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On a Detection Method for Known Finite Sequences

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A method of detection of known finite-sequence signals is investigated possessing the CFAR (constant false alarm rate) property.

1. INTRODUCTION

Let a signal be represented by a real finite sequence $\{b_0, \ldots, b_h\}$, h given, $b_0 \neq 0$, $b_h \neq 0$, let $\{a_0, \ldots, a_N\}$ $(N \geqq h,\ a_0 \neq 0,\ a_N \neq 0)$ be the weighting sequence of the respective inversion filter (e.g. matched or minimum mean square error, [1]), let T $(0 \leqq T \leqq N + h)$ be given [1]. Then using in the receiver a simple averaging CFAR circuit is not justified.

In what follows, another method of CFAR detection will be investigated.

2. FUNDAMENTAL RELATIONS

At the receiver, the sequence $\{b_0, \ldots, b_h\}$ is corrupted by noise, so that it is $\{y_0, \ldots, y_h\}$.

Let the output of the inversion filter in the ideal case be

$$\begin{aligned} \alpha c_0 &= \alpha b_0 a_0\,, \\ \alpha c_1 &= \alpha(b_1 a_0 + b_0 a_1)\,, \\ &\vdots \end{aligned} \tag{1}$$

α being an arbitrary coefficient. In the nonideal case, there is

$$C_0 = y_0 a_0\,, \tag{2}$$

$$C_1 = y_1 a_0 + y_0 a_1 ,$$
$$\vdots$$

Now, one will seek α so that

$$\sum_{i=0}^{N+h} (\alpha c_i - C_i)^2 = \min . \tag{3}$$

One finds

$$\alpha = \frac{\sum_{i=0}^{N+h} c_i C_i}{\sum_{i=0}^{N+h} c_i^2} \tag{4}$$

and from (3), (4) one obtains

$$\min_\alpha \frac{\sum(\alpha c_i - C_i)^2}{\sum C_i^2} = 1 - \frac{(\sum c_i C_i)^2}{\sum c_i^2 \cdot \sum C_i^2} \geqq 0 \tag{5}$$

or

$$1 \geqq \frac{(\sum c_i C_i)^2}{\sum c_i^2 \cdot \sum C_i^2} \geqq 1 - \frac{\sum(\alpha c_i - C_i)^2}{\sum C_i^2} . \tag{6}$$

The left inequality follows immediately from the Cauchy inequality. Moreover, there follows from the same inequality that the equality holds precisely if $C_i = \beta c_i$ $(i = 0, \ldots, N + h)$, $\beta \neq 0$ being arbitrary.

Thus, one sees that the middle term in (6) can be used for detection and that it possesses the CFAR property.

3. SOME SIMPLIFYING SUPPOSITIONS

In what follows we will suppose that for a given T, $c_T \gg c_i (i \neq T)$. E.g. for the minimum mean square inversion filter, there is [1]

$$\sum_{i=0}^{N+h} c_i^2 = c_T \tag{7}$$

and $c_T \doteq 1$.

With this supposition, we neglect in (6) all c_i's for $i \neq T$, and obtain

$$0 \leqq \frac{C_T^2}{\sum_{i=0}^{N+h} C_i^2} \leqq 1 . \tag{8}$$

This expression is much simpler than that in (6), but it may be expected to retain its properties.

Suppose further that the input noise of the inversion filter is a white Gaussian sequence $N(0, \sigma)$. The output sequence of the filter is also Gaussian but not white. Denoting it $\{\eta_n\}$, one finds the autocorrelation

$$(9) \qquad \varrho(\eta_n \eta_{n+k}) = \frac{E(\eta_n \eta_{n+k})}{E(\eta_n)^2} = \frac{a_0 a_k + \ldots + a_{N-k} a_N}{\sum_{i=0}^{N} a_i^2} .$$

One will now suppose that a "good" sequence $\{b_0, \ldots, b_h\}$ is "practically uncorrelated". Then, the sequence $\{a_0, \ldots, a_N\}$ is also "practically uncorrelated". Thus from (9), the output of the inversion filter is "practically independent".

