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Additivity in General Coalition-Games

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The concept of the general-coalition-game was introduced in [3] and [4] without the super-addivity or additivity assumption. However, the importance of that assumptions in the classical coalition-games theory is so great that it is useful to investigate their sense and consequences in the general coalition-game model, too. This is done in the presented paper, where their influence on the general coalition-game model and, especially, on the stability of solution of such game is investigated. Moreover, the concept of subadditivity is introduced and studied here.

0. INTRODUCTION

The concept of the general coalition-game, introduced in [3] and further investigated in [4], includes a relatively wide scale of more special cooperative and coalition games, due to various applications. For some of those special games, known from the literature, the superadditivity assumption is usual and natural. For some other, this assumption is neither necessary nor desirable, as it unconveniently limits the range of their possible applications. It is the main reason why the superadditivity of pay-offs was not generally assumed in the model presented in [3].

However, as the superadditivity and additivity assumptions are important in so many classical game models investigated in the known literature (namely in coalition-games with side-payments and in games with mixed strategies mentioned in [3] and well known e.g. from [1], [6] or [7]) it is useful to study their sense in the considered general game model, too. Moreover, even the subadditivity assumption is useful in some (usually economical) applications.

It means that it could be useful to mention, at least briefly, the influence of different forms of additivity on the general coalition-game model, especially on the existence and mutual connections between the strongly and dynamicaly stable solutions of such game. It is not so difficult to see that the classes of (strongly or dynamicaly) stable coalition structures will be influenced to a greater extent. The consequences of the additivity assumptions for the classes of stable configurations (which are derived from the stable coalition structures) are closely connected with the analogous consequences for the coalition structures. In accordance with this fact, this paper is subjected to the properties of stable coalition structures in superadditive general coalition-games only, and the configurations are not considered here. The typical properties of configurations in such games may be easily derived from the properties of coalition structures presented in this paper, by means of methods and results introduced in [3] and [4].

1. THE CONSIDERED GAME MODEL

In this section, the definition of the general coalition-game given in [3] is briefly repeated, and a few of useful auxiliary notions, introduced also in [3] and [4] are reminded.

In the whole paper, the symbol R denotes the set of all real numbers.

Let us suppose that I is a non-empty and finite set, and that $\mathscr{I}$ is the class of all non-empty subsets of I. Let us suppose, further, that there exists a mapping V from the class 2^I into the class of subsets of R^I, such that for all $K \in 2^I$

(1.1) $V(K)$ *is closed;*

(1.2) *if* $\mathbf{x} = (x_i)_{i\in I} \in V(K)$, $\mathbf{y} = (y_i)_{i\in I} \in R^I$ *and* $x_i \geqq y_i$ *for all* $i \in K$, *then* $\mathbf{y} \in V(K)$;

(1.3) $V(K) \neq \emptyset\,, \quad V(K) = R^I \Leftrightarrow K = \emptyset\,.$

Then the pair $\Gamma = (I, V)$ is called a *general coalition-game.* The elements of the sets I and $\mathscr{I}$ are called *players* and *coalitions*, respectively. The mapping V is the *general characteristic function* of the game Γ.

Any partition of the set I into disjoint subsets is called a *coalition structure.* The class of all coalition structures in the given game will be denoted by $\mathbf{K}$. If $\mathscr{K} \in \mathbf{K}$, $\mathscr{L} \in \mathbf{K}$ then $\mathscr{K}$ is called a *subpartition* of $\mathscr{L}$ iff for every coalition $K \in \mathscr{K}$ there exists a coalition $L \in \mathscr{L}$ such that $K \subset L$. If $\mathbf{M} \subset \mathbf{K}$ is a class of coalition structures then the symbol $\bigcup \mathbf{M}$ denotes the set of coalitions

$$\bigcup \mathbf{M} = \{M \in \mathscr{I} \colon M \in \mathscr{M} \textit{ for some } \mathscr{M} \in \mathbf{M}\}\,.$$

If $K \in \mathscr{I}$ is a coalition then the set

$$\begin{aligned}(1.4)\quad V^*(K) &= \{\mathbf{x} = (x_i)_{i\in I} \colon \textit{if } \mathbf{y} = (y_i)_{i\in I} \in R^I,\ y_i \geqq x_i \textit{ for all } i \in K \textit{ and } y_j > x_j \\ &\quad \textit{for some } j \in K \textit{ then } \mathbf{y} \notin V(K)\} = \\ &= \{\mathbf{x} = (x_i)_{i\in I} \colon \textit{for all } \mathbf{z} = (z_i)_{i\in I} \in V(K) \textit{ is either } x_i > z_i \textit{ for some} \\ &\quad i \in K \textit{ or } x_i = z_i \textit{ for all } i \in K\}\end{aligned}$$

is called the *superoptimum* of the coalition K.

If $\mathscr{M} \subset \mathscr{I}$ is a non-empty set of coalitions then

$$V(\mathscr{M}) = \bigcap_{M \in \mathscr{M}} V(M), \quad V^*(\mathscr{M}) = \bigcap_{M \in \mathscr{M}} V^*(M).$$

For the empty set of coalitions $\emptyset \subset \mathscr{I}$ it is, by (1.3),

$$V(\emptyset) = V^*(\emptyset) = R^I. \tag{1.5}$$

It was shown in [3] that for every coalition $K \in \mathscr{I}$ and for every coalition structure $\mathscr{K} \in \mathbf{K}$ is

$$V(K) \cup V^*(K) = R^I, \tag{1.6}$$

and

$$V(K) \cap V^*(K) \neq \emptyset, \quad V(\mathscr{K}) \cap V^*(\mathscr{K}) \neq \emptyset. \tag{1.7}$$

Every real valued vector $\mathbf{x} \in R^I$ is called an *imputation*. If there exists a coalition structure $\mathscr{K} \in \mathbf{K}$ such that $\mathbf{x} \in V(\mathscr{K})$ then $\mathbf{x}$ is an *admissible imputation*.

The general properties of the notions introduced above were presented in [3] and, partially, also in [4]. The following statements describe two of them and introduce two further properties, useful for the derivation of some results presented in the following sections.

Lemma 1. If $\mathbf{x} = (x_i)_{i \in I} \in R^I$, $\mathbf{y} = (y_i)_{i \in I} \in R^I$, $x_i \leqq y_i$ for all $i \in I$, and if $K \in \mathscr{I}$, then $\mathbf{x} \in V^*(K) \Rightarrow \mathbf{y} \in V^*(K)$, and $\mathbf{y} \in V(K) \Rightarrow \mathbf{x} \in V(K)$.

The preceding result was introduced also in [4], and partially in [3] as Remark 1. The following lemma was proved in [3] as Lemma 1.

