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Polynomial Approach to Conversion between Laplace and Z Transforms

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In this paper a new numerical approach to some problems in the Laplace and $\mathscr{Z}$ transformations is presented. The mathematical background is the congruence of analytic functions modulo a polynomial [1] and hence we obtain fewer operations in comparison to known matrix methods for similar problems.

INTRODUCTION

The procedures given below can be used for the inverse Laplace transformation, for finding suitable sampling period and time delay between the input and output of continuous linear systems in digital control by using digital simulation etc. All procedures have been tested on the IBM 370/135 computet using the PL/I language. The main results are summarized in Theorems 3, 4, 7, 8, 10, 13.

Now we summarize some notation and topics which are mentioned more precisely in [1]. References to theorems and definitions in [1] will be written as "Theorem 3/I".

Consider a polynomial $m(z) = m_0 + m_1 z + \ldots m_{\partial m} z^{\partial m}$ with real coefficients and degree $\partial m > 0$. Then the set $\mathscr{M} = \{z : m(z) = 0\}$ is called the spectrum of the polynomial m.

Let functions f, g be analytic on $\mathscr{M}$. Then f and g are congruent modulo m, written $f = g \bmod m$, if there exists a function h analytic on $\mathscr{M}$ such that $f = g + hm$. For any function f analytic on $\mathscr{M}$ only one polynomial r exists such that

$$f = r \bmod m \, , \quad \partial r < \partial m \, .$$

The polynomial m is called the modulus. The operation which yields such polynomial r is called the reduction of f modulo m and it is denoted as

$$[f]_m = r \, .$$

Denote $\mathscr{F}_m$ the set of all functions analytic or having at worst removable singularities on $\mathscr{M}$.

PROPERTIES OF REDUCTION MODULO m

Let a modulus m, $\partial m > 0$, and functions f, g analytic on $\mathscr{M}$ be given. Then for $[f]_m = a$, $[g]_m = b$ and any complex number λ the next equations hold:

(1) $$[f + g]_m = [f]_m + [g]_m = a + b\,,$$

(2) $$[\lambda f]_m = \lambda[f]_m = \lambda a\,,$$

(3) $$[fg]_m = [[f]_m [g]_m]_m = [ab]_m\,,$$

(4) if f/g is analytic on $\mathscr{M}$ then

$$\left[\frac{f}{g}\right]_m = \left[\frac{[f]_m}{[g]_m}\right]_m = \left[\frac{a}{b}\right]_m .$$

If the function f is a polynomial then the reduction of f modulo m produces the remainder after dividing f by m.

Procedures for the computation of $[f]_m$ for the functions $\ln(z)$, e^{kx}, $\sqrt{x}$, x^k with k real, and for b/a, $b \,.\, a$ with a, b polynomial are described in [1].

Consider a modulus m, $\partial m > 0$, and a function f analytic on $\mathscr{M}$. The anihilating polynomial of the function f modulo m, denoted by $\mathscr{A}[f]_m$, is a nonzero polynomial $p = p_0 + p_1 x + \ldots p_k x^k$ with minimal degree for which $[p(f)]_m = 0$. It is evident that an anihilating polynomial exists and that $\partial p \leqq \partial m$.

Now we introduce a new operation:

(5) $$\left\langle \frac{x[f]_a}{a} \right\rangle_x = \lim_{x \to \infty} \frac{x[f]_a}{a} = \frac{c_{k-1}}{a_k}\,,$$

where $k = \partial a$,

$$[f]_a = c_0 + c_1 x + \ldots c_{k-1} x^{k-1}$$

and the subscript x denotes the variable with respect to which the operation $\langle \cdot \rangle_x$ is performed. The definition of this new operation corresponds to Theorem 13/I which gives the following equality

(6) $$\frac{1}{2\pi j} \int_{(a)} \frac{f}{a}\, dx = \left\langle \frac{x[f]_a}{a} \right\rangle_x ,$$

where $\int_{(a)}$ denotes the contour integral $\int_{\mathscr{C}}$, where $\mathscr{C}$ is a closed curve such that all zeros of a lie inside $\mathscr{C}$ and the function f is analytic inside and on $\mathscr{C}$ (see Example 3/I).

