 1, \ldots, m , \quad k = 1, \ldots, r .$$

Let ξ be an $\mathscr{F}_0$-measurable m-dimensional random variable. Random process $\bar{X} = \{\bar{X}_t, t \in [0, T]\}$ is a solution of linear stochastic differential equation

(15) $$d\bar{X}_t = (\bar{a}(t) + A(t)\bar{X}_t)\, dt + B(t)\, d\bar{W}_t, \quad t \in [0, T],$$

with initial condition ξ, if it has stochastic differential (15), and if $\bar{X}_0 = \xi$.

***40.** Prove that equation (15) with initial condition ξ has unique solution

$$\bar{X}_t = F(t)\left(\xi + \int_0^t F(s)^{-1}\, \bar{a}(s)\, ds + \int_0^t F(s)^{-1}\, B(s)\, d\bar{W}_s\right), \quad t \in [0, T],$$

where matrix F satisfies $(d/dt) F = A(t) F$, $F(0) = I$. (I is the unit matrix.)

***41.** Let $\bar{X} = \{\bar{X}_t, t \in [0, T]\}$, ξ, be as in Ex. 40, and let $E|\xi|^2 < \infty$. Introduce

$$\bar{e}(t) = E\bar{X}_t , \quad R(u, t) = E(\bar{X}_u - \bar{e}(u))(\bar{X}_t - \bar{e}(t))' , \quad R(t, t) = Q(t) , \quad u, t \in [0, T] .$$

Prove

$$\frac{d}{dt}\bar{e} = A(t)\bar{e} + \bar{a}(t), \quad t \in [0, T], \quad \bar{e}(0) = E\xi ,$$

$$\frac{d}{dt} Q = A(t) Q + QA(t)' + B(t) B(t)' , \quad t \in [0, T], \quad Q(0) = E(\xi - E\xi)(\xi - E\xi)' ,$$

$$\frac{d}{du} R = A(u) R , \quad u \in [t, T], \quad R(t, t) = Q(t) .$$

42. Solve

$$dX_t = -lX_t\,dt + lb\,dW_t\,, \quad t \geqq 0\,, \quad X_0 = \xi\,,$$

where l, b are positive constants, ξ is $\mathscr{F}_0$-measurable, $E\xi = 0$, $E\xi^2 = b^2 l/2$. Compute

$$EX_t\,, \quad E\int_0^t X_s\,ds\,, \quad EX_u X_t\,, \quad EX_u \int_0^t X_s\,ds\,, \quad E\int_0^u X_s\,ds \int_0^t X_s\,ds\,, \quad u, t \geqq 0\,,$$

and prove

$$\operatorname*{l.i.m.}_{l\to\infty} \int_0^t X_s\,ds = bW_t\,, \quad t \geqq 0\,. \qquad \square$$

43. Let $\bar{X} = \{({}^1X_t, {}^2X_t, {}^3X_t)', t \geqq 0\}$ satisfy

$$d\bar{X}_t = A\bar{X}_t\,dt + d\bar{W}_t\,, \quad t \geqq 0\,,$$

where

$$A = \begin{pmatrix} -3, & 1, & -1 \\ -2, & 0, & -1 \\ -1, & 1, & -2 \end{pmatrix}.$$

Define $\bar{e}(t)$, $Q(t)$, $t \geqq 0$, as in Ex. 41. Given $\bar{e}(0)$, compute $\bar{e}(t)$, $t \geqq 0$. Find $Q(0)$, for which $Q(t)$, $t \geqq 0$, is constant. $\square$

44. Let $\bar{X} = \{({}^1X_t, {}^2X_t, {}^3X_t)', t \geqq 0\}$ satisfy

$$d\bar{X}_t = A\bar{X}_t\,dt + d\bar{W}_t\,, \quad t \geqq 0\,,$$

where

$$A = \begin{pmatrix} 1, & 0, & 0 \\ 0, & 0, & 1 \\ 0, & 1, & 0 \end{pmatrix}.$$

Define $\bar{e}(t)$, $Q(t)$, $t \geqq 0$, as in Ex. 41. Given $\bar{e}(0)$, $Q(0)$, compute $\bar{e}(t)$ and $Q(t)$, $t \geqq 0$. $\square$

45. Let $\bar{X} = \{({}^1X_t, {}^2X_t)', t \geqq 0\}$ satisfy

$$d\,{}^1X_t = {}^2X_t\,dt\,,$$
$$d\,{}^2X_t = -{}^1X_t\,dt + b\,dW_t\,, \quad t \geqq 0\,,$$

where b is a constant. Given $E[({}^1X_0)^2 + ({}^2X_0)^2]$, compute $E[({}^1X_t)^2 + ({}^2X_t)^2]$, $t \geqq 0$. $\square$

***46.** Let $\bar{X} = \{({}^1X_t, {}^2X_t)', t \geqq 0\}$ satisfy

$$d\,{}^1X_t = -l\,{}^1X_t\,dt - k\,{}^2X_t\,dt + b\,d\,{}^1W_t\,,$$
$$d\,{}^2X_t = k\,{}^1X_t\,dt - l\,{}^2X_t\,dt + b\,d\,{}^2W_t\,, \quad t \geqq 0\,,$$

where l, k, b are positive constants. Define $\bar{e}(t)$, $Q(t)$, $R(u, t)$, $u, t \geqq 0$, as in Ex. 41. Given $\bar{e}(0)$, $Q(0)$, compute $\bar{e}(t)$, $Q(t)$, $R(u, t)$, $u, t \geqq 0$. □

47. Let $\bar{X} = \{({}^1X_t, {}^2X_t, {}^3X_t)', t \geqq 0\}$ satisfy

$$\begin{aligned} \mathrm{d}\, {}^1X_t &= ({}^2X_t + {}^3X_t)\,\mathrm{d}t\,, \\ \mathrm{d}\, {}^2X_t &= 0\,, \\ \mathrm{d}\, {}^3X_t &= -a\, {}^3X_t\,\mathrm{d}t + b\,\mathrm{d}W_t\,, \quad t \geqq 0\,, \end{aligned}$$

where a, b are positive constants. Define $\bar{e}(t)$, $Q(t)$, $R(u, t)$, $u, t \geqq 0$, as in Ex. 41. Given $\bar{e}(0)$, $Q(0) = \|q_{ij}(0)\|_{i,j=1}^{3}$ with $q_{23}(0) = q_{32}(0) = 0$, compute $\bar{e}(t)$, $Q(t)$, $R(u, t)$, $u, t \geqq 0$. □

48. Let ${}^lX = \{{}^lX_t, t \in [0, T]\}$, $l \geqq 0$, satisfy

$$\mathrm{d}X_t = a(t)\,X_t\,\mathrm{d}t + lb(t)\,\mathrm{d}W_t\,, \quad t \in [0, T]\,, \quad X_0 = 1\,,$$

where

$$\int_0^T |a(t)|\,\mathrm{d}t < \infty\,, \quad \int_0^T b(t)^2\,\mathrm{d}t < \infty\,.$$

Prove

$$\underset{l \to 0}{\mathrm{l.\,i.\,m.}}\; l^{-1}({}^lX_t - {}^0X_t) = \exp\left\{\int_0^t a(s)\,\mathrm{d}s\right\} \int_0^t b(s)\,\mathrm{d}W_s\,, \quad t \in [0, T]\,.$$

49. Let $X = \{X_t, t \in [0, T]\}$, $Y = \{Y_t, t \in [0, T]\}$ satisfy

$$(16) \qquad \begin{aligned} \mathrm{d}X_t &= a(t)\,X_t\,\mathrm{d}t + b(t)\,\mathrm{d}\,{}^1W_t\,, \\ \mathrm{d}Y_t &= c(t)\,X_t\,\mathrm{d}t + g(t)\,\mathrm{d}\,{}^2W_t\,, \quad t \in [0, T]\,, \quad Y_0 = 0\,. \end{aligned}$$

$a(t)$, $b(t)$, $c(t)$, $g(t)$ are functions on $[0, T]$ such that

$$\int_0^T |a(t)|\,\mathrm{d}t < \infty\,, \quad \int_0^T b(t)^2\,\mathrm{d}t < \infty\,, \quad \int_0^T c(t)^2\,\mathrm{d}t < \infty\,,$$

$$\int_0^T g(t)^2\,\mathrm{d}t < \infty\,, \quad g(t)^2 \geqq g_0 > 0\,.$$

Let $EX_0 = e_0$, $E(X_0 - e_0)^2 = q_0 < \infty$.