4. THE DISTRIBUTION OF $C_T^2/\sum C_i^2$ UNDER SIMPLIFYING SUPPOSITIONS

There is

$$(10) \qquad C_T^2/\sum C_i^2 < Z < 1$$

the same as

$$(11) \qquad C_T^2/(C_0^2 + \ldots + C_{T-1}^2 + C_{T+1}^2 + \ldots + C_{N+h}^2) < Z/(1 - Z),$$

Z being a threshold. For the noise alone, the random variables in (11) are independent $N(0, \sigma)$. Then, (11) is equivalent to

$$(12) \qquad \chi_1^2(1)/\chi_2^2(N + h) < Z/(1 - Z),$$

that is to

$$(13) \qquad \mathscr{F}(1, N + h) < (N + h)\, Z/(1 - Z),$$

where the random variable at the left posseses the F-distribution. Denoting γ the critical value to P_f, a given probability of false alarm, one obtains for the threshold Z

$$(14) \qquad Z = \gamma/(\gamma + N + h).$$

Further, supposing that the noise is additive, one has in the case of signal plus noise:

$$(15) \qquad C_T/\sigma \,.\, \sqrt{\sum a_i^2} \quad \text{is} \quad N(c_T/\sigma \,.\, \sqrt{\sum a_i^2}, 1)$$

and for $i \neq T$

$$(16) \qquad C_i/\sigma \,.\, \sqrt{\sum a_i^2} \quad \text{is} \quad N(0, 1).$$

Thus, the probability $1 - P_d$ of missing the signal is, for the given Z,

$$(17)\quad 1 - P_d = P\left(C_T^2 < \frac{Z}{1-Z}\left(C_0^2 + \ldots + C_{T-1}^2 + C_T^2 + \ldots + C_{N+h}^2\right)\right).$$

For this probability, no standard tables exist.

Let an arbitrary $y > 0$ be given. Then, clearly, the probability density of the second member of the inequality in (17) is

$$(18)\qquad f(y) = \frac{1-Z}{Z} \cdot \frac{\left(\frac{1-Z}{Z} \cdot y\right)^{((N+h)/2)-1} \cdot \mathrm{e}^{-(1-Z)y/2Z}}{2^{((N+h)/2)} \cdot \Gamma\left(\frac{N+h}{2}\right)}.$$

Further

$$(19)\qquad P\left(\frac{C_T^2}{\sigma^2 \sum a_i^2} < y\right) = g(y) = \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-t^2/2}\, \mathrm{d}t$$

where the integration interval at the right is

$$(20)\qquad \left\langle -\sqrt{y} - \frac{c_T}{\sigma \sqrt{\sum a_i^2}},\ \sqrt{y} - \frac{c_T}{\sigma \sqrt{\sum a_i^2}} \right\rangle.$$

Finally,

$$(21)\qquad 1 - P_d = \int_0^\infty f(y)\, g(y)\, \mathrm{d}y$$

have been computed by numerical integration for various values of the input signal/noise ratio

$$(22)\qquad s/n = \sum_{i=0}^{h} b_i^2 / \sigma^2 (h+1).$$

5. EXAMPLES

Example 1. Let $\{b_0, b_1, b_2\} = \{1, 1, -1\}$ be the known Barker sequence. Let

$$(23)\qquad \{a_0, \ldots, a_6\} = \{-1, 1, -2, 3, 2, 1, 1\}$$

and thus

$$(24)\qquad \{c_0, \ldots, c_8\} = \{-1, 0, 0, 0, 7, 0, 0, 0, -1\}.$$

From (9)

$$(25)\qquad \{\varrho_0, \ldots, \varrho_6\} = \{1; 0; 0{\cdot}3; 0; -0{\cdot}15; 0; -0{\cdot}05\},$$

so that we adopt the supposition of independence of C_i's.

Let $P_f = 0{\cdot}01$ be justified by the subsequent second-threshold evaluation of repeated signal sequences. Then, from the table of the F-distribution [2]

(26) $$Z = \frac{11{\cdot}3}{11{\cdot}3 + 8} = 0{\cdot}58\,.$$

Now, the results of computing (21) are in Table 1.

Table 1.

s/n(dB)	σ^2	$1 - P_d$	P_d	P_{dc}
0	1	0·91	0·09	0·20
3	$\frac{1}{2}$	0·79	0·21	0·45
6	$\frac{1}{4}$	0·55	0·45	0·81
9	$\frac{1}{8}$	0·20	0·80	0·99
12	$\frac{1}{16}$	0·01	0·99	1·00

The sense of the last column will be explained later.