Assumption. In applications only polynomials with real coefficients are considered.

Theorem 1. Let a, b, c be polynomials in variable z such that

$$\left\langle \frac{z}{a} [bz^n]_a \right\rangle_z = \left\langle \frac{z}{a} [cz^n]_a \right\rangle_z \quad \text{for} \quad n = 0, 1, 2, \ldots, (\partial a - 1).$$

Then $[b]_a = [c]_a$.

Proof. Let

$$[b]_a = \beta = \beta_0 + \beta_1 z + \ldots \beta_k z^k ,$$
$$[c]_a = \gamma = \gamma_0 + \gamma_1 z + \ldots \gamma_k z^k , \quad k = \partial a - 1 .$$

Since the operations $[\cdot]_a$, $\langle \cdot \rangle_z$ are linear, we can write

$$\left\langle \frac{z}{a} [(b - c) z^n]_a \right\rangle_z = 0 ,$$

$$\left\langle \frac{z}{a} [([b]_a - [c]_a) z^n]_a \right\rangle_z = 0 ,$$

$$I = \left\langle \frac{z}{a} [(\beta - \gamma) z^n]_a \right\rangle_z = 0 .$$

Consider $n = 0$. Then from (1) we obtain

$$I = \frac{\beta_k - \gamma_k}{a_{k+1}} = 0 , \quad a_{k+1} \neq 0$$

and hence $\beta_k = \gamma_k$.

Using this procedure for $n = 1, 2, \ldots, k$ we obtain $\beta_{k-1} = \gamma_{k-1}, \beta_{k-2} = \gamma_{k-2}, \ldots$
$\ldots, \beta_0 = \gamma_0$. Hence $\beta = \gamma$.

Theorem 2. (Inverse function). Let a polynomial $\hat{a}$, $\partial \hat{a} > 0$, and a function $F \in \mathscr{F}_{\hat{a}}$ be given. If the anihilating polynomial of the function $F(z)$ modulo $\hat{a}(z)$, denoted by $a = \mathscr{A}[F]_{\hat{a}}$, has the property $\partial a = \partial \hat{a}$, then the inverse function Φ exists such that

$$[\Phi(F(z))]_{\hat{a}(z)} = [z]_{\hat{a}(z)} . \tag{7}$$

Proof. Denote

$$[F(z)]_{\hat{a}} = f_0 + f_1 z + \ldots + f_k z^k , \quad k = \partial a - 1$$

and

$$[F^i(z)]_{\hat{a}} = f_{i,0} + f_{i,1} z + \ldots + f_{i,k} z^k , \quad i = 0, 1, \ldots, k , \quad k + 1 . \tag{8}$$

Then the polynomial $a = \mathscr{A}[F(s)]_{\hat{a}(s)} = a_0 + a_1 s + \ldots + a_k s^k + a_{k+1} s^{k+1}$ is given by the equation (see COMPUTING THE ANIHILATING POLYNOMIAL/1)

$$\text{(i)} \qquad \begin{bmatrix} 1 & f_{1,0} & \cdots & f_{k,0} & f_{k+1,0} \\ 0 & f_{1,1} & \cdots & f_{k,1} & f_{k+1,1} \\ 0 & f_{1,2} & \cdots & f_{k,2} & f_{k+1,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & f_{1,k} & \cdots & f_{k,k} & f_{k+1,k} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdots \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ 0 \end{bmatrix}.$$

It is evident that $\partial a = \partial \hat{a}$ if and only if the rank of the above matrix is equal to $k + 1 = \partial \hat{a}$. Write $\Phi = \varphi_0 + \varphi_1 s + \ldots + \varphi_k s^k$. Then the property $[\Phi(F(z))]_{\hat{a}} = [z]_{\hat{a}}$ gives for $\partial \hat{a} > 1$ the next equation

$$(9) \qquad \begin{bmatrix} 1 & f_{1,0} & \cdots & f_{k,0} \\ 0 & f_{1,1} & \cdots & f_{k,1} \\ 0 & f_{1,2} & \cdots & f_{k,2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & f_{1,k} & \cdots & f_{k,k} \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \cdot \\ \varphi_k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ 0 \end{bmatrix}.$$

If $\partial \hat{a} = 1$ then $\Phi = \varphi_0$ only. Consider $\partial a = \partial \hat{a}$. Then (i) has no nontrivial solution for $a_{k+1} = 0$. Hence the rank of the matrix in (9) is equal to $k + 1$ and the inverse function Φ is defined as the polynomial with degree less than $\partial \hat{a}$.