By solving a) – g) derive the Kalman-Bucy estimate of X_t from the observation of Y_s, $s \in [0, t]$.

a) Prove that

$$X_t^* = X_t - e_0 \exp\left\{\int_0^t a(s)\,\mathrm{d}s\right\},$$

$$Y_t^* = Y_t - \int_0^t c(s)\, e_0 \exp\left\{\int_0^s a(y)\, \mathrm{d}y\right\} \mathrm{d}s\,, \quad t \in [0, T]\,,$$

also satisfy (16), and $EX_t^* = EY_t^* = 0$, $t \in [0, T]$, $EX_0^{*2} = q_0$.

b) Let $t \in [0, T]$, and let $u(s)$, $s \in [0, t]$, be such function that $\int_0^t u(s)^2\, g(s)^2\, \mathrm{d}s < \infty$. Prove: If function $z(s)$, $s \in [0, t]$, fulfils

$$z'(s) + a(s)\, z(s) - u(s)\, c(s) = 0\,, \quad s \in [0, t]\,, \quad z(t) = 1\,,$$

then

$$E\left(X_t^* - \int_0^t u(s)\, \mathrm{d}Y_s^*\right)^2 = z(0)^2\, q_0 + \int_0^t z(s)^2\, b(s)^2\, \mathrm{d}s + \int_0^t u(s)^2\, g(s)^2\, \mathrm{d}s\,.$$

Hint. $X_t^* = z(0)\, X_0^* + \int_0^t \mathrm{d}(z(s)\, X_s^*)$.

c) Prove: Let $r(s)$, $s \in [0, t]$, be solution of the Riccati equation

$$r'(s) = 2a(s)\, r(s) + b(s)^2 - c(s)^2\, g(s)^{-2}\, r(s)^2\,, \quad s \in [0, t]\,,$$
$$r(0) = q_0\,.$$

Then

$$z(0)^2\, q_0 + \int_0^t z(s)^2\, b(s)^2\, \mathrm{d}s + \int_0^t u(s)^2\, g(s)^2\, \mathrm{d}s =$$
$$= \int_0^t (u(s) - c(s)\, g(s)^{-2}\, r(s)\, z(s))^2\, g(s)^2\, \mathrm{d}s + r(t)\,.$$

Hint. $z(0)^2\, q_0 = r(t)\, z(t)^2 - \int_0^t \mathrm{d}(r(s)\, z(s))^2$.

d) Prove: $E(X_t^* - \int_0^t u(s)\, \mathrm{d}Y_s^*)^2$ is minimal for

$$u(s) = c(s)\, g(s)^{-2}\, r(s)\, z(s)\,, \quad s \in [0, t]\,,$$

i.e.

$$u(s) = k(s) \exp\left\{\int_s^t (a(y) - k(y)\, c(y))\, \mathrm{d}y\right\},$$

where

$$k(s) = c(s)\, g(s)^{-2}\, r(s)\,, \quad s \in [0, t]\,.$$

Hint. Use c).

e) Set

$$\hat{X}_t^* = \int_0^t k(s) \exp\left\{\int_s^t (a(y) - k(y)\, c(y))\, \mathrm{d}y\right\} \mathrm{d}Y_s^*\,,$$

$$\hat{X}_t = \hat{X}_t^* + e_0 \exp\left\{\int_0^t a(s)\, \mathrm{d}s\right\}, \quad t \in [0, T]\,.$$

Prove

(17) $$d\hat{X}_t = a(t)\,\hat{X}_t\,dt + k(t)\,(dY_t - c(t)\,\hat{X}_t\,dt)\,, \quad t \in [0, T]\,, \quad \hat{X}_0 = e_0\,,$$
$$E(\hat{X}_t - X_t)^2 = r(t)\,, \quad t \in [0, T]\,.$$

(17) is the equation of the Kalman-Bucy filter.

f) Prove

(18) $$E(X_t - \hat{X}_t)^2 = \min_{v,u(s)} E\left(X_t - v - \int_0^t u(s)\,dY_s\right)^2, \quad t \in [0, T]\,,$$

i.e. $\hat{X}_t$ is the state estimate of process X on the basis of Y_s, $s \in [0, t]$.

g) Prove the following generalization of (18): For $0 \leqq t \leqq t' \leqq T$,

$$\min_{v,u(s)} E\left(X_{t'} - v - \int_0^t u(s)\,dY_s\right)^2 = E\left(X_{t'} - \exp\left\{\int_t^{t'} a(s)\,ds\right\}\hat{X}_t\right)^2.$$

Set

$$r(t', t) = E\left(X_{t'} - \exp\left\{\int_t^{t'} a(s)\,ds\right\}\hat{X}_t\right)^2.$$

Prove

$$\frac{d}{dt'}\,r = 2a(t')\,r + b(t')^2\,, \quad t \leqq t' \leqq T\,, \quad r(t, t) = r(t)\,.$$

50. Let X_0 be an $\mathscr{F}_0$-measurable random variable, $EX_0 = e_0$, $E(X_0 - e_0)^2 = q_0$. Let $Y = \{Y_t, t \geqq 0\}$ satisfy

$$dY_t = X_0\,dt + dW_t\,, \quad t \geqq 0\,, \quad Y_0 = 0\,.$$

For $t \geqq 0$ determine $\hat{X}_t$, the estimate of X_0 on the basis of Y_s, $s \in [0, t]$. □

51. Let $X = \{X_t, t \geqq 0\}$, $Y = \{Y_t, t \geqq 0\}$ satisfy

$$dX_t = aX_t\,dt + b\,d\,{}^1W_t\,, \quad t \geqq 0\,, \quad EX_0 = e_0\,, \quad E(X_0 - e_0)^2 = q_0\,,$$
$$dY_t = X_t\,dt + g\,d\,{}^2W_t\,, \quad t \geqq 0\,, \quad Y_0 = 0\,.$$

a, b, g are constants, $g \neq 0$. Then, using the denotations of Ex. 49e,

$$d\hat{X}_t = a\hat{X}_t\,dt + k(t)\,(dY_t - \hat{X}_t\,dt)\,, \quad t \geqq 0\,.$$

Determine $k(t)$. □

The notion of the *state estimate of process* X on the basis of Y_s, $s \in [0, t]$, as introduced in Ex. 49*f*, can be extended to multidimensional processes. Let

$$A(t) = \|a_{ij}(t)\|_{i,j=1}^m\,, \qquad B(t) = \|b_{ij}(t)\|_{i,j=1}^m\,,$$
$$C(t) = \|c_{ij}(t)\|_{i=1,j=1}^{r\;\;m}\,, \quad G(t) = \|g_{ij}(t)\|_{i,j=1}^r\,, \quad t \in [0, T]\,,$$

be matrix functions. Let the elements of $A(t)$ be integrable, and those of $B(t)$, $C(t)$, $G(t)$ be quadratically integrable on $[0, T]$. Further, let the matrix $(G(t)\,G(t)')^{-1}$ be bounded on $[0, T]$. Consider processes $\bar{X} = \{({}^1X_t, \ldots, {}^mX_t)', t \in [0, T]\}$, $\bar{Y} = \{({}^1Y_t, \ldots, {}^rY_t)', t \in [0, T]\}$, satisfying

$$d\bar{X}_t = A(t)\,\bar{X}_t\,dt + B(t)\,d\,{}^1\bar{W}_t\,, \quad E\bar{X}_0 = \bar{e}_0\,, \quad E(\bar{X}_0 - \bar{e}_0)(\bar{X}_0 - \bar{e}_0)' = Q_0\,,$$
$$d\bar{Y}_t = C(t)\,\bar{X}_t\,dt + G(t)\,d\,{}^2\bar{W}_t\,, \quad \bar{Y}_0 = 0\,,$$

where ${}^1\bar{W}$, ${}^2\bar{W}$ are mutually independent Wiener processes of dimension m and r, respectively.