Lemma 2. If $K \in \mathscr{I}$ and $\mathbf{x} = (x_i)_{i \in I} \in V(K) - V^*(K)$ is internal point of $V(K)$ then there exists $\mathbf{y} = (y_i)_{i \in I} \in V(K) \cap V^*(K)$ such that $y_i > x_i$ for all $i \in I$.

Lemma 3. If $K, L \in \mathscr{I}$ are coalitions such that $K \cap L = \emptyset$ then

$$V(K) \cap V(L) = [V(K) \cap R^K] \times [V(L) \cap R^L] \times R^{I-(K \cup L)},$$

$$V^*(K) \cap V^*(L) = [V^*(K) \cap R^K] \times [V^*(L) \cap R^L] \times R^{I-(K \cup L)}.$$

Proof. Let $\mathbf{x} = (x_i)_{i \in I} \in R^I$. Then $\mathbf{x} \in V(K) \cap V(L)$ if and only if $(x_i)_{i \in K} \in V(K) \cap R^K$ and $(x_i)_{i \in L} \in V(L) \in R^L$. The same is true for V^*, as follows from (1.5), (1.4) and from the assumption of disjointness of K and L.

Lemma 4. Let $\mathscr{K}, \mathscr{L} \in \mathbf{K}$ be coalition structures and let $\mathscr{K}$ be a subpartition of $\mathscr{L}$ such that $V(\mathscr{K}) \supset V(\mathscr{L})$. Then

$$V^*(\mathscr{K}) \subset V^*(\mathscr{L}).$$

Proof. If $\mathbf{x} \in V^*(\mathscr{K})$ then for all coalitions $K \in \mathscr{K}$ is $\mathbf{x} \in V^*(K)$. It means that for all $K \in \mathscr{K}$ and all $\mathbf{y} \in V(K)$ is either $y_i \leqq x_i$ for all $i \in K$, or $y_j < x_j$ for some $j \in K$. Every coalition $L \in \mathscr{L}$ is a union of some coalitions from $\mathscr{K}$, let us denote $L = K_1 \cup \ldots \cup K_r$. Then the assumed inclusion $V(\mathscr{K}) \supset V(\mathscr{L})$ implies that

$$V(L) \subset \bigcap_{s=1}^{r} V(K_s),$$

as the coalitions in any coalition structure are disjoint. Then for all imputations $\mathbf{z} \in V(L)$ is either $x_i \geqq z_i$ for all $i \in L$, or there exists $j \in L$ such that $x_j > z_j$. Consequently, $\mathbf{x} \in V^*(L)$, and the procedure described above may be used for all coalitions $L \in \mathscr{L}$. It means that $\mathbf{x} \in V^*(\mathscr{L})$.

2. AUXILIARY NOTIONS

In this section, the definitions of a few auxiliary notions introduced in [4] are briefly repeated, and their properties which are necessary for the further explanation are mentioned.

A coalition $K \in \mathscr{I}$ is called *effective* iff

$$V(K) \cap \bigcap_{J \in \mathscr{I}, J \subset K} V^*(J) \neq \emptyset .$$

A coalition structure $\mathscr{K} \in \mathbf{K}$ is called *effective from below* iff all coalitions in $\mathscr{K}$ are effective.

Further, $\mathscr{K} \in \mathbf{K}$ is called *effective from above* iff for every coalition structure $\mathscr{L}$, which is effective from below and such that $\mathscr{K}$ is a subpartition of $\mathscr{L}$, there exists an imputation $\mathbf{x} \in V(\mathscr{K})$ such that $\mathbf{x} \notin V(\mathscr{L}) - V^*(\mathscr{L})$.

A coalition structure $\mathscr{K} \in \mathbf{K}$ is called *effective* iff it is effective from below and effective from above.

The classes of coalition structures which are effective from below, effective from above and effective are denoted by $\mathbf{K}_{\text{ef}}$, $\mathbf{K}^{\text{ef}}$ and $\mathbf{K}_{\text{ef}}^{\text{ef}}$, respectively.

It was proved in [4] that there always exists an effective coalition structure in every general coalition-game. The following important result was proved in [4] as Lemma 3.

Lemma 5. If $L \in \mathscr{I}$ is not effective then there exists a set of effective coalitions

$$\mathscr{I}^*(L) = \{J\colon J \in \mathscr{I},\ J \subset L,\ J \textit{ is effective}\}$$

such that

$$V^*(L) \supset \bigcap_{J \in \mathscr{I}^*(L)} V^*(J) .$$

If $\mathscr{K} \in \mathbf{K}$ is a coalition structure and $\mathscr{M} \subset \mathscr{I}$ is a set of coalitions then we say that $\mathscr{K}$ *is safe against* $\mathscr{M}$ and write $\mathscr{K}\ \sigma\ \mathscr{M}$ iff $V(\mathscr{K}) \cap V^*(\mathscr{M}) \neq \emptyset$. If $\mathscr{K}$ is not safe against $\mathscr{M}$ then we write $\mathscr{K}$ non $\sigma\ \mathscr{M}$.

Lemma 6. If $\mathscr{K} \in \mathbf{K}$ is a coalition structure then

(1) for any $\mathscr{M} \subset \mathscr{I}$, exactly one of the relations $\mathscr{K}\ \sigma\ \mathscr{M}$ and $\mathscr{K}$ non $\sigma\ \mathscr{M}$ is true;

(2) if $\mathscr{K} \in \mathbf{K} - \mathbf{K}_{\text{ef}}$ then there exists a coalition $K \in \mathscr{K}$ and a set of coalitions $\mathscr{M} \subset \{J\colon J \in \mathscr{I},\ J \subset K,\ J \textit{ is effective}\}$ such that $\mathscr{K}$ non $\sigma\ \mathscr{M}$;

(3) if $\mathscr{K} \in \mathbf{K} - \mathbf{K}^{\text{ef}}$ then there exists $\mathscr{L} \in \mathbf{K}_{\text{ef}}^{\text{ef}}$ such that $\mathscr{K}$ is a subpartition of $\mathscr{L}$, $\mathscr{K}$ non $\sigma\ \mathscr{L}$ and $\mathscr{L}\ \sigma\ \mathscr{K}$;

(4) if $\mathscr{M} \subset \mathscr{N} \subset \mathscr{I}$ and $\mathscr{K}\ \sigma\ \mathscr{N}$ then $\mathscr{K}\ \sigma\ \mathscr{M}$;

(5) if $\mathscr{M} \subset \mathscr{I}$ and $\mathscr{K}\ \sigma\ \mathscr{M}$ then $\mathscr{K}\ \sigma\ (\mathscr{M} \cup \mathscr{K})$;

(6) if $\emptyset = \mathscr{M} \subset \mathscr{I}$ then $\mathscr{K}\ \sigma\ \mathscr{M}$.

The preceding lemma was proved in [4] as Lemma 5.