Theorem 3. Let real polynomials $\hat{b}$, $\hat{a}$ and a function $F(z)$ be given such that $\partial \hat{b} < \partial \hat{a}$ and $[F(z)]_{\hat{a}}$ exists as a polynomial with real coefficients. Then for the real sequence $\hat{h}_0, \hat{h}_1, \hat{h}_2, \ldots$ given by

$$(10) \qquad \hat{h}_i = \left\langle \frac{z}{\hat{a}} [\hat{b}\, F^i(z)]_{\hat{a}} \right\rangle_z, \quad i = 0, 1, 2, \ldots$$

the next formulas hold.

$$(11) \qquad h_i = \left\langle \frac{p}{a} [b p^i]_a \right\rangle_p, \quad i = 0, 1, 2, \ldots,$$

$$h_i = \hat{h}_i \quad \text{for} \quad i = 0, 1, 2, \ldots,$$

where

$$a = \mathscr{A}[F]_{\hat{a}} = a_0 + a_1 p + \ldots a_n p^n + a_{n+1} p^{n+1},$$

$$b = b_0 + b_1 p + \ldots + b_n p^n,$$

a_i, b_i are real coefficients,

$$(12) \qquad b_{n-j} = \sum_{k=0}^{n-j} a_{k+j+1} \hat{h}_k, \quad \text{for} \quad j = 0, 1, \ldots, n,$$

$$\frac{b}{a} = \sum_{i=0}^{\infty} \frac{h_i}{p^{i+1}}\,. \tag{13}$$

Proof. First, let us establish the correspondence between (11) and (13). Consider

$$\sum_{i=0}^{\infty} \frac{h_i}{s^{i+1}} = \sum_{i=0}^{\infty} \left\langle \frac{p}{a} \left[b \frac{p^i}{s^{i+1}} \right]_a \right\rangle_p = \left\langle \frac{p}{a} \left[\frac{b}{s-p} \right]_a \right\rangle_p$$

using

$$\frac{1}{s} \sum_{i=0}^{\infty} \left(\frac{p}{s} \right)^i = \frac{1}{s-p}$$

for $|p/s| < 1$. Second, we show that

$$\sum_{i=0}^{\infty} \frac{h_i}{s^{i+1}} = \frac{b(s)}{a(s)}$$

using simple algebraic operations and the next properties

$$[a]_a = 0\,, \quad \left\langle \frac{p}{a} [b]_a \right\rangle_p = 0 \quad \text{if} \quad \partial b < \partial a - 1\,.$$

Write

$$\frac{b(s)}{s-p} = \frac{b(s)}{s-p} + \frac{b(p) - b(s)}{s-p} = \frac{b(s)}{a(s)} \left(\frac{a(s) - a(p)}{s-p} + \frac{a(p)}{s-p} \right) + \frac{b(p) - b(s)}{s-p}\,.$$

Hence

$$\left\langle \frac{p}{a} \left[\frac{b}{s-p} \right]_a \right\rangle_p = \frac{b(s)}{a(s)} \left\langle \frac{p}{a} \left[\frac{a(s) - a(p)}{s-p} \right]_a \right\rangle_p = \frac{b(s)}{a(s)}\,.$$

Third equations (12) in matrix notation read

$$\begin{bmatrix} a_1 a_2 & \dots & a_n & a_{n+1} \\ a_2 a_3 & & a_{n+1} & 0 \\ \vdots & & & \vdots \\ a_n & a_{n+1} & 0 & \\ a_{n+1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{n-1} \\ h_n \end{bmatrix} = \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}. \tag{14}$$