For $t \in [0, T]$ denote by $\hat{\bar{X}}_t$ the estimate of $\bar{X}_t$ from the observation of $\bar{Y}_s$, $s \in [0, t]$. In the same way as in Exercises 49a-f it can be proved that

$$d\hat{\bar{X}}_t = A(t)\,\hat{\bar{X}}_t\,dt + K(t)\,(d\bar{Y}_t - C(t)\,\hat{\bar{X}}_t\,dt)\,, \quad t \in [0, T]\,, \quad \bar{X}_0 = \bar{e}_0\,,$$

(Kalman-Bucy filter), where

(19) $$K(t) = R(t)\,C(t)'\,(G(t)\,G(t)')^{-1}\,,$$

(20) $$\frac{d}{dt}R = A(t)\,R + R\,A(t)' + B(t)\,B(t)' - R\,C(t)'\,(G(t)\,G(t)')^{-1}\,C(t)\,R\,,$$
$$t \in [0, T]\,, \quad R(0) = Q_0\,.$$

It holds

$$E(\bar{X}_t - \hat{\bar{X}}_t)(\bar{X}_t - \hat{\bar{X}}_t)' = R(t) = \|r_{ij}(t)\|_{i,j=1}^m\,, \quad t \in [0, T]\,.$$

If A, B, C, G are constant matrices, equating $(d/dt)\,R$ to 0 in (20), the equation for the *steady value* of R is obtained. Inserting that value into (19), one obtains the matrix K of the steady filter.

52. Let $\bar{X} = \{({}^1X_t, {}^2X_t)', t \geqq 0\}$, $Y = \{Y_t, t \geqq 0\}$ satisfy

$$d\,{}^1X_t = {}^2X_t\,dt + d\,{}^1W_t\,,$$
$$d\,{}^2X_t = b\,d\,{}^2W_t\,,$$
$$dY_t = {}^1X_t\,dt + d\,{}^3W_t\,, \quad t \geqq 0\,,$$

where b is a constant. From (19), (20) find the steady value of R and of K. □

53. Let $\bar{X} = \{({}^1X_t, {}^2X_t)', t \geqq 0\}$, $Y = \{Y_t, t \geqq 0\}$ satisfy

$$d\,{}^1X_t = -({}^1X_t + {}^2X_t)\,dt + d\,{}^1W_t\,,$$
$$d\,{}^2X_t = {}^1X_t\,dt\,,$$
$$dY_t = {}^1X_t\,dt + d\,{}^2W_t\,, \quad t \geqq 0\,.$$

From (19), (20) find the steady value of R and of K. □

54. Let $\bar{X} = \{({}^1X_t, {}^2X_t)', t \geqq 0\}$, $Y = \{Y_t, t \geqq 0\}$ satisfy

$$\begin{aligned} d\,{}^1X_t &= -({}^1X_t + a\,{}^2X_t)\,dt + d\,{}^1W_t\,, \\ d\,{}^2X_t &= -{}^2X_t\,dt + d\,{}^2W_t\,, \\ dY_t &= {}^1X_t\,dt + d\,{}^3W_t\,, \quad t \geqq 0\,, \end{aligned}$$

where a is constant. Find a for which the steady value of r_{11} is 1. □

4. $d\bar{X} = (\bar{a}(t) + A(t)\,\bar{X})\,dt + \sum_k {}^kB(t)\,\bar{X}\,d\,{}^kW$

Let

$$\begin{aligned} \bar{a}(t) &= (a^1(t), \ldots, a^m(t))'\,, \quad A(t) = \|a_{ij}(t)\|_{i,j=1}^m\,, \\ {}^kB(t) &= \|{}^kb_{ij}(t)\|_{i,j=1}^m\,, \quad k = 1, \ldots, r\,, \end{aligned}$$

be bounded Borel measurable vector and matrix functions on $[0, T]$, $T < \infty$. Let ξ be an $\mathscr{F}_0$-measurable m-dimensional random variable. Then the equation

$$d\bar{X}_t = (\bar{a}(t) + A(t)\,\bar{X}_t)\,dt + \sum_{k=1}^r {}^kB(t)\,\bar{X}_t\,d\,{}^kW_t\,, \quad t \in [0, T]\,, \tag{21}$$

with initial condition $\bar{X}_0 = \xi$ has unique solution $\bar{X} = \{({}^1X_t, \ldots, {}^mX_t)', t \in [0, T]\}$. If $E|\xi|^{2n} < \infty$, then $E|\bar{X}_t|^{2n}$ is bounded on $[0, T]$, $n = 1, 2, \ldots$

55. Let $\bar{X} = \{\bar{X}_t, t \in [0, T]\}$ satisfy (21). Assume $E|\bar{X}_0|^2 < \infty$, denote $\bar{e}(t) = E\bar{X}_t$ $t \in [0, T]$, and prove

$$\frac{d}{dt}\,\bar{e} = \bar{a}(t) + A(t)\,\bar{e}\,.$$

Assume $E|\bar{X}_0|^4 < \infty$, denote $Q(t) = E\bar{X}_t\bar{X}_t'$, $t \in [0, T]$, and prove

$$\begin{aligned} \frac{d}{dt}\,Q(t) &= \bar{a}(t)\,\bar{e}(t)' + \bar{e}(t)\,\bar{a}(t)' + A(t)\,Q(t) + Q(t)\,A(t)' + \\ &+ \sum_{k=1}^r {}^kB(t)\,Q(t)\,{}^kB(t)'\,, \quad t \in [0, T]\,. \end{aligned}$$

Hint. $\bar{X}_t\bar{X}_t' = \bar{X}_0\bar{X}_0' + \int_0^t d(\bar{X}_s\bar{X}_s')$.

***56.** Let $a(t)$, $b(t)$ be Borel measurable functions on $[0, T]$,

$$\int_0^T |a(t)|\,dt < \infty\,, \quad \int_0^T b(t)^2\,dt < \infty\,.$$

Let ξ be an $\mathscr{F}_0$-measurable random variable. Prove that the unique solution of the equation

$$dX_t = a(t)\,X_t\,dt + b(t)\,X_t\,dW_t\,,\quad t\in[0,T]\,,\quad X_0=\xi\,,$$

is

$$X_t = \xi\exp\left\{\int_0^t (a(s) - \tfrac{1}{2}b(s)^2)\,ds + \int_0^t b(s)\,dW_s\right\},\quad t\in[0,T]\,.$$

57. Let $a(t)$, $b(t)$, ξ be as in Ex. 56, and let $\Phi = \{\Phi_t, t\in[0,T]\}$, $\Psi = \{\Psi_t, t\in[0,T]\}$ be nonanticipative and such that

$$\int_0^T |\Phi_t|\,dt < \infty\,,\quad \int_0^T \Psi_t^2\,dt < \infty \text{ a. s.}$$

Prove that

$$X_t = \xi\exp\left\{\int_0^t (a(s) - \tfrac{1}{2}b(s)^2)\,ds + \int_0^t b(s)\,dW_s\right\} +$$
$$+ \int_0^t \exp\left\{\int_u^t (a(s) - \tfrac{1}{2}b(s)^2)\,ds + \int_u^t b(s)\,dW_s\right\}\,.$$
$$.\,[(\Phi_u - b(u)\,\Psi_u)\,du + \Psi_u\,dW_u]\,,\quad t\in[0,T]\,,$$

is the unique solution of the inhomogeneous equation

$$dX_t = a(t)\,X_t\,dt + b(t)\,X_t\,dW_t + \Phi_t\,dt + \Psi_t\,dW_t\,,\quad t\in[0,T]\,,$$

with initial condition $X_0 = \xi$.