A mapping Δ from the class of coalition structures $\mathbf{K}$ into the family of the subclasses of the class of coalition structures which are effective from below,

$$\Delta\colon \mathbf{K} \to 2^{\mathbf{K}_{\text{ef}}},$$

such that for every $\mathscr{K} \in \mathbf{K}$ is

$$\Delta(\mathscr{K}) = \{\mathscr{J} \in \mathbf{K}_{\text{ef}}\colon \textit{there exists } \mathbf{M} \subset \mathbf{K}_{\text{ef}} \textit{ such that } \mathscr{K}\ \sigma\ (\bigcup \mathbf{M}) \textit{ and } \mathscr{K} \text{ non } \sigma\ (\bigcup \mathbf{M} \cup \mathscr{J})\}$$

is called a *domination structure* in the given general coalition-game.

It is obvious that for any $\mathscr{K} \in \mathbf{K}$ and $\mathscr{J} \in \mathbf{K}_{\text{ef}}$ is $\mathscr{J} \in \Delta(\mathscr{K})$ iff there exists a class of coalition structures $\mathbf{M} \subset \mathbf{K}_{\text{ef}}$ such that

$$V(\mathscr{K}) \cap (\bigcap_{\mathscr{M} \in \mathbf{M}} V^*(\mathscr{M})) \neq \emptyset$$

and

$$V(\mathscr{K}) \cap V^*(\mathscr{J}) \cap (\bigcap_{\mathscr{M} \in \mathbf{M}} V^*(\mathscr{M})) = \emptyset\,.$$

The notions and concepts briefly mentioned in this section and investigated in [4] enable us to introduce the concept of stability of coalition structures in general coalition-game.

3. STABILITY OF COALITION STRUCTURES

Even the concept of stability, as well as all preceding notions, was already defined in [3] and [4]. Its definition is briefly repeated in this section, and some properties of it, useful for the next sections, are introduced.

A coalition structure $\mathcal{K} \in \mathbf{K}$ is *strongly stable* iff $\Delta(\mathcal{K}) = \emptyset$. The class of all strongly stable coalition structures in the considered general coalition-game is denoted by $\mathbf{S}^*$.

A coalition structure $\mathcal{K} \in \mathbf{K}$ is *dynamicaly stable* iff for every finite sequence of coalition structures

$$\{\mathcal{K}_1, \ldots, \mathcal{K}_n\} \subset \mathbf{K}_{\mathrm{ef}}$$

such that

$$\mathcal{K}_1 = \mathcal{K}\,, \quad \mathcal{K}_r \in \Delta(\mathcal{K}_{r-1})\,, \quad r = 2, \ldots, n\,,$$

there exists a finite sequence of coalition structures

$$\{\mathcal{L}_1, \ldots, \mathcal{L}_m\} \subset \mathbf{K}_{\mathrm{ef}}$$

such that

$$\mathcal{L}_1 \in \Delta(\mathcal{K}_n)\,, \quad \mathcal{L}_r \in \Delta(\mathcal{L}_{r-1})\,, \quad r = 2, \ldots, m\,, \quad \mathcal{K} \in \Delta(\mathcal{L}_m)\,.$$

The class of all dynamicaly stable coalition structures is denoted by $\mathbf{S}$.

The classes of coalition structures which represent the results of rational bargaining in a general coalition-game are the classes

$$\mathbf{S}^* \quad \text{and} \quad \mathbf{S} \cap \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}\,.$$

The motivation of their definition was discussed in [4] and in [3], where also their general properties were derived. It was shown there, besides other results, that the class $\mathbf{S}^*$ may be empty for some games, but the classes $\mathbf{S}$ and $\mathbf{S} \cap \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}$ are always non-empty, and

$$\mathbf{S}^* \subset \mathbf{S} \subset \mathbf{K}_{\mathrm{ef}}\,, \quad \mathbf{S}^* \subset \mathbf{K}_{\mathrm{ef}}^{\mathrm{ef}}\,.$$

The following lemmas were proved in [4] as Lemma 11 and Lemma 12.

Lemma 7. If $\mathcal{K}, \mathcal{L} \in \mathbf{K}$, if $\mathcal{K} \in \mathbf{S}^*$, and if there exists a finite sequence of coalition structures

$$\{\mathcal{K}_1, \ldots, \mathcal{K}_n\} \subset \mathbf{K}_{\mathrm{ef}}$$

such that $\mathcal{K}_1 = \mathcal{L}$, $\mathcal{K}_r \in \Delta(\mathcal{K}_{r-1})$, $r = 2, \ldots, n$. $\mathcal{K} \in \Delta(\mathcal{K}_n)$, then $\mathcal{L} \notin \mathbf{S}$.

Lemma 8. If $\mathcal{K}, \mathcal{L} \in \mathbf{K}$ are coalition structures such that $\mathbf{V}(\mathcal{K}) \subset \mathbf{V}(\mathcal{L}) - \mathbf{V}^*(\mathcal{L})$, then $\mathcal{K}$ non $\sigma\, \mathcal{L}$ and $\mathcal{L}\, \sigma\, \mathcal{K}$. If, moreover, is $\mathcal{L} \in \mathbf{S}^*$ then $\mathcal{K} \notin \mathbf{S}$.

Corollary. If the assumptions of the previous lemma are fulfilled and if $\mathcal{L} \in \mathbf{K}_{\mathrm{ef}}$ then $\mathcal{K} \in \mathbf{K} - \mathbf{K}^{\mathrm{ef}}$ and $\mathcal{L} \in \Delta(\mathcal{K})$ as follows from the definitions of the effectivity from above and of the domination structure Δ.

The following useful result was proved in [4] as Theorem 2.

Lemma 9. A coalition structure $\mathcal{K}$ is strongly stable iff it is safe against the class of all effective coalitions.

4. SUPERADDITIVE GENERAL COALITION-GAME

The general coalition-game model presented in [3], further developed in [4], and briefly mentioned in the preceding sections of this paper, does not include any form of the superadditivity of the payments or utilities obtained by players. These utilities are represented by the general characteristic function V. It was shown in [3], that the superadditivity, assumed for some well known special cases of coalition-games, influences the properties of the general characteristic function. This influence was shown, especially, for the coalition-games with side-payments and, partly, also for the coalition-games with mixed strategies and utilities.

As the superadditivity assumption is of a great importance in so many special coalition-game models, it is useful to investigate its influence on the general coalition-game and on the stability of coalition structures in such game. In this section, the superadditivity concept is defined for the general coalition-game, and some results, following from it and concerning the notions introduced above, are derived.

The superadditivity in usual coalition-games reflects certain ability of disjoint coalitions to increase (or not decrease) their profit by uniting their endeavour. Formally, it is expressed as a property of pay-off functions or, if more detailed models are used, as a property of sets of strategies. It is easy to see that in case of the general coalition-game the superadditivity is a property of the general characteristic function, and Section 4 in [3], namely Theorem 4, shows the way how it should be formally described.