It is evident that the equation (14) defines the first part of long division of b by a and hence a polynomial b can be found such that $h_i = \hat{h}_i$ for $i = 0, 1, \dots n$. To prove $h_i = \hat{h}_i$ for $i > n$ we use the next properties

$$\sum_{i=0}^{\partial a} h_{i+j} a_i = \left\langle \frac{p}{a} [b \sum_{i=0}^{\partial a} p^{i+j} a_i]_a \right\rangle_p = \left\langle \frac{p}{a} [b p^j a]_a \right\rangle_p = 0\,,$$

for $j = 0, 1, \ldots$, and similarly

$$\sum_{i=0}^{\partial a} \hat{h}_{i+j} a_i = \left\langle \frac{z}{\hat{a}} [\hat{b} \sum_{i=0}^{\partial a} a_i F^{i+j}]_{\hat{a}} \right\rangle_z = \left\langle \frac{z}{\hat{a}} [\hat{b} F^j [\sum_{i=0}^{\partial a} a_i F^i]_{\hat{a}}]_{\hat{a}} \right\rangle_z = 0\,,$$

for $j = 0, 1, \ldots,$
because

$$[\sum_{i=0}^{\partial a} a_i F^i]_{\hat{a}} = 0$$

due to $a = \mathscr{A}[F]_{\hat{a}}$.

Theorem 4. Let polynomials $\hat{b}$, $\hat{a}$ and a function $F(z) \in \mathscr{F}_{\hat{a}}$ be given such that $\partial \hat{b} < \partial \hat{a}$ and the anihilating polynomial of $F(z)$ modulo $\hat{a}$, denoted by $a = \mathscr{A}[F(z)]_{\hat{a}}$, has the property $\partial a = \partial \hat{a}$. Then there exists the polynomial b and the function $\Phi(p)$ such that

(15) $$\partial b < \partial a\,,$$

(16) $$\left\langle \frac{z}{a} [\hat{b}\, F^i(z)_{\hat{a}} \right\rangle_z = \left\langle \frac{p}{a} [b p^i]_a \right\rangle_p\,, \quad i = 0, 1, 2, \ldots,$$

(17) $$\left\langle \frac{z}{a} [\hat{b} z^i]_{\hat{a}} \right\rangle_z = \left\langle \frac{p}{a} [b\, \Phi(p)]_a \right\rangle_p\,, \quad i = 0, 1, 2, \ldots,$$

(18) $$[\Phi(F(z))]_{\hat{a}} = [z]_{\hat{a}}\,.$$

Proof. Properties (15) and (16) follows from Theorem 3. Using Theorem 2, the function Φ is defined as the inverse function of F by (8) and (9). Consider

$$[\Phi^j(p)]_a = \sum_{r=0}^{\partial a - 1} \varphi_{j,r} p^r\,.$$

Then using (18)

$$[\Phi^j(F(z))]_{\hat{a}} = [z^j]_{\hat{a}}\,.$$

Put

$$g_j = \sum_{r=0}^{\partial a - 1} h_r \varphi_{j,r}\,, \quad h_r = \left\langle \frac{z}{\hat{a}} [\hat{b}\, F^r(z)]_{\hat{a}} \right\rangle_z\,, \quad r = 0\ 1, 2, \ldots\ \partial a - 1\,, \quad j = 0, 1, 2, \ldots$$

then

$$g_j = \left\langle \frac{z}{\hat{a}} [\hat{b} \sum_{r=0}^{\partial a - 1} \varphi_{j,r}\, F^r(z)]_{\hat{a}} \right\rangle_z = \left\langle \frac{z}{\hat{a}} [\hat{b} z^j]_{\hat{a}} \right\rangle_z$$

and from (16)

$$g_j = \left\langle \frac{p}{a} [b \sum_{i=0}^{\partial a - 1} \varphi_{j,r} p^r]_a \right\rangle_p = \left\langle \frac{p}{a} \, b \, \Phi^j(p)]_a \right\rangle_p .$$

Hence the formula (17) follows and the proof is complete.