58. Let $X = \{X_t, t \geqq 0\}$ satisfy

$$dX_t = aX_t\,dt + b\,dW_t\,,\quad t\geqq 0\,,\quad X_0 = x_0 > 0\,,$$

where a, b are constants. Let p be a positive number. Prove that $\lim_{t\to\infty} EX_t^p = 0$ holds if and only if $a + \tfrac{1}{2}b^2(p-1) < 0$.

59. Let $X = \{X_t, t\geqq 0\}$ be as in Ex. 58. Prove that $\log(X_t/x_0)$ has normal distribution with mean $(a - \tfrac{1}{2}b^2)\,t$ and variance b^2t.

60. Let $Z = \{Z_t, t\in[0,T]\}$ satisfy

$$dZ_t = -lZ_t\,dt + lb\,dW_t\,,\quad t\in[0,T]\,,$$

where l, b are positive constants, $EZ_0 = 0$, $EZ_0^2 = b^2l/2$. Further let

$$\frac{d}{dt}\,Y_t = Z_tY_t\,,\quad t\in[0,T]\,,\quad Y_0 = 1\,.$$

Prove that

$$\underset{l\to\infty}{\text{l. i. m.}}\ Y_t = X_t\,, \quad t \in [0, T]\,,$$

where $X = \{X_t, t \in [0, T]\}$ is the solution of

$$\mathrm{d}X_t = \tfrac{1}{2}b^2 X_t\,\mathrm{d}t + bX_t\,\mathrm{d}W_t\,, \quad t \in [0, T]\,, \quad X_0 = 0\,.$$

61. Let $a(t)$, $b(t)$ be Borel measurable functions on $[0, T]$,

$$\int_0^T |a(t)|\,\mathrm{d}t < \infty\,, \quad \int_0^T b(t)^2\,\mathrm{d}t < \infty\,.$$

For $l \geqq 0$ consider the solution ${}_lX = \{{}_lX_t, t \in [0, T]\}$ of

$$\mathrm{d}\,{}_lX_t = a(t)\,{}_lX_t\,\mathrm{d}t + lb(t)\,{}_lX_t\,\mathrm{d}W_t\,, \quad t \in [0, T]\,, \quad X_0 = 1\,.$$

Prove that

$$\lim_{l\to 0} l^{-2}E({}_lX_t - {}_0X_t)^2 = \int_0^t b(s)^2\,\mathrm{d}s\,\exp\left\{2\int_0^t a(s)\,\mathrm{d}s\right\}, \quad t \in [0, T]\,.$$

62. Let A, ${}^1B, \ldots, {}^rB$ be matrices of type $m \times m$, commuting with each other. Let ξ be an m-dimensional $\mathscr{F}_0$-measurable random variable. Find a matrix C such that

$$\bar{X}_t = \exp\{Ct + \sum_{k=1}^r {}^kB\,{}^kW_t\}\,\xi\,, \quad t \geqq 0\,,$$

satisfies

$$\mathrm{d}\bar{X}_t = A\bar{X}_t\,\mathrm{d}t + \sum_{k=1}^r {}^kB\bar{X}_t\,\mathrm{d}\,{}^kW_t\,, \quad t \geqq 0\,.$$

□

63. Let A, ${}^1B, \ldots, {}^rB$, ξ and $\bar{X} = \{\bar{X}_t, t \geqq 0\}$ be as in Ex. 62, $E|\xi|^2 < \infty$. Assume that A, ${}^1B, \ldots, {}^rB$, A', ${}^1B', \ldots, {}^rB'$, and $Q(0) = E\xi\xi'$ commute with each other. Prove

$$E\bar{X}_t\bar{X}'_t = \exp\{(A + A' + \sum_{k=1}^r {}^kB\,{}^kB')\,t\}\,Q(0)\,.$$

Hint. Use Ex. 55.

64. Let

$$A(t) = \|a_{ij}(t)\|_{i,j=1}^m\,, \quad {}^kB(t) = \|{}^kb_{ij}(t)\|_{i,j=1}^m\,, \quad k = 1, \ldots, r\,,$$

be bounded Borel measurable functions on $[0, T]$. Denote by $X = \{X_t, t \in [0, T]\}$ the $m \times m$-matrix valued process, satisfying

(22) $$\mathrm{d}X_t = A(t)\,X_t\,\mathrm{d}t + \sum_{k=1}^r {}^kB(t)\,X_t\,\mathrm{d}\,{}^kW_t\,, \quad t \in [0, T]\,, \quad X_0 = I\,.$$

(X is the fundamental matrix of (22).) Set

$$D_t = \det(X_t), \quad t \in [0, T].$$

Prove

$$\mathrm{d}D_t = [\mathrm{tr}(A(t)) + \tfrac{1}{2}\sum_{k=1}^{r}\sum_{i,j=1}^{m}({}^kb_{ii}(t)\,{}^kb_{jj}(t) - {}^kb_{ij}(t)\,{}^kb_{ji}(t))] \,.$$

$$.\, D_t\,\mathrm{d}t + D_t\sum_{k=1}^{r}\mathrm{tr}({}^kB(t))\,\mathrm{d}\,{}^kW_t\,, \quad t \in [0, T], \quad D_0 = 1\,,$$

and hence,

$$D_t = \exp\left\{\int_0^t \mathrm{tr}(A(s))\,\mathrm{d}s - \tfrac{1}{2}\sum_{k=1}^{r}\int_0^t \mathrm{tr}({}^kB(s)^2)\,\mathrm{d}s + \sum_{k=1}^{r}\int_0^t \mathrm{tr}({}^kB(s))\,\mathrm{d}\,{}^kW_s\right\},$$
$$t \in [0, T].$$

(A generalization of the *Liouville formula*.)

65. Let

$$A(t)\,, \quad {}^kB(t)\,, \quad k = 1, \ldots, r\,, \quad \text{and} \quad X = \{X_t, t \in [0, T]\}$$

be as in Ex. 64. Further, let

$$\bar{\Phi} = \{\bar{\Phi}_t, t \in [0, T]\}\,, \quad {}^k\bar{\Psi} = \{{}^k\bar{\Psi}_t\,, \quad t \in [0, T]\}\,, \quad k = 1, \ldots, r\,,$$

be nonanticipative and such that

$$\int_0^T |\bar{\Phi}_t|\,\mathrm{d}t < \infty\,, \quad \int_0^T |{}^k\bar{\Psi}_t|^2\,\mathrm{d}t < \infty\,, \quad k = 1, \ldots, r\,, \quad \text{a. s.}$$

Prove that for any $\mathscr{F}_0$-measurable m-dimensional random variable ξ

$$\bar{Y}_t = X_t\xi + X_t\int_0^t X_s^{-1}(\bar{\Phi}_s - \sum_{k=1}^{r}{}^kB(s)\,{}^k\bar{\Psi}_s)\,\mathrm{d}s + \sum_{k=1}^{r}X_t\int_0^t X_s^{-1}\,{}^k\bar{\Psi}_s\,\mathrm{d}\,{}^kW_s\,, \quad t \in [0, T]$$

is the unique solution of the inhomogeneous equation

$$\mathrm{d}\bar{Y}_t = A(t)\,\bar{Y}_t\,\mathrm{d}t + \sum_{k=1}^{r}{}^kB(t)\,\bar{Y}_t\,\mathrm{d}\,{}^kW_t + \bar{\Phi}_t\,\mathrm{d}t + \sum_{k=1}^{r}{}^k\bar{\Psi}_t\,\mathrm{d}\,{}^kW_t\,, \quad t \in [0, T]\,,$$

with initial condition $\bar{Y}_0 = \xi$. Consider the case when $A, {}^kB, k = 1, \ldots, r,$ are constant matrices commuting with each other, and compute X as in Ex. 62.