Definition 1. Let $\Gamma = (I, V)$ be a general coalition-game. The general characteristic function V is called *superadditive* iff for any pair of disjoint coalitions $K, L \in \mathcal{I}$ the inclusion

$$V(K \cup L) \supset V(K) \cap V(L) \tag{4.1}$$

holds. The game Γ is called superadditive iff its general characteristic function is superadditive.

Remark 1. It follows from Definition 1 immediately that if $\mathcal{K}, \mathcal{L} \in \mathbf{K}$ are coalition structures in a superadditive coalition-game, and if $\mathcal{K}$ is a subpartition of $\mathcal{L}$, then $V(\mathcal{K}) \subset V(\mathcal{L})$.

Special cases of the superadditive coalition-games were deeply investigated in literature. Interesting results are presented e.g. in [7] and [6], many results are summarized in [1]. The model of strongly or dynamicaly stable solution of the general coalition-games, introduced in [3] and [4] and briefly repeated in the preceding sections of this paper, is an analogy of a similar stability model suggested in [2] for superadditive coalition-games with side-payments. It is shown by Theorem 4 of this work that the superadditivity introduced by Definition 1 is a generalization of the superadditivity concept used in the classical coalition-game models, especially in case of coalition-games with side-payments and with the von Neumann-Morgenstern characteristic function.

It was already claimed above that the superadditivity assumption influences the properties of the stability and of the objects connected with it. Some results following from that influence are given in the remaining part of this section.

Lemma 10. In a superadditive coalition-game, a coalition structure $\mathcal{K} \in \mathbf{K}$ is effective from above iff for every $\mathcal{L} \in \mathbf{K}_{\text{ef}}$, such that $\mathcal{K}$ is a subpartition of $\mathcal{L}$, is

$$V(\mathcal{K}) \cap V^*(\mathcal{K}) \cap V^*(\mathcal{L}) \neq \emptyset .$$

Proof. It follows from (4.1) that for any pair of coalition structures $\mathcal{K}, \mathcal{L} \in \mathbf{K}$ such that $\mathcal{K}$ is a subpartition of $\mathcal{L}$ is

$$V(\mathcal{K}) \subset V(\mathcal{L})$$

(cf. Remark 1) and, consequently, also

$$V(\mathcal{K}) \cap V^*(\mathcal{K}) \subset V(\mathcal{L}) . \tag{4.2}$$

According to the definition of the effectivity from above, $\mathcal{K} \in \mathbf{K}^{\text{ef}}$ iff there exists an imputation $\boldsymbol{x} \in V(\mathcal{K})$ such that

$$\boldsymbol{x} \notin V(\mathcal{L}) - V^*(\mathcal{L}) ,$$

and (4.2) implies that this condition is fulfilled iff

$$V(\mathcal{K}) \cap V^*(\mathcal{K}) \cap V(\mathcal{L}) \cap V^*(\mathcal{L}) \neq \emptyset .$$

This inequality immediately implies the statement of this lemma.

Lemma 11. If $\mathcal{J}, \mathcal{K} \in \mathbf{K}$ are coalition structures in a superadditive general coalition-game, and if $\mathcal{J}$ is a subpartition of $\mathcal{K}$, then

$$\mathcal{J} \in \mathbf{K}^{\text{ef}} \Rightarrow \mathcal{K} \in \mathbf{K}^{\text{ef}} ,$$

$$\mathcal{K} \in \mathbf{K}_{\text{ef}} \quad \text{and} \quad V(\mathcal{J}) = V(\mathcal{K}) \Rightarrow \mathcal{J} \in \mathbf{K}_{\text{ef}} .$$

Proof. If $\mathcal{J} \in \mathbf{K}^{\text{ef}}$ and $\mathcal{K} \in \mathbf{K} - \mathbf{K}^{\text{ef}}$ then there exists $\mathcal{L} \in \mathbf{K}_{\text{ef}}$ such that $\mathcal{K}$ is a subpartition of $\mathcal{L}$ and

$$V(\mathcal{K}) \subset V(\mathcal{L}) - V^*(\mathcal{L}) ,$$

as follows from (4.1) and from Lemma 9. Then $\mathscr{J}$ is also a subpartition of $\mathscr{L}$ and Remark 1 implies that

$$V(\mathscr{J}) \subset V(\mathscr{K}) \subset V(\mathscr{L}) - V^*(\mathscr{L}) .$$

It contradicts to the assumption that $\mathscr{J} \in \mathbf{K}^{ef}$, and, consequently, $\mathscr{K} \in \mathbf{K}^{ef}$, too. Let $\mathscr{K} \in \mathbf{K}_{ef}$, $V(\mathscr{J}) = V(\mathscr{K})$, and let us denote

$$\mathbf{K}(\mathscr{K}) = \{\mathscr{M} \in \mathbf{K}\colon \mathscr{M} \textit{ is a subpartition of } \mathscr{K}\} ,$$
$$\mathbf{K}(\mathscr{J}) = \{\mathscr{M} \in \mathbf{K}\colon \mathscr{M} \textit{ is a subpartition of } \mathscr{J}\} .$$

The assumption $\mathscr{K} \in \mathbf{K}_{ef}$ implies that there exists an imputation $\boldsymbol{x}$ such that

$$\boldsymbol{x} \in V(\mathscr{K}) \cap \Big(\bigcap_{\mathscr{M} \in \mathbf{K}(\mathscr{K})} V^*(\mathscr{M})\Big) .$$

But, $V(\mathscr{K}) = V(\mathscr{J})$ by assumption, and

$$\bigcap_{\mathscr{M} \in \mathbf{K}(\mathscr{K})} V^*(\mathscr{M}) \subset \bigcap_{\mathscr{M} \in \mathbf{K}(\mathscr{J})} V^*(\mathscr{M}) ,$$

as $\mathbf{K}(\mathscr{J}) \subset \mathbf{K}(\mathscr{K})$. (If the class $\mathbf{K}(\mathscr{J})$ or both, $\mathbf{K}(\mathscr{J})$ and $\mathbf{K}(\mathscr{K})$, are empty then the respective intersections are equal to R^I, as follows from (1.5).) It means that also

$$\boldsymbol{x} \in V(\mathscr{J}) \cap \Big(\bigcap_{\mathscr{M} \in \mathbf{K}(\mathscr{J})} V^*(\mathscr{M})\Big) ,$$

and $\mathscr{J} \in \mathbf{K}_{ef}$.

The superadditivity assumption in a general coalition-game implies some special results concerning the properties of the maximal coalition $I \in \mathscr{I}$ of all players, and the coalition structure $\{I\} \in \mathbf{K}$ formed by this single coalition.