Remark 1. Considering b, a, Φ instead of $\hat{b}$, $\hat{a}$, F we have

$$\hat{a} = \mathscr{A}[\Phi]_a, \quad [F(\Phi(p))]_a = [p]_a .$$

LAPLACE TRANSFORMATION($\mathscr{L}$-TRANSFORMATION)

Define the sequence $g_0, g_1, g_2 \ldots$ by

$$g_i = \left\langle \frac{p}{a} [bp^i]_a \right\rangle, \quad i = 0, 1, 2, \ldots$$

where b, a are polynomials and $\partial b < \partial a$. Then by Theorem 3

$$\sum_{i=0}^{\infty} \frac{g_i}{p^{i+1}} = \frac{b}{a} .$$

Consider the function $f(t)$ for which $\mathrm{d}^i f(t)/\mathrm{d}t^i|_{t=0} = g_i$, $i = 0, 1, 2, \ldots$. Using the Taylor series at $t = 0$ we have

$$f(t) = \sum_{i=0}^{\infty} g_i \frac{t^i}{i!} = \left\langle \frac{p}{a} \left[b \sum_{i=0}^{\infty} \frac{p^i t^i}{i!} \right]_a \right\rangle_p = \left\langle \frac{p}{a} [b \, \mathrm{e}^{pt}]_a \right\rangle_p .$$

Hence the Laplace transform of $f(t)$, denoted by $\mathscr{L}(f(t))$, is equal to $b(p)/a(p)$ because $\mathscr{L}(t^i/i!) = 1/p^{i+1}$.

Theorem 5. Let polynomials b, a, $\partial b < \partial a$, with real coefficients be given. Then the inverse Laplace transform of b/a, denoted as $\mathscr{L}^{-1}(b/a)$, is the real-valued function $f(t)$ given by

$$f(t) = \left\langle \frac{p}{a} [b \, \mathrm{e}^{pt}]_a \right\rangle_p . \tag{19}$$

Moreover

$$\frac{\mathrm{d}^i f(t)}{\mathrm{d}t^i} = \left\langle \frac{p}{a} [p^i b \, \mathrm{e}^{pt}]_a \right\rangle_p , \quad i = 0, 1, 2, \ldots . \tag{20}$$

Proof. The formula (19) follows from the above discussion. To proove (20) the linearity of operations $[\cdot]_a$ and $\langle \cdot \rangle$ is sufficient. The function $f(t)$ is real valued since $[\mathrm{e}^{pt}]_a$ is a polynomial with real coefficients.

Define the sequence $g_0, g_1, g_2 \ldots$ by

$$g_i = \left\langle \frac{z}{\hat{a}} [\hat{b} z^i]_{\hat{a}} \right\rangle, \quad i = 0, 1, 2, \ldots,$$

where $\hat{b}$, $\hat{a}$ are polynomials and $\partial \hat{b} < \partial \hat{a}$. Then from the definition of the $\mathscr{Z}$-transformation and from Theorem 3 it follows that

$$\sum_{i=0}^{\infty} g_i z^{-i} = \frac{z\hat{b}}{\hat{a}}.$$

Theorem 6. Let polynomials $\hat{b}$, $\hat{a}$, $\partial \hat{b} < \partial \hat{a}$, with real coefficients be given. Then the inverse $\mathscr{Z}$-transform of $z\hat{b}/\hat{a}$, denoted as $\mathscr{Z}^{-1}(z\hat{b}/\hat{a})$, is the real sequence given by

$$g_i = \left\langle \frac{z}{\hat{a}} [\hat{b} z^i]_{\hat{a}} \right\rangle, \quad i = 0, 1, 2, \ldots. \tag{21}$$

CONVERSIONS BETWEEN $\mathscr{L}$ AND $\mathscr{Z}$ TRANSFORM (SAMPLING PROCESS)

Only the polynomials with real coefficients are considered.