5. INEQUALITIES

Random process $\bar{X} = \{\bar{X}_t, t \in [0, T]\}$ is a *diffusion process*, if it has stochastic differential

(23) $$\mathrm{d}\bar{X}_t = \bar{a}(t, \bar{X}_t)\,\mathrm{d}t + B(t, \bar{X}_t)\,\mathrm{d}\bar{W}_t\,, \quad t \in [0, T]\,,$$

where

$$\bar{a}(t, \bar{x}) = (a^1(t, \bar{x}), \ldots, a^m(t, \bar{x}))',$$
$$B(t, \bar{x}) = \|b_{ij}(t, \bar{x})\|_{i=1,j=1}^{m\ \ r}, \quad (t, \bar{x}) \in [0, T] \times R^m,$$

are a vector function and a matrix function, Borel measurable on $[0, T] \times R^m$.

Introduce the matrix

$$\|c_{ij}(t, \bar{x})\|_{i,j=1}^{m} = C(t, \bar{x}) = B(t, \bar{x})\, B(t, \bar{x})', \quad (t, \bar{x}) \in [0, T] \times R^m.$$

The operator

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{m} a^i(t, \bar{x}) \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{m} c_{ij}(t, \bar{x}) \frac{\partial^2}{\partial x^i\, \partial x^j}$$

is called the *differential generator* associated with (23) (with the diffusion process having stochastic differential (23)). Let function $f(t, \bar{x})$, $(t, \bar{x}) \in [0, T] \times R^m$, be as in Section 2, and let (23) hold. Then

$$\mathrm{d}f(t, \bar{X}_t) = Lf(t, \bar{X}_t)\, \mathrm{d}t + (\nabla f(t, \bar{X}_t))'\, B(t, \bar{X}_t)\, \mathrm{d}\bar{W}_t, \quad t \in [0, T],$$

where

$$(\nabla f(t, \bar{x}))' = (\partial f(t, \bar{x})/\partial x^1, \ldots, \partial f(t, \bar{x})/\partial x^m).$$

66. Let $\bar{X} = \{({}^1X_t, {}^2X_t)', t \geqq 0\}$ have Stratonovich differential

$$\mathrm{d}^*\, {}^1X_t = l\, {}^2X_t\, \mathrm{d}t,$$
$$\mathrm{d}^*\, {}^2X_t = -l\, {}^1X_t\, \mathrm{d}t - k\, {}^2X_t\, \mathrm{d}t + b\, {}^2X_t\, \mathrm{d}^* W_t, \quad t \geqq 0,$$

where l, k, b are constants. Find the corresponding differential generator L. □

67. Let $\bar{X} = \{({}^1X_t, {}^2X_t)', t \geqq 0\}$ have stochastic differential

$$\mathrm{d}\, {}^1X_t = {}^1X_t\, \mathrm{d}\, {}^1W_t + {}^2X_t\, \mathrm{d}\, {}^2W_t,$$
$$\mathrm{d}\, {}^2X_t = -{}^2X_t\, \mathrm{d}\, {}^1W_t + {}^1X_t\, \mathrm{d}\, {}^2W_t, \quad t \geqq 0.$$

Define

$${}^1Y_t = \mathrm{e}^{{}^1W_t}({}^1X_0 \cos {}^2W_t + {}^2X_0 \sin {}^2W_t),$$
$${}^2Y_t = \mathrm{e}^{{}^1W_t}(-{}^1X_0 \sin {}^2W_t + {}^2X_0 \cos {}^2W_t), \quad t \geqq 0.$$

Prove that $\bar{X}$ and $\bar{Y}$ have same differential generators and that $\bar{X}_t = \bar{Y}_t$, $t \geqq 0$, a. s., if and only if $\bar{X}_0 = 0$ a. s.

68. Let $X = \{X_t, t \in [0, T]\}$ have stochastic differential

$$\mathrm{d}X_t = a(X_t)\, \mathrm{d}t + b(X_t)\, \mathrm{d}W_t, \quad t \in [0, T],$$

where $a(x)$, $b(x)$, $x \in (-\infty, \infty)$, are Borel measurable functions. Let there exist a function $v(x) \geqq 0$, $x \in (-\infty, \infty)$, twice continuously differentiable, $\lim_{x \to \pm\infty} v(x) = \infty$, and constants $c \geqq 0$, $k > 0$ such that

$$\tfrac{1}{2}b(x)^2 \frac{d^2}{dx^2} v(x) + a(x) \frac{d}{dx} v(x) + k\, v(x) \leqq c\,, \quad x \in (-\infty, \infty)\,.$$

Prove

$$E\, v(X_t) \leqq E\, v(X_0) + \frac{c}{k}\,, \quad t \in [0, T]\,.$$

Hint. $v(X_t) - e^{-kt} v(X_0) = \int_0^t d(e^{-k(t-s)} v(X_s))$.

69. Let $X = \{X_t, t \geqq 0\}$ have stochastic differential

$$dX_t = -aX_t\, dt + bX_t\, dW_t\,, \quad t \geqq 0\,,$$

where a, b are constants $a > \frac{1}{2}b^2$. Use Ex. 68 to prove

$$EX_t^2 \leqq EX_0^2\,, \quad t \geqq 0\,.$$

70. Let $X = \{X_t, t \geqq -\Delta\}$, $\Delta \geqq 0$, fulfil

$$E \sup_{t \in [-\Delta, 0]} X_t^2 < \infty\,,$$

$$dX_t = (a_1 X_t + a_2 X_{t-\Delta})\, dt + bX_t\, dW_t\,, \quad t \geqq 0\,,$$

where a_1, a_2, b are constants. Set

$$q = 2(a_1 + a_2)(1 - |a_2|\, \Delta) + b^2\,,$$

and assume

(24) $$q < 0\,, \quad |a_2|\, \Delta < 1\,.$$

Prove that $\int_0^\infty EX_t^2\, dt < \infty$, and that EX_t^2, $t \geqq 0$, is bounded.

Hint. Define

$$V_t = \left(X_t + a_2 \int_{t-\Delta}^t X_s\, ds\right)^2 - (a_1 + a_2)\, |a_2| \int_{t-\Delta}^t \int_s^t X_u^2\, du\, ds\,,$$

prove

$$dV_t \leqq 2\left(X_t + a_2 \int_{t-\Delta}^t X_s\, ds\right) bX_t\, dW_t + qX_t^2\, dt\,,$$

and hence,

$$q \int_0^t EX_s^2\, ds \geqq EV_t - EV_0\,, \quad E\left(X_t + a_2 \int_{t-\Delta}^t X_s\, ds\right)^2 \leqq EV_0\,.$$

71. Let $X = \{X_t, t \geqq -\Delta\}$ be as in Ex. 70, and let (24) hold. Prove that

$$\lim_{t\to\infty} EX_t^2 = 0 .$$

Hint. Use the Itô formula and Ex. 70 to prove $|EX_t^2 - EX_s^2| \leqq \text{const.}\, |t - s|$.

72. Let $\bar{X}$ and L be as in Ex. 66. Find matrix V such that

$$\tfrac{1}{2}L\bar{x}'V\bar{x} = -|\bar{x}|^2 , \quad \bar{x} \in R^2 .$$

Prove that V is positively definite if and only if $b^2 < k$, and that

$$\lim_{t\to\infty} E|\bar{X}_t|^2 = 0$$

if $b^2 < k$. □

73. $X = \{X_t, t \geqq 0\}$, $X_0 = x$, have stochastic differential

$$\mathrm{d}X_t = -A_t X_t \,\mathrm{d}t + b \,\mathrm{d}W_t , \quad t \geqq 0 ,$$

where $A = \{A_t, t \geqq 0\}$ is nonanticipative, $A_t \geqq a$, $t \geqq 0$, a. s. a and b are positive constants. Prove that

$$EX_t^2 \leqq \frac{b^2}{2a} + \left(x^2 - \frac{b^2}{2a}\right)\mathrm{e}^{-2at} , \quad t \geqq 0 , \tag{25}$$

where equality holds if and only if $A_t = a$, $t \geqq 0$, a. s.