Remark 2. It follows from the definition of superadditivity and from Remark 1 that for every $\mathscr{K} \in \mathbf{K}$ is $V(\mathscr{K}) \subset V(I)$.

Remark 3. It follows from the definition of effectivity that the coalition $I \in \mathscr{I}$ is effective iff the coalition structure $\{I\} \in \mathbf{K}$ is effective from below. Moreover, $\{I\}$ is always effective from above.

Remark 4. It follows from Lemma 8 that if the coalition I is effective and for some $\mathscr{K} \in \mathbf{K}$ is $V(\mathscr{K}) \subset V(I) - V^*(I)$ then $\mathscr{K}$ is not effective from above.

Lemma 12. The coalition structure $\{I\}$ in a superadditive general coalition-game is strongly stable iff the coalition I is effective.

Proof. If I is effective then $V(I) \cap V^*(\mathscr{I}) \neq \emptyset$, as follows from the definition of effectivity. It means that

$$V(I) \cap \Big(\bigcap_{\mathscr{K} \in \mathbf{K}} V^*(\mathscr{K})\Big) \neq \emptyset$$

and, consequently, $\Delta(\{I\}) = \emptyset$. On the other hand, if $\{I\} \in \mathbf{S}^*$ then, according to Lemma 9

$$V(I) \cap \Big(\bigcap_{K \in \mathscr{I}^*(I)} V^*(K) \Big) \neq \emptyset ,$$

where $\mathscr{I}^*(I) = \{K \colon K \in \mathscr{I}, K \text{ is effective}\}$. Lemma 5 implies that also

$$V(I) \cap \Big(\bigcap_{K \in \mathscr{I}} V^*(K) \Big) \neq \emptyset$$

and the statement is proved.

Theorem 1. Let us consider a superadditive general coalition-game. Then there exists at least one strongly stable coalition structure in such game if and only if the all-players coalition is effective, i.e. if and only if the coalition structure formed by this single coalition is effective from below; in symbols:

$$\mathbf{S}^* \neq \emptyset \Leftrightarrow \{I\} \in \mathbf{K}_{\mathrm{ef}} .$$

Proof. According to the definition of effectivity from below

$$V(I) \cap \Big(\bigcap_{K \in \mathscr{I}} V^*(K) \Big) \neq \emptyset ,$$

iff I is effective (cf. Remark 3). It means that

$$V(I) \cap \Big(\bigcap_{\mathscr{K} \in \mathbf{K}} V^*(\mathscr{K}) \Big) \neq \emptyset$$

and, consequently, $\{I\}\ \sigma\ (\bigcap \mathbf{K})$. It means that $\Delta(\{I\}) = \emptyset$ and $\{I\} \in \mathbf{S}^* \neq \emptyset$. Let us suppose, now, that the coalition I is not effective. According to Remark 3, it is equivalent to the assumption that $\{I\} \notin \mathbf{K}_{\mathrm{ef}}$. Then, according to Lemma 6, there exists a set of effective coalitions $\mathscr{M} \subset \mathscr{I}$ such that

$$V(I) \cap V^*(\mathscr{M}) = V(I) \cap \Big(\bigcap_{M \in \mathscr{M}} V^*(M) \Big) = \emptyset .$$

Hence, for any coalition structure $\mathscr{K} \in \mathbf{K}$ is $V(\mathscr{K}) \cap V^*(\mathscr{M}) = \emptyset$, as follows from Remark 2. It is not difficult to verify that for every effective coalition L there exists at least one coalition structure $\mathscr{L} \in \mathbf{K}_{\mathrm{ef}}$ such that $L \in \mathscr{L}$. It is sufficient to put

$$\mathscr{L} = \{K\} \cup \{\{i\}\}_{i \in I - K} ,$$

i.e. to construct $\mathscr{L}$ consisting of the coalition K and of all one-player coalitions of players from $I - K$; the one-player coalitions are always effective. Consequently, there exists a class of coalition structures $\mathbf{M} \subset \mathbf{K}_{\mathrm{ef}}$ such that for every coalition structure $\mathscr{K} \in \mathbf{K}$ is

$$\mathscr{K} \text{ non } \sigma\ (\bigcup \mathbf{M}) .$$

On the other hand, $\mathscr{K}\ \sigma\ \emptyset$, as follows from Lemma 6. It means that for every coalition structure $\mathscr{K} \in \mathbf{K}$ necessarily exists a class of coalition structures $\mathbf{N} \subset \mathbf{M}$ and a coalition structure $\mathscr{L} \in \mathbf{M} - \mathbf{N}$, such that

$$\mathscr{K}\ \sigma\ (\bigcup \mathbf{N}) \quad \text{and} \quad \mathscr{K}\ \text{non}\ \sigma\ (\bigcup \mathbf{N} \cup \mathscr{L}) .$$

Then $\mathscr{L} \in \Delta(\mathscr{K})$. As such $\mathbf{N}$ and $\mathscr{L}$ may be found for every $\mathscr{K} \in \mathbf{K}$, it is proved that $\mathbf{S}^* = \emptyset$.

Corollary. It follows from Theorem 1 and Lemma 12 immediately that $\mathbf{S}^* \neq \emptyset \Leftrightarrow \Leftrightarrow \{I\} \in \mathbf{S}^*$, in every superadditive general coalition-game.

Theorem 2. Let us consider a superadditive general coalition-game. If there exists a strongly stable coalition structure in the considered game and if a coalition structure $\mathscr{K} \in \mathbf{K}$ is such that

$$V(\mathscr{K}) \subset V(I) - V^*(I)$$

then $\mathscr{K}$ is neither strongly nor dynamically stable. If, on the other hand, it is

$$V(\mathscr{K}) \cap V^*(I) = V(\mathscr{K}) \cap V^*(\mathscr{K})$$

then $\mathscr{K}$ is strongly and dynamically stable.

Proof. If $\mathbf{S}^* \neq \emptyset$ then, according to Theorem 1, it is $\{I\} \in \mathbf{S}^*$. Lemma 8 implies that in such case $\mathscr{K}$ is not dynamically stable, i.e. $\mathscr{K} \notin \mathbf{S}$, if

$$V(\mathscr{K}) \subset V(I) - V^*(I) .$$

As $\mathbf{S} \supset \mathbf{S}^*$, the first statement is proved. Let be

$$V(\mathscr{K}) \cap V^*(I) = V(\mathscr{K}) \cap V^*(\mathscr{K}) . \tag{4.3}$$

As we suppose that $\{I\} \in \mathbf{S}^*$, the relation

$$V(I) \cap (\bigcap_{L \in \mathscr{I}} V^*(L)) \neq \emptyset$$

holds, and there exists $\mathbf{x} \in V(I)$ such that

$$\mathbf{x} \in V(I) \cap V^*(\mathscr{K}) \quad \text{and} \quad \mathbf{x} \in V^*(L) \quad \text{for all} \quad L \in \mathscr{I} .$$

Then it is also $\mathbf{x} \in V(\mathscr{K})$, because of (4.3), and $\mathscr{K} \in \mathbf{S}^* \subset \mathbf{S}$.