Theorem 7 ($\mathscr{L} \Rightarrow \mathscr{Z}$). Let polynomials b, a, $\partial b < \partial a$, and a real number T, the sampling period, be given. Denote $f(t) = \mathscr{L}^{-1}(b/a)$. Then the sequence $\{f(nT)\}_{n=0}^{\infty}$ has the $\mathscr{Z}$-transform in the form $z\hat{b}/\hat{a}$, where

$$\begin{aligned} \hat{a} &= \mathscr{A}[e^{pT}]_a, \\ \hat{b} &= \hat{b}_0 + \hat{b}_1 z + \ldots + \hat{b}_n z^n, \quad n = \partial \hat{a} - 1, \\ b_{n-j} &= \sum_{k=0}^{n-j} \hat{a}_{k+j+1} h_k, \quad j = 0, 1, \ldots, n, \\ h_k &= \left\langle \frac{p}{a} [b\, e^{pkT}]_a \right\rangle_p, \quad k = 0, 1, \ldots, n. \end{aligned} \tag{22}$$

The proof of this theorem follows from Theorem 3 on using $F(p) = e^{pT}$ and b, a instead of $\hat{b}$, $\hat{a}$.

Theorem 8 ($\mathscr{Z} \Rightarrow \mathscr{L}$). Let polynomials $\hat{b}$, $\hat{a}$, $\partial \hat{b} < \partial \hat{a}$ and a real number T, the sampling period, be given. Denote $\{g_0, g_1, \ldots\} = \mathscr{Z}^{-1}(z\hat{b}/\hat{a})$. Then the sufficient condition for the existence of real polynomials b, a, $\partial b < \partial a$ such that $f(t) = \mathscr{L}^{-1}(b/a)$ and $f(nT) = g_n$, $n = 0, 1, 2, \ldots$ is $\partial a = \partial \hat{a}$, $a = \mathscr{A}[\ln z/T]_{\hat{a}}$.

Proof. Consider the sequence $\{h_0, h_1, \ldots\}$ given by

$$\hat{h}_i = \left\langle \frac{z}{\hat{a}} \left[\hat{b} \left(\frac{\ln z}{T} \right)^i \right]_{\hat{a}} \right\rangle_z, \quad i = 0, 1, 2, \ldots. \tag{23}$$

It is evident that h_i is real if $[\ln z]_{\hat{a}}$ is real. Using Theorem 3 for $F = \ln z/T$ we obtain

$$\sum_{i=0}^{\infty} \frac{\hat{h}_i}{p^{i+1}} = \frac{b}{a},$$

where b, a are given by $a = \mathscr{A}[\ln z/T]_{\hat{a}}$ and (10), (12).

Due to the fact that $\partial a = \partial \hat{a}$. Theorem 4 can be used with $F(z) = \ln z/T$. Then from (18) $\Phi(p) = e^{pT}$ follows. The equation (16) gives b, a (see Theorem 3), and (17) satisfies $g_n = f(nT)$, $n = 0, 1, 2, \ldots$.

Remark 2. The condition $\partial a = \partial \hat{a}$, $a = \mathscr{A}[\ln z/T]_{\hat{a}}$ is equivalent to the condition $\hat{a}(\lambda) \neq 0$ for $\lambda \leqq 0$. The principal value of $\ln z$ is used and this single-valued function is analytic in the complex plane with the nonpositive real axis deleted. From this and from the property of the anihilating polynomial the above equivalence follows. (see Remark 1/I and $\mathrm{d} \ln z/\mathrm{d}z \neq 0$).

ONE-TO-ONE CORRESPONDENCE BETWEEN $\mathscr{L}$ AND $\mathscr{Z}$ TRANSFORMS

Find the condition under which $\partial \mathscr{A}[F(s)]_{\hat{a}} = \partial \hat{a}$ if a polynomial $\hat{a}$ is given. Denote

$$\hat{a} = \prod_{i=1}^{l} (z - z_i)^{n_i},$$

where z_i is a zero of $\hat{a}$ and n_i is its multiplicity. By Theorem 1/I and Remark 1/I we have the following conditions

$$F(z_i) \neq F(z_j) \quad \text{for} \quad z_i \neq z_j, \tag{24}$$

$$\text{if} \quad n_i > 1 \quad \text{then} \quad \left. \frac{\mathrm{d}F}{\mathrm{d}z} \right|_{z=z_i} \neq 0. \tag{25}$$

Under these conditions

$$a = \prod_{i=1}^{l} (z - F(z_i))^{n_i}$$

and $\partial a = \partial \hat{a}$.