Hint. Denote the right-hand side of (25) $v(t, x)$ and use

$$EX_t^2 - v(t, x) = E\int_0^t \mathrm{d}v(t - s, X_s) .$$

74. Let $x \in (-\infty, \infty)$, and let a, b be constants. Consider $X = \{X_t, t \in [0, T]\}$ satisfying

$$\mathrm{d}X_t = u(t, x)\,(a \,\mathrm{d}t + b \,\mathrm{d}W_t) , \quad t \in [0, T] , \quad X_0 = x ,$$

and find $u(t, x)$, $t \in [0, T]$, so that EX_T^2 is minimal. □

75. Let $x \in (-\infty, \infty)$, and let a, b be constants $b \neq 0$. Consider $X = \{X_t, t \in [0, T]\}$ having Stratonovich differential

$$\mathrm{d}^*X_t = u(t)\, X_t(a \,\mathrm{d}t + b \,\mathrm{d}^*W_t) , \quad t \in [0, T], X_0 = x ,$$

and find $u(t)$, $t \in [0, T]$, so that EX_T^2 is minimal. □

Hint. Use Exs. 37, 56.

76. Let $X = \{X_t, t \in [0, T]\}$, $T < \infty$, have stochastic differential

$$dX_t = a(t) X_t \, dt + k(t) U_t \, dt + b(t) \, dW_t , \quad t \in [0, T] ,$$

where the Borel measurable functions $a(t)$, $k(t)$, $b(t)$, $t \in [0, T]$, fulfil

$$\int_0^T |a(t)| \, dt < \infty , \quad \int_0^T k(t)^2 \, dt < \infty , \quad \int_0^T b(t)^2 \, dt < \infty ,$$

and $U = \{U_t, t \in [0, T]\}$ is nonanticipative, $\int_0^T U_t^2 \, dt < \infty$ a. s. Further let $EX_0 = e_0$, $E(X_0 - e_0)^2 = q_0 < \infty$. Think of U as of a control signal. Define

$$Z_T = l(T) X_T^2 + \int_0^T X_t^2 r(t) \, dt + \int_0^T U_t^2 s(t) \, dt ,$$

where $l(T) \geqq 0$, and $r(t)$, $s(t)$, $t \in [0, T]$, are Borel measurable functions such that

$$\int_0^T r(t) \, dt < \infty , \quad r(t) \geqq 0 , \quad s(t) \geqq s_0 > 0 , \quad t \in [0, T] .$$

By solving *a*), *b*) determine U for which EZ_T is minimal.

a) Let $p(t)$, $t \in [0, T]$, be the solution of the Riccati equation

$$- \frac{d}{dt} p = 2a(t) p + r(t) - s(t)^{-1} k(t)^2 p^2 , \quad t \in [0, T] ,$$

with terminal condition $p(T) = l(T)$. Set

$$v(t, x) = p(t) x^2 + \int_t^T b(s)^2 p(s) \, ds , \quad (t, x) \in [0, T] \times (-\infty, \infty) .$$

Prove

$$\min_{u \in (-\infty, \infty)} \left\{ \frac{\partial}{\partial t} v + \frac{b(t)^2}{2} \frac{\partial^2}{\partial x^2} v + (a(t) x + k(t) u) \frac{\partial}{\partial x} v + x^2 r(t) + u^2 s(t) \right\} = 0 ,$$

$$(t, x) \in [0, T] \times (-\infty, \infty) ,$$

$$v(T, x) = l(T) x^2 , \quad x \in (-\infty, \infty) .$$

The expression in curly brackets is minimal for

$$u = - \frac{k(t) \partial}{2s(t) \partial x} v .$$

b) Prove that

(26) $$EZ_T \geqq p(0) (e_0^2 + q_0) + \int_0^T b(t)^2 p(t) \, dt ,$$

and that equality in (26) is attained for

$$U_t = -s(t)^{-1}\, k(t)\, p(t)\, X_t\,, \quad t \in [0, T]\,.$$

Hint.

$$E l(T)\, X_T^2 - p(0)\,(e_0^2 + q_0) - \int_0^T b(t)^2\, p(t)\,\mathrm{d}t = E \int_0^T \mathrm{d}v(t, X_t)\,.$$

77. Let the hypotheses of Ex. 76 hold. Moreover, let $a(t) = a$, $k(t) = k \neq 0$, $b(t) = b$, $r(t) = r$, $s(t) = 1$, $t \in [0, T]$, $l(T) = l$, $T \geqq 0$.

Compute $p(t)$, $t \in [0, T]$, prove that

$$p_\infty = \lim_{T \to \infty} p(t) = k^{-2}(a + \sqrt{(a^2 + rk^2)})\,,$$

and that for each $U = \{U_t,\ t \in [0, T]\}$, $EZ_T \geqq b^2 p_\infty T + o(T)$ as $T \to \infty$. □

78. Under the hypotheses of Ex. 77 let

$$U_t = -k p_\infty X_t = -k^{-1}(a + \sqrt{(a^2 + rk^2)})\, X_t\,, \quad t \in [0, T]\,.$$

Compute EZ_T, and prove

$$EZ_T = b^2 p_\infty T + O(1) \quad \text{as} \quad T \to \infty\,.$$ □

The result of Ex. 76 can be directly generalized to the multidimensional case. Let

$$\begin{aligned} A(t) &= \|a_{ij}(t)\|_{i,j=1}^m\,, & K(t) &= \|k_{ij}(t)\|_{i=1,j=1}^{m\ \ h}\,, \\ B(t) &= \|b_{ij}(t)\|_{i=1,j=1}^{m\ \ r}\,, & R(t) &= \|r_{ij}(t)\|_{i,j=1}^m\,, \\ S(t) &= \|s_{ij}(t)\|_{i,j=1}^h\,, & t &\in [0, T]\,, \quad T < \infty\,, \end{aligned}$$

be Borel measurable matrix functions, $R(t)$ nonnegatively definite, $S(t)$ uniformly positively definite, $t \in [0, T]$. The elements of A, R are assumed to be integrable, those of K, B quadratically integrable on $[0, T]$. Let $L(T)$ be a nonnegatively definite matrix function.

Denote by $P(t) = \|p_{ij}(t)\|_{i,j=1}^m$, $t \in [0, T]$, the solution of the matrix Riccati equation

$$-\frac{\mathrm{d}}{\mathrm{d}t} P = A(t)'\, P + PA(t) + R(t) - PK(t)\, S(t)^{-1}\, K(t)'\, P\,, \quad t \in [0, T]\,,$$

with terminal condition $P(T) = L(T)$. Then

$$v(t, \bar{x}) = \bar{x}' P(t)\, \bar{x} + \int_t^T \operatorname{tr}\,(B(s)\, B(s)'\, P(s))\,\mathrm{d}s\,, \quad (t, \bar{x}) \in [0, T] \times R^m\,,$$

satisfies

$$\min_{\bar{u}\in R^h}\left\{\frac{\partial}{\partial t}v+\tfrac{1}{2}\operatorname{tr}\left(B(t)\,B(t)'\,(\nabla\nabla' v)\right)+\left(A(t)\,\bar{x}+K(t)\,\bar{u}\right)'\,\nabla v+\bar{x}'R(t)\,\bar{x}+\bar{u}'S(t)\,\bar{u}\right\}=0\,,$$

$$(t,\bar{x})\in[0,T]\times R^m\,,\quad v(T,\bar{x})=\bar{x}'L(T)\,\bar{x}\,,\quad \bar{x}\in R^m\,.$$

$\nabla=(\partial/\partial x^1,\ldots,\partial/\partial x^m)'$, and tr means trace. The expression in curly brackets attains its minimum for

$$\bar{u}=-\tfrac{1}{2}S(t)^{-1}\,K(t)'\,\nabla v\,.$$

79. Let $\bar{X}=\{({}^1X_t,\ldots,{}^mX_t)',\ t\in[0,T]\}$, $E|\bar{X}_0|^2<\infty$, satisfy

$$\mathrm{d}\bar{X}_t=A(t)\,\bar{X}_t\,\mathrm{d}t+K(t)\,\bar{U}_t\,\mathrm{d}t+B(t)\,\mathrm{d}\bar{W}_t\,,\quad t\in[0,T]\,,$$

where $\bar{U}=\{({}^1U_t,\ldots,{}^hU_t)',\ t\in[0,T]\}$ is nonanticipative.