It was mentioned in the introductory paragraphs of this section that the definition of superadditivity given here should be a generalization of the usual concept of superadditivity used especially in the theory of coalition-games with side-payments. The following theorem implies that it is really so.

Theorem 3. If the considered game $\Gamma = (I, V)$ is a coalition-game with side-payments, i.e. if for every coalition $K \in \mathscr{I}$ there exists a real number $v(K)$ such that

$$V(K) = \{\mathbf{x} = (x_i)_{i \in I}: \sum_{i \in K} x_i \leqq v(K)\}$$

then the game Γ is superadditive if and only if

$$v(K \cup L) \geqq v(K) + v(L) \quad \text{for all} \quad K, L \in \mathscr{I}, \quad K \cap L = \emptyset . \tag{4.4}$$

Proof. Let (4.4) be true for $K, L \in \mathscr{I}$, $K \cap L = \emptyset$. Let us choose

$$\mathbf{x} = (x_i)_{i \in I} \in V(K) \cap V(L) .$$

Then

$$\sum_{i \in K} x_i \leqq v(K) \quad \text{and} \quad \sum_{i \in L} x_i \leqq v(L) .$$

Hence

$$\sum_{i \in K \cup L} x_i \leqq v(K) + v(L) \leqq v(K \cup L)$$

and $\mathbf{x} \in V(K \cup L)$. Let, on the other hand, (4.4) be not true for some $K, L \in \mathscr{I}$, $K \cap L = \emptyset$. Then there exists $\mathbf{x} = (x_i)_{i \in I} \in R^I$ such that

$$\sum_{i \in K} x_i = v(K) , \quad \sum_{i \in L} x_i = v(L) , \quad \sum_{i \in K \cup L} x_i > v(K \cup L) .$$

Then $\mathbf{x} \in V(K)$ and $\mathbf{x} \in V(L)$ but $\mathbf{x} \notin V(K \cup L)$.

The connection between the strong and dynamic stability may be specified for the superadditive coalition-games with side-payments (cf. [3], Section 4) in the following way.

Theorem 4. Let us suppose that the considered superadditive general coalition-game $\Gamma = (I, V)$ is a coalition-game with side-payments, i.e. for every coalition $K \in \mathscr{I}$ there exists a number $v(K) \in R$ such that

$$v(K \cup L) \geqq v(K) + v(L) \quad \text{for} \quad K, L \in \mathscr{I}, \quad K \cap L = \emptyset ,$$

and

$$V(K) = \{\mathbf{x} = (x_i)_{i \in I}: \sum_{i \in K} x_i \leqq v(K)\} , \quad K \in \mathbf{K} .$$

If there exists at least one strongly stable coalition structure in such game Γ then every dynamically stable coalition structure is also strongly stable, and a coalition structure $\mathscr{K}$ is strongly stable if and only if the sum of the values $v(K)$ for all $K \in \mathscr{K}$ is equal to $v(I)$. In symbols

$$\mathbf{S}^* \neq \emptyset \Leftrightarrow \mathbf{S}^* = \mathbf{S} = \{\mathscr{K} \in \mathbf{K}: \sum_{K \in \mathscr{K}} v(K) = v(I)\} .$$

Proof. In the assumed type of games, the following equation holds for every $K \in \mathscr{I}$:

$$V^*(K) = \{\mathbf{x} = (x_i)_{i \in I}: \sum_{i \in K} x_i \geqq v(K)\}.$$

If $\mathbf{S}^* \neq \emptyset$ then $\{I\} \in \mathbf{S}^*$, as follows from Theorem 1 and Lemma 12. Let us consider an arbitrary coalition structure $\mathscr{K} \in \mathbf{K}$. If

$$\sum_{K \in \mathscr{K}} v(K) < v(I)$$

then necessarily

$$V(\mathscr{K}) \subset V(I) - V^*(I)$$

and, according to Theorem 2, $\mathscr{K} \notin \mathbf{S}$. If, on the other hand

$$\sum_{K \in \mathscr{K}} v(K) = v(I)$$

then

$$V(\mathscr{K}) \cap V^*(\mathscr{K}) = \{\mathbf{x} = (x_i)_{i \in I}: \sum_{i \in K} x_i = v(K) \textit{ for all } K \in \mathscr{K}\} =$$

$$= \{\mathbf{x} = (x_i)_{i \in I}: \sum_{i \in K} x_i \leqq v(K) \textit{ for all } K \in \mathscr{K} \textit{ and } \sum_{i \in I} x_i = v(I)\} =$$

$$= V(\mathscr{K}) \cap V^*(I).$$

It follows from Theorem 2 that then is $\mathscr{K} \in \mathbf{S}^*$. It follows from the previous steps of this proof that

$$\mathbf{S}^* = \{\mathscr{K} \in \mathbf{K}: \sum_{K \in \mathscr{K}} v(K) = v(I)\}.$$

The preceding theorem is an analogy of a similar result proved for the superadditive coalition-games with side-payments in [2] in Theorem 5.2.

5. SUBADDITIVE GENERAL COALITION-GAMES

The concept of subadditivity is, in certain sense, an opposite one to the super-additivity concept. Its special cases for the classical coalition-games, namely for the games with side-payments, contradict the usual interpretation of the game model and of the von Neumann-Morgenstern characteristic function of the coalition-game. It is the cause why it was not considered and investigated in the classical theory of coalition-games.

However, the notion of subadditivity has its sense in case of some special coalition-games, namely the market games mentioned in [3], and for some games investigated in [5]. In this section, the main properties of the subadditive general coalition-games are derived, and it is shown that the subadditivity assumption essentially simplifies some properties of stability in general coalition-games.

Definition 2. Let $\Gamma = (I, V)$ be a general coalition-game. The general characteristic function V is called *subadditive* iff for any pair of disjoint coalitions $K, L \in \mathcal{I}$ the inclusion

$$V^*(K \cup L) \supset V^*(K) \cap V^*(L) \tag{5.1}$$

holds. The game Γ is called subadditive iff its general characteristic function is subadditive.

Lemma 13. If the inclusion $V(K \cup L) \subset V(K) \cap V(L)$ holds for all $K, L \in \mathcal{I}$, $K \cap L = \emptyset$, then the considered game is subadditive.