Put $F(p) = e^{pT}$. Then the condition (25) is satisfied for all complex z_i. Condition (24) gives

$$e^{z_i T} \neq e^{z_j T} \quad \text{for} \quad z_i \neq z_j$$

hence

$$z_j \neq z_i + \frac{2k\pi_j}{T}, \quad k = \ldots, -2, -1, 0, 1, 2, \ldots.$$

Theorem 9. Let a polynomial $a(p) = \prod_{i=1}^{l} (p - p_i)^{n_i}$, $p_i \neq p_j$ for $i \neq j$ be given such that $|\text{Im}\, p_i| < \pi/T$. Then no zero of $\hat{a} = \mathscr{A}[e^{pT}]_a$ lies on the nonpositive real axis and $a = \mathscr{A}[\ln z/T]_{\hat{a}}$.

Proof. By the assumptions, $e^{p_i T} \neq e^{p_j T}$ for $p_i \neq p_j$ follows and using Remark 1/I we can write

$$a = \prod_{i=1}^{l} (z - e^{p_i T})^{n_i}.$$

Since $e^{p_i T} \neq \lambda$, $\lambda \leqq 0$ the principal value of $\ln z$ gives $\ln e^{p_i T}/T = p_i$ and hence $a = \mathscr{A}[\ln z/T]_{\hat{a}}$.

Theorem 10 ($\mathscr{L} \Leftrightarrow \mathscr{Z}$). Let a real T and a function $f(t)$ be given such that $\mathscr{L}(f(t)) = b/a$, b, a polynomials, and for any zero p_i of $a(p)$ the inequality $|\text{Im}\, p_i| < \pi/T$ holds. Then there exist polynomials $\hat{b}$, $\hat{a}$, $\partial\hat{b} < \partial\hat{a} = \partial a$ given by (22) such that

$$\mathscr{Z}(\{f(nT)\}_{n=0}^{\infty}) = \frac{z\hat{b}}{\hat{a}}, \tag{26}$$

$$\left\langle \frac{p}{a} [b e^{pt}]_a \right\rangle_p = f(t) = \left\langle \frac{z}{\hat{a}} \hat{b} z^{t/T}]_{\hat{a}} \right\rangle_z, \tag{27}$$

$$\left\langle \frac{p}{a} [p^i b e^{pt}]_a \right\rangle_p = \frac{d^i f(t)}{dt^i} = \left\langle \frac{z}{\hat{a}} \left[\hat{b} \left(\frac{\ln z}{T} \right)^i z^{t/T} \right]_{\hat{a}} \right\rangle_z. \tag{28}$$

Proof. The expression (26) follows from Theorem 7. The left-hand equality in (27) and (28) follow from Theorem 5. Now we prove

$$f(t) = \left\langle \frac{z}{\hat{a}} \hat{b} z^{t/T}]_{\hat{a}} \right\rangle_z.$$

Put

$$z^{t/T} = \sum_{n=0}^{\infty} \left(\frac{\ln z}{T} \right)^n \frac{t^n}{n!}.$$

Define the function $g(t)$ by

$$g(t) = \sum_{n=0}^{\infty} \left\langle \frac{z}{\hat{a}} \left[\hat{b} \left(\frac{\ln z}{T} \right)^i \right]_{\hat{a}} \right\rangle_z \frac{t^i}{i!}.$$

Then

$$\left\langle \frac{z}{\hat{a}} \left[\hat{b} \left(\frac{\ln z}{T} \right)^i \right]_{a} \right\rangle_z = \left. \frac{\mathrm{d}^i g(t)}{\mathrm{d}t^i} \right|_{t=0}$$

is evident by the property of the Taylor series. It is clear from the property of the principal value of $\ln z$ that $\mathscr{L}(g(t)) = d/c$, where c, d are given by Theorem 3, and for any zero p^i of $c(p) = \mathscr{A}[\ln p/T]_a$ the inequality $|\mathrm{Im}\, p_i| < \pi/T$ holds. Hence $c = a$ and it follows from (27) with $t = nT$ that $d = b$. Moreover, $f(t) = g(t)$ on using Theorem 8. Differentiating (27) with respect to t, (28) is obtained and the proof is complete.