Denote $E\bar{X}_0=\bar{e}_0$, $E(\bar{X}_0-\bar{e}_0)(\bar{X}_0-\bar{e}_0)'=Q_0$. Introduce

$$Z_T=\bar{X}_T'L(T)\,\bar{X}_T+\int_0^T\bar{X}_t'R(t)\,\bar{X}_t\,\mathrm{d}t+\int_0^T\bar{U}_t'S(t)\,\bar{U}_t\,\mathrm{d}t\,.$$

Prove that

(27) $$EZ_T\geqq\bar{e}_0'P(0)\,\bar{e}_0+\operatorname{tr}\left(P(0)\,Q_0\right)+\int_0^T\operatorname{tr}\left(B(s)\,B(s)'\,P(s)\right)\mathrm{d}s\,,$$

and that equality in (27) holds for

$$\bar{U}_t=-S(t)^{-1}\,K(t)'\,P(t)\,\bar{X}_t\,,\quad t\in[0,T]\,.$$

80. Let the hypotheses of Ex. 79 hold. Moreover, let matrices A, K, B, R, S, and L be constant. Assume that the stationary Riccati equation

$$0=A'P+PA+R-PKS^{-1}K'P$$

has solution P_∞, and denote $g=\operatorname{tr}(BB'P_\infty)$.

a) Prove that for each control $\bar{U}=\{\bar{U}_t,\ t\in[0,T]\}$

$$EZ_T\geqq gT+E\bar{X}_T'(P_\infty+L)\,\bar{X}_T-\bar{e}_0'P_\infty\bar{e}_0-\operatorname{tr}(P_\infty Q_0)\,.$$

b) If all characteristic roots of $A-KS^{-1}K'P_\infty$ have negative real parts, then for

(28) $$\bar{U}_t=-S^{-1}K'P_\infty\bar{X}_t\,,\quad t\in[0,T]\,,$$

holds

(29) $$EZ_T=gT+O(1)\quad\text{as}\quad T\to\infty\,.$$

(Because of (29), $\bar{U}$ defined as in (28) is an *optimal stationary control*.)

81. Let in Ex. 80

$$A = \begin{pmatrix} -1, & 1 \\ -1, & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} \sqrt{2}, & 0 \\ 0, & \sqrt{2} \end{pmatrix},$$

$$R = \begin{pmatrix} \frac{1}{2}, & 0 \\ 0, & 0 \end{pmatrix}, \quad S = (\tfrac{1}{2}).$$

Determine the optimal stationary control (28), and the corresponding mean value g. □

82. Replace A in Ex. 81 by

$$A = \begin{pmatrix} -1, & 1 \\ 0, & -1 \end{pmatrix}.$$ □

6. SOLUTIONS

2. T, $T^2/2$, $1 \,.\, 3 \ldots (2n-1)\, T^{n+1}/(n+1)$, $n = 2, 3, \ldots$; $\frac{1}{2}(e^{2T} - 1)$.

15.
$$a^1(\bar{x}) = 0\,, \quad a^2(\bar{x}) = \tfrac{1}{2} f''(x^1)\,,$$
$$b^1(\bar{x}) = 1\,, \quad b^2(\bar{x}) = f'(x^1)\,.$$

22.
$$\frac{\partial b(t,u)}{\partial t} + a(t,u)\frac{\partial b(t,u)}{\partial x} + \tfrac{1}{2} b^2(t,u)\frac{\partial^2 b(t,u)}{\partial x^2} = b(t,u)\frac{\partial a(t,u)}{\partial x}\,.$$

23.
$$X_t = (\sqrt{x_0} + \tfrac{1}{2} W_t)^2\,.$$

24.
$$X_t = \operatorname{tg}(W_t + t + \operatorname{arctg} x_0)\,.$$

25.
$$X_t = 2 \operatorname{arctg}\left[\left(\operatorname{tg}\frac{x_0}{2}\right) e^{W_t}\right].$$

26.
$$X_t = (1+t)^2\,[x_0 + a(W_t - t)]\,.$$

29.
$$f(t,x) = \int_0^x \frac{dy}{b(t,y)} + \text{const}\,.$$

30.
$$f(x) = C_1 + C_2 \int_0^x \exp\left\{-\int_0^y \frac{2a(u)}{b^2(u)}\,du\right\} dy\,,$$

where C_1, C_2 are constants.

31.
$$C_1 \int_0^x \frac{dy}{b(y)} + \frac{a(x)}{b(x)} - \tfrac{1}{2} b'(x) = C_2\,,$$

where C_1, C_2 are constants;

$$f(t,x) = e^{C_1 t}\int_0^x \frac{dy}{b(y)}.$$

34. $$Y_t = \int_0^t e^{c(t-s)}\left(b\cos(t-s) - a\sin(t-s)\right) dW_s .$$

38.

(4) $d^*X_t = -\sin^2 X_t\, dt - \sin^2 X_t\, d^*W_t$,

(7) $d^*X_t = \sqrt{X_t}\, d^*W_t$,

(8) $d^*X_t = (1 + X_t^2)\, dt + (1 + X_t^2)\, d^*W_t$,

(9) $d^*X_t = \sin X_t\, d^*W_t$.

42. $$EX_t = 0\,;\quad E\int_0^t X_s\, ds = 0\,;\quad EX_uX_t = \frac{b^2 l}{2} e^{-l(t-u)},\quad t \geqq u\,;$$

$$EX_u\int_0^t X_s\, ds = \frac{b^2 e^{-lu}}{2}(e^{lt} - 1),\quad u > t,$$

$$= \frac{b^2}{2}(2 - e^{-lu} - e^{-l(t-u)}),\quad t \geqq u\,;$$

$$E\int_0^u X_s\, ds\int_0^t X_s\, ds = \frac{b^2}{2}\left(2u + \frac{e^{-lu} + e^{-lt} - e^{-l(t-u)} - 1}{l}\right),\quad t \geqq u\,.$$

43. $$F(t) = \begin{pmatrix} (1-t)e^{-2t}, & t e^{-2t}, & -t e^{-2t} \\ (1-t)e^{-2t} - e^{-t}, & e^{-t} + t e^{-2t}, & -t e^{-2t} \\ e^{-2t} - e^{-t}, & e^{-t} - e^{-2t}, & e^{-2t} \end{pmatrix},$$

$$\bar{e}(t) = F(t)\,\bar{e}(0),$$

$$Q(0) = \begin{pmatrix} \frac{333}{8.161}, & \frac{843}{24.161}, & \frac{-111}{12.161} \\ \frac{843}{24.161}, & \frac{697}{8.161}, & \frac{123}{12.161} \\ \frac{-111}{12.161}, & \frac{123}{12.161}, & \frac{50}{161} \end{pmatrix}.$$