Proof. Let us consider $K, L \in \mathcal{I}$, $K \cap L = \emptyset$. Let $\mathbf{x} = (x_i)_{i \in I} \notin V^*(K \cup L)$. Then there exists $\mathbf{y} = (y_i)_{i \in I} \in V(K \cup L)$ such that $y_i \geqq x_i$ for all $i \in K \cup L$, and $y_j > x_j$ for some $j \in K \cup L$. If

$$V(K \cup L) \subset V(K) \cap V(L)$$

then $\mathbf{y} \in V(K)$ and $\mathbf{y} \in V(L)$ and, consequently, $\mathbf{x} \notin V^*(K)$ or $\mathbf{x} \notin V^*(L)$. It means that also $\mathbf{x} \notin V^*(K) \cap V^*(L)$.

Lemma 14. A coalition $K \in \mathcal{I}$ in a subadditive general coalition-game is effective iff

$$V(K) \cap (\bigcap_{i \in K} V^*(\{i\})) \neq \emptyset .$$

Proof. For every $L \in \mathcal{I}$, it is

$$V^*(L) \supset \bigcap_{i \in L} V^*(\{i\})$$

as follows from Definition 2. It means that if

$$V(K) \cap (\bigcap_{i \in K} V^*(\{i\})) \neq \emptyset$$

then also

$$V(K) \cap (\bigcap_{L \in \mathcal{I}, L \subset K} V^*(L)) \neq \emptyset ,$$

and K is effective. The opposite implication follows from the effectivity definition immediately.

Lemma 15. A coalition structure $\mathcal{K} \in \mathbf{K}$ in a subadditive general coalition-game is effective from below iff

$$V(\mathcal{K}) \cap (\bigcap_{i \in I} V^*(\{i\})) \neq \emptyset . \tag{5.2}$$

Proof. The statement of this lemma follows from Lemma 14. All coalitions in $\mathcal{K}$ are effective iff (5.2) is true.

Lemma 16. Every coalition structure in a subadditive general coalition-game is effective from above, i.e. $\mathscr{K} = \mathbf{K}^{ef}$.

Proof. Let $\mathscr{K}$, $\mathscr{L} \in \mathbf{K}$, and let $\mathscr{K}$ be a subpartition of $\mathscr{L}$. Then

$$V^*(\mathscr{K}) \subset V^*(\mathscr{L})$$

as follows from Definition 2. Relation (1.7) implies

$$V^*(\mathscr{K}) \cap V(\mathscr{K}) \neq \emptyset$$

and, consequently,

$$V(\mathscr{K}) \cap V^*(\mathscr{L}) \neq \emptyset .$$

Then the inclusion $V(\mathscr{K}) \subset V(\mathscr{L}) - V^*(\mathscr{L})$ cannot be true and $\mathscr{K}$ is necessarily effective from above.

Let us denote, now, by the symbol $\mathscr{I}_0$ the coalition structure of exactly all one-player coalitions

$$\mathscr{I}_0 = \{\{i\}\}_{i \in I} .$$

This coalition structure plays an important role in the subadditive general coalition-games theory. Its importance reminds of the role of the all-players coalition I and of the corresponding coalition structure $\{I\}$ in the superadditive general coalition-games.

Theorem 5. A coalition structure $\mathscr{K} \in \mathbf{K}$ in a subadditive general coalition-game is strongly stable if and only if it is safety against the coalition structure $\mathscr{I}_0$ of all one-player coalitions, $\mathscr{K} \; \sigma \; \mathscr{I}_0$, i.e. iff $V(\mathscr{K}) \cap V^*(\mathscr{I}_0) \neq \emptyset$.

Proof. It follows from Definition 2 that

$$V^*(\mathscr{L}) \supset V^*(\mathscr{I}_0) \quad \text{for any} \quad \mathscr{L} \in \mathbf{K} .$$

If $V(\mathscr{K}) \cap V^*(\mathscr{I}_0) \neq \emptyset$ then

$$V(\mathscr{K}) \cap \left(\bigcap_{\mathscr{L} \in \mathbf{K}} V^*(\mathscr{L}) \right) \neq \emptyset$$

and $\mathscr{K} \in \mathbf{S}^*$. The opposite implication follows directly from the definition of the strong stability.

Corollary. The coalition structure $\mathscr{I}_0$ is always strongly stable, if the considered general coalition-game is subadditive, as follows from (1.7) and from Theorem 5. Consequently, the class $\mathbf{S}^*$ of strongly stable coalition structures is non-empty if the considered game is subadditive.

The mutual relation between the strong and dynamic stability of coalition structures is, in subadditive general coalition-games, even stronger than in case of the super-additive ones, as follows from the next theorem.

Theorem 6. A coalition structure in a subadditive general coalition-game is dynamically stable if and only if it is strongly stable, i.e. $\mathbf{S} = \mathbf{S}^*$.

Proof. It was proved in [4] and mentioned in Section 3 of this paper that $\mathbf{S}^* \subset \mathbf{S}$. Let us suppose a coalition structure $\mathscr{K} \in \mathbf{K} - \mathbf{S}^*$. It means that

$$\mathscr{K} \text{ non } \sigma \mathscr{I}_0$$

as follows from Theorem 5. As, according to Lemma 6, is $\mathscr{K} \sigma \emptyset$, the relation $\mathscr{I}_0 \in \Delta(\mathscr{K})$ is true. It follows from Lemma 7 and from the previous Corollary of Theorem 5 that $\mathscr{K} \notin \mathbf{S}$.

Corollary. It follows from the previous theorems that

$$\mathbf{K}_{\text{ef}} = \mathbf{K}_{\text{ef}}^{\text{ef}} = \mathbf{S} = \mathbf{S}^*$$

in every subadditive general coalition-game.

Even if the concept of subadditivity is not usually considered in the coalition-games with side-payments, it may be expressed by means of conditions fulfilled by the von Neumann-Morgenstern characteristic function.

Theorem 7. If the considered game $\Gamma = (I, \mathbf{V})$ is a coalition-game with side-payments, i.e. if for every coalition $K \in \mathscr{I}$ there exists a real number $\mathbf{v}(K)$ such that

$$\mathbf{V}(K) = \{\mathbf{x} = (x_i)_{i \in I}: \sum_{i \in K} x_i = \mathbf{v}(K)\} \tag{5.3}$$

then the game Γ is subadditive if and only if

$$\mathbf{v}(K \cup L) \leqq \mathbf{v}(K) + \mathbf{v}(L) \quad \text{for every} \quad K, L \in \mathscr{I}, \quad K \cap L = \emptyset .$$

Proof. The proof of this theorem is analogous to the one of Theorem 3. It follows from (5.3) immediately that for every coalition $K \in \mathscr{I}$ is

$$\mathbf{V}^*(K) = \{\mathbf{x} = (x_i)_{i \in I}: \sum_{i \in K} x_i \geqq \mathbf{v}(K)\} .$$