Example 2. Let

(27) $$\{b_0, \ldots, b_{10}\} = \{1, 1, 1, 1, 1, -1, -1, 1, -1, 1, -1\}$$

be the known Golay-Schroeder sequence. Let for the corresponding inversion filter (normalized $\times$ 1000)

(28) $$\{a_0, \ldots, a_{30}\} = \{-7, 0, -2, 22, -14, -8, -13, 22, 25, -5, -87, 62, -18, 139, -118, -154, 118, 139, 18, 62, 87, -5, -25, 22, 13, -8, 14, 22, 2, 0, 7\}\,.$$

One may find $c_{20} = 0{\cdot}999$, other c_i's will not be published here, but (normalizing $\times$ $\times$ 100)

(29) $$\{\varrho_0, \ldots, \varrho_{30}\} = \{100, 0, -32, 0, 17, 0, 0, 0, 2, 0, 8, 0, -4, 0, 2, \text{all 0's}\}\,.$$

Further,

(30) $$Z = \frac{7{\cdot}31}{7{\cdot}31 + 40} = 0{\cdot}154$$

and from (28), $\sum a_i^2 = 0{\cdot}11788$. As in the preceding example, Table 2 has been computed:

Table 2.

s/n(dB)	σ^2	$1 - P_d$	P_d	P_{dc}
-6	4	0·88	0·12	0·18
-3	2	0·72	0·28	0·41
0	1	0·41	0·59	0·77
3	$\frac{1}{2}$	0·08	0·92	0·98
6	$\frac{1}{4}$	0·00	1·00	1·00

The values in the last columns in both tables have been computed for a pulse signal of the same height and length as the respective sequences, for $P_f = 0{\cdot}01$ and a fixed threshold, and for the absolute value of signal plus Gaussian noise exceeding the threshold.

The loss of the signal/noise ratio of both inversion filters compared with the matched ones is about 1·1 dB, so that the net loss of the detection method (computing with the simplifying suppositions) is about 2 dB in the first example and about 0·3 dB in the second one.

6. RESULTS OF SIMULATION

To see the influence of the simplifying suppositions, the detection with the aid of (8) has been simulated for both examples of the preceding section. The frequencies of 100 experiments are contained in Tables 3 and 4 for various signal/noise ratios.

Table 3.

	s/n(dB)					
	$-\infty$	0	3	6	9	12
⟨0 −0·20⟩	82	43	24	3	0	0
(0·20−0·40⟩	14	34	32	20	4	0
(0·40−0·60⟩	3	18	25	39	18	1
(0·60−0·80⟩	1	4	19	33	55	46
(0·80−1⟩	0	1	0	5	23	53

Table 4.

	s/n(dB)					
	$-\infty$	-6	-3	0	3	6
⟨0 −0·15⟩	100	79	63	36	7	0
(0·15−0·30⟩	0	19	31	42	38	7
(0·30−0·45⟩	0	2	6	20	48	36
(0·45−0·60⟩	0	0	0	2	7	50
(0·60−0·75⟩	0	0	0	0	0	7
(0,75−1⟩	0	0	0	0	0	0

The frequencies above 0·60 in the first case and above 0·15 in the second one show a very good agreement of simplified calculation and simulation in the first case and a good agreement (on the 95% confidence level) in the second one.

7. ANOTHER WAY HOW TO ARRIVE AT THE MATCHED FILTER

Consider the situation where in contrast to the introduction no inversion filter is used and instead of (3) one postulates

$$\sum_{i=0}^{h}(\alpha b_i - y_i)^2 = \min . \tag{31}$$

One finds easily that instead of the expression in the middle of (6) the expression

$$0 \leqq \frac{(\sum b_i y_i)^2}{\sum b_i^2 \sum y_i^2} \leqq 1 \tag{32}$$

may be used for detection. One sees that the sum in the numerator is formed from $\{y_i\}$ by a linear filter with the weighting sequence

$$\{w_{h-i}\} = \{b_i\} \tag{33}$$

i.e. by the matched filter. The inequalities in (32) follow from the Cauchy inequality.

Moreover, with the normalization $\sum b_i^2 = 1$,

$$\varrho(\sum b_i y_i, y_j) = b_j , \tag{34}$$

so that the mathematical treatment of (32) seems to be somewhat more difficult than that of (8).

8. CONCLUDING REMARKS

For longer sequences, the method described in the preceding sections may be modified, e.g. by considering only a part of summands in the denominator in (8). An extension to complex sequences is straightforward, with a generalized Siebert CFAR detector resulting.

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