44. $$\bar{e}(t) = \begin{pmatrix} 0 \\ \frac{e^2(0) - e^3(0)}{2} \\ \frac{e^3(0) - e^2(0)}{2} \end{pmatrix} e^{-t} + \begin{pmatrix} e^1(0) \\ \frac{e^2(0) + e^3(0)}{2} \\ \frac{e^2(0) + e^3(0)}{2} \end{pmatrix} e^t\,;$$

$$q_{11}(t) = (q_{11}(0) + \tfrac{1}{2})\,e^{2t} - \tfrac{1}{2},$$
$$q_{12}(t) = \tfrac{1}{2}(q_{12}(0) - q_{13}(0) + e^{2t}(q_{12}(0) + q_{13}(0))),$$
$$q_{13}(t) = \tfrac{1}{2}(q_{13}(0) - q_{12}(0) + e^{2t}(q_{12}(0) + q_{13}(0))),$$
$$q_{22}(t) = q_{33}(t) + q_{22}(0) - q_{33}(0),$$
$$q_{33}(t) = \frac{q_{22}(0) - 2q_{23}(0) + q_{33}(0) - 1}{4}\,e^{-2t} + \frac{q_{22}(0) + 2q_{23}(0) + q_{33}(0) + 1}{4}\,e^{2t} - \frac{q_{22}(0) - q_{33}(0)}{2},$$
$$q_{23}(t) = \frac{-q_{22}(0) + 2q_{23}(0) - q_{33}(0) + 1}{4}\,e^{-2t} + \frac{q_{22}(0) + 2q_{23}(0) + q_{33}(0) + 1}{4}\,e^{2t} - \tfrac{1}{2}.$$

45. $$E[({}^1X_t)^2 + ({}^2X_t)^2] = E[({}^1X_0)^2 + ({}^2X_0)^2] + b^2 t.$$

46. $$\bar{e}(t) = e^{-lt}\begin{pmatrix}\cos kt, & -\sin kt\\ \sin kt, & \cos kt\end{pmatrix}\bar{e}(0);$$

$$Q(t) = \begin{pmatrix}q_{11}(t), & q_{12}(t)\\ q_{21}(t), & q_{22}(t)\end{pmatrix}, \quad q_{12}(t) = q_{21}(t),$$

$$\begin{pmatrix}q_{11}(t)\\ q_{22}(t)\\ q_{12}(t)\end{pmatrix} = e^{-2lt}\begin{pmatrix}\cos^2 kt, & \sin^2 kt, & -\sin 2kt\\ \sin^2 kt, & \cos^2 kt, & \sin 2kt\\ \tfrac{1}{2}\sin 2kt, & -\tfrac{1}{2}\sin 2kt, & \cos 2kt\end{pmatrix}\begin{pmatrix}q_{11}(0)\\ q_{22}(0)\\ q_{12}(0)\end{pmatrix} + \begin{pmatrix}\frac{b^2}{2l}(1 - e^{-2lt})\\ \frac{b^2}{2l}(1 - e^{-2lt})\\ 0\end{pmatrix}.$$

47. $$\bar{e}(t) = \begin{pmatrix}1, & t, & (1 - e^{-at})/a\\ 0, & 1, & 0\\ 0, & 0, & e^{-at}\end{pmatrix}\bar{e}(0);$$

$$q_{11}(t) = q_{11}(0) + 2q_{12}(0)\,t + q_{13}(0)\frac{2}{a}(1 - e^{-at}) + q_{22}(0)\,t^2 + \frac{b^2}{a^2}t + \left(q_{33}(0) - \frac{3b^2}{2a}\right)\frac{1}{a^2} - \left(q_{33}(0) - \frac{b^2}{a}\right)\frac{2}{a^2}e^{-at} + \left(q_{33}(0) - \frac{b^2}{2a}\right)\frac{1}{a^2}e^{-2at},$$

$$q_{12}(t) = q_{12}(0) + q_{22}(0)\,t\,;$$

$$q_{13}(t) = q_{13}(0)\,e^{-at} + \frac{b^2}{2a^2} + \left(q_{33}(0) - \frac{b^2}{a}\right)\frac{1}{a}\,e^{-at} -$$

$$- \left(q_{33}(0) - \frac{b^2}{2a}\right)\frac{1}{a}\,e^{-2at},$$

$$q_{22}(t) = q_{22}(0),$$

$$q_{23}(t) = 0,$$

$$q_{33}(t) = \left(q_{33}(0) - \frac{b^2}{2a}\right)e^{-2at} + \frac{b^2}{2a}\,;$$

$$R(u, t) = \begin{pmatrix} 1, & u - t, & (1 - e^{-a(u-t)})/a \\ 0, & 1, & 0 \\ 0, & 0, & e^{-a(u-t)} \end{pmatrix} Q(t), \quad u \geqq t\,.$$

50. $$\hat{X}_t = (e_0 + q_0 Y_t)/(1 + q_0 t)\,.$$

51. $$k(t) = \frac{(\alpha g^2 + q_0 - a g^2)(a + \alpha)(e^{2\alpha t} - 1) + 2\alpha q_0}{(\alpha g^2 + q_0 - a g^2)(e^{2\alpha t} - 1) + 2\alpha},$$

$$\alpha = \sqrt{(a^2 + b^2/g^2)}\,.$$

52. $$R = \begin{pmatrix} \sqrt{(1 + 2b)}, & b \\ b, & b\sqrt{(1 + 2b)} \end{pmatrix},$$

$$K = \begin{pmatrix} \sqrt{(1 + 2b)} \\ b \end{pmatrix}.$$

53. $$R = \begin{pmatrix} \sqrt{2} - 1, & 0 \\ 0, & \sqrt{2} - 1 \end{pmatrix},$$

$$K = \begin{pmatrix} \sqrt{2} - 1 \\ 0 \end{pmatrix}.$$

54. $$a = \pm\sqrt{7}\,.$$

62. $$C = A - \tfrac{1}{2}\sum_{k=1}^{r} {}^{k}B^2\,.$$

66. $$L = \frac{\partial}{\partial t} + l x^2 \frac{\partial}{\partial x^1} + \left[\left(\frac{b^2}{2} - k\right)x^2 - l x^1\right]\frac{\partial}{\partial x^2} + \frac{b^2}{2}(x^2)^2 \frac{\partial^2}{(\partial x^2)^2}\,.$$

72. $$V = \begin{pmatrix} \dfrac{2}{k - b^2} + \dfrac{1}{l^2}\left(k - \dfrac{b^2}{2}\right), & \dfrac{1}{l} \\ \dfrac{1}{l} & , \dfrac{2}{k - b^2} \end{pmatrix}.$$

74. $$u(t, x) = \frac{-ax}{b^2 + a^2 T}, \quad t \in [0, T].$$

75. $$u(t) = -\frac{a}{2b^2}, \quad t \in [0, T].$$

77. $$p(t) = k^{-2} \frac{(\alpha + \alpha l k^2 - a)(a + \alpha)(\exp\{2\alpha(T - t)\} - 1) + 2\alpha l k^2}{(\alpha + \alpha l k^2 - a)(\exp\{2\alpha(T - t)\} - 1) + 2\alpha k^2},$$

$$\alpha = \sqrt{(a^2 + rk^2)}.$$

78. $$EZ_T = \frac{b^2(a + \alpha)}{k^2} T + l \frac{b^2}{2\alpha} + \left(q_0 + e_0^2 - \frac{b^2}{2\alpha}\right).$$

$$\cdot \left(l\,\mathrm{e}^{-2\alpha T} + \frac{a + \alpha}{k^2}(1 - \mathrm{e}^{-2\alpha T})\right),$$

$$\alpha = \sqrt{(a^2 + rk^2)}.$$

81. $$\bar{U}_t = (1 - \sqrt{2}, 0)\, \bar{X}_t,$$

$$g = 2(\sqrt{2} - 1).$$

82. $$\bar{U}_t = (1 - \sqrt{2}, 2\sqrt{2} - 3)\, \bar{X}_t,$$

$$g = 5\sqrt{2} - \frac{13}{2}.$$

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