Let us consider $K, L \in \mathscr{I}$, $K \cap L = \emptyset$, and let

$$\mathbf{v}(K \cup L) \leqq \mathbf{v}(K) + \mathbf{v}(L) .$$

Let us choose an imputation

$$\mathbf{x} = (x_i)_{i \in I} \in \mathbf{V}^*(K) \cap \mathbf{V}^*(L) .$$

Then

$$\sum_{i \in K} x_i \geqq \mathbf{v}(K) \quad \text{and} \quad \sum_{i \in L} x_i \geqq \mathbf{v}(L) .$$

Hence

$$\sum_{i \in K \cup L} x_i \geqq v(K) + v(L) \geqq v(K \cup L)$$

and $\boldsymbol{x} \in V^*(K \cup L)$. Let us suppose, now, that

$$v(K) + v(L) < v(K \cup L) \quad \text{for some} \quad K, L \in \mathscr{I}, \quad K \cap L = \emptyset .$$

Then there exists $\boldsymbol{x} = (x_i)_{i \in I} \in R^I$ such that

$$\sum_{i \in K} x_i = v(K) \quad \text{and} \quad \sum_{i \in L} x_i = v(L) .$$

It means that $\boldsymbol{x} \in V^*(K)$ and $\boldsymbol{x} = V^*(L)$ but $\boldsymbol{x} \notin V^*(K \cup L)$.

6. ADDITIVE GENERAL COALITION-GAMES

The concept of additivity is considered, as an extremal case of superadditivity, in the coalition-games theory, e.g. in [6] or [7]. It is not difficult to generalize it for the case of general coalition-games and to investigate its influence on the properties of stability in such games.

Definition 3. Let $\Gamma = (I, V)$ be a general coalition-game. The general characteristic function V is called *additive* iff it is superadditive and subadditive. The game Γ is called additive iff its general characteristic function V is additive.

Remark 5. It follows from Definition 3 that V is additive iff for every pair of disjoint coalitions $K, L \in \mathscr{I}$ is

(6.1) $$V(K \cup L) \supset V(K) \cap V(L)$$

and

(6.2) $$V^*(K \cup L) \supset V^*(K) \cap V^*(L) .$$

Remark 6. If for every pair of disjoint coalitions $K, L \in \mathscr{I}$ is

$$V(K \cup L) = V(K) \cap V(L)$$

then the general characteristic function V is additive, as follows from Definition 3 Definition 2 and Lemma 13.

The additivity of a general coalition-game represents an extremal degeneration of the profitability of the cooperation among players. In such game, the cooperation and the coalition forming does not practically influence the utility obtained by the formed coalitions. It means that all coalition structures have equivalent possibilities to satisfy the rational demands of players. It is formulated, more exactly, in the next statement.

Theorem 8. Every coalition structure in an additive general coalition-game is strongly stable.

Proof. It follows from (1.2) that for every one-element coalition $\{j\} \in \mathscr{I}$, $j \in I$, there exists a real number $x_j^* \in R$ such that

$$V(\{j\}) = \{\mathbf{x} = (x_i)_{i\in I} : x_j \leqq x_j^*\}, \tag{6.3}$$

$$V^*(\{j\}) = \{\mathbf{x} = (x_i)_{i\in I} : x_j \geqq x_j^*\}.$$

It means that

$$\mathbf{x}^* = (x_i^*)_{i\in I} \in V(\{i\}) \cap V^*(\{i\})$$

for all $i \in I$. The superadditivity of V implies that

$$\mathbf{x}^* \in V(K) \quad \text{for all} \quad K \in \mathscr{I},$$

and then

$$\mathbf{x}^* \in V(\mathscr{K}) \quad \text{for all} \quad \mathscr{K} \in \mathbf{K}.$$

If $\mathscr{I}_0$ is the coalition structure of exactly all one-player coalitions,

$$\mathscr{I}_0 = \{\{i\}\}_{i\in I},$$

then

$$\mathbf{x}^* \in V^*(\mathscr{I}_0)$$

and

$$\mathbf{x}^* \in V(\mathscr{K}) \cap V^*(\mathscr{I}_0) \quad \text{for all} \quad \mathscr{K} \in \mathbf{K}.$$

The subadditivity of V allows to use Theorem 5. It means that $\mathscr{K} \in \mathbf{S}^*$ for all $\mathscr{K} \in \mathbf{K}$.

Corollary. It follows from the preceding theorem that

$$\mathbf{K} = \mathbf{K}_{\text{ef}} = \mathbf{K}^{\text{ef}} = \mathbf{K}_{\text{ef}}^{\text{ef}} = \mathbf{S} = \mathbf{S}^*$$

in every additive general coalition-game.

Remark 7. If $\mathbf{x}^* = (x_i^*)_{i\in I} \in R^I$ is the imputation defined by (6.3) then Theorem 8, Theorem 6 and (6.3) imply that for every $K \in \mathscr{I}$ and every $\mathscr{K} \in \mathbf{K}$ is

$$\mathbf{x}^* \in V(K) \cap V^*(K) \quad \text{and} \quad \mathbf{x}^* \in V(\mathscr{K}) \cap V^*(\mathscr{K}).$$

Morevover,

$$\{\mathbf{x}^*\} = V(\mathscr{I}_0) \cap V^*(\mathscr{I}_0)$$

where $\mathscr{I}_0 = \{\{i\}\}_{i\in I}$ is the coalition structure of exactly all one-player coalitions.

The definition of additivity, introduced in this section, is really a generalization of the additivity of coalition-games with side-payments, as follows from the following theorem.

Theorem 9. If the considered game $\Gamma = (I, V)$ is a coalition-game with side-payments, i.e. if for every coalition $K \in \mathscr{I}$ there exists a real number $v(K)$ such that

$$V(K) = \{\mathbf{x} = (x_i)_{i \in I} : \sum_{i \in K} x_i \leqq v(K)\}$$

then the game Γ is additive if and only if

$$v(K \cup L) = v(K) + v(L) \quad \text{for all} \quad K, L \in \mathscr{I}, \quad K \cap L = \emptyset .$$

Proof. The theorem follows from Definition 3 and from Theorem 3 and Theorem 7.

7. CONCLUSION

The concepts of superadditivity and additivity belong to the most important and frequented notions of the classical coalition-games theory. Even the concept of the subadditivity, which is not usually investigated in the known literature, has its sense in some special modifications and applications of the coalition-game models. The consequences of these concepts for the (strong or dynamic) stability of coalition structures derived in this paper may help us to see their sense for a much wider class of coalition-games.

Further results, concerning especially the stability of configurations (see [4]), obviously follow from the results introduced here and from the definitions of the stability of configurations. Some other, more detailed, results could be surely derived for particular special cases of general coalition-games. It concerns, especially, the coalition-games with restricted side-payments and the market games (see [3] and [5]), or other non-classical modifications of the general coalition-game model.

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