Remark 3. The conditions of Theorem 10 are the conditions for one-to-one correspondence between the $\mathscr{L}$ and $\mathscr{Z}$ transforms in the sampling process.

HALF AND DOUBLE SAMPLING PERIOD

Theorem 11. ($\mathscr{Z} \Rightarrow \mathscr{Z}$). Let polynomials b, a, $\partial b < \partial a$ be given. Denote

$$\{f_n\}_{n=0}^{\infty} = \mathscr{Z}^{-1}\left(\frac{zb}{a}\right).$$

Then

$$\mathscr{Z}(\{f_{2n}\}_{n=0}^{\infty}) = \frac{zd}{c}$$

where

$$c(z^2) = a(z)\, a(-z)\,, \tag{29}$$

$$z^2\, d(z^2) = \frac{z}{2}\left(b(z)\, a(-z) - b(-z)\, a(z)\right). \tag{30}$$

Proof. Write

$$\frac{z\, b(z)}{a(z)} = h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots,$$

$$\frac{z\, b(-z)}{a(-z)} = h_0 - h_1 z^{-1} + h_2 z^{-2} - \ldots$$

then

$$\frac{1}{2}\left(\frac{z\,b(z)}{a(z)} + \frac{-z\,b(-z)}{a(-z)}\right) = \frac{z\,b(z)\,a(-z) - z\,b(-z)\,a(z)}{2\,a(z)\,a(-z)} = h_0 + h_2 z^{-2} + h_4 z^{-4} + \dots$$

and the proof is complete.

Theorem 12. Consider a function $f(t)$ and a sampling period T by Theorem 10. Let $\mathscr{Z}(\{f(nT)\}_{n=0}^{\infty}) = zd/c$, $\partial d < \partial c$. Then $\mathscr{Z}(\{f(n(T/2))\}_{n=0}^{\infty}) = zb/a$, where a is the spectral factor of $c(z^2)$ (all zeros of $a(z)$ lie in the right half of the complex plane) and b is the solution of the "symmetric" polynomial equation (30). Algorithms for spectral factorization (29) and the solution of "symmetric" polynomial equation (30) are given in [3].

Proof. Consider Theorem 4 with $\hat{b} = d$, $\hat{a} = c$, $F(z) = \sqrt{z}$. Write

$$\frac{zb}{a} = \sum_{i=0}^{\infty} h_i z^{-i}\,, \quad \frac{zd}{c} = \frac{z\hat{b}}{\hat{a}} = \sum_{i=0}^{\infty} e_i z^{-i}$$

then $e_i = h_{2i}$ by (16) and $\Phi(z) = z^2$ is the inverse function of $F(z) = \sqrt{z}$ by (18) with respect to the properties of the function $f(t)$ which implies that no zero of $c(z)$ lies on the nonpositive real axis and hence $\partial\mathscr{A}[\sqrt{z}]_c = \partial c$.

From this polynomial $a = \mathscr{A}[\sqrt{z}]_c$ has no zero on the imaginary axis and equation (30) has only one solution (see [3]). Denoting $c = \prod_{i=1}^{l} (z - z_i)^{n_i}$ we obtain $a = \prod_{i=1}^{l} (z - \sqrt{z_i})^{n_i}$ by the property of the anihilating polynomial of $\sqrt{z}$ modulo c. Hence $a(z)$ coincides with the above mentioned spectral factor of $c(z^2)$.

To complete this proof denote $\mathscr{L}(f(t)) = s/r$. Then $c = \mathscr{A}[e^{pT}]_r$ and $a = \mathscr{A}[e^{p(T/2)}]_r$. If the conditions of Theorem 10 are satisfied for T, then they are satisfied for $T/2$, too.

Theorem 13. Consider a function $f(t)$ and a sampling period T by Theorem 9. Let $\mathscr{Z}(\{f(nT)\}_{n=0}^{\infty}) = zb/a$, $\partial b < \partial a$. Then $\mathscr{Z}\{f((n + \varepsilon)\,T)\}_{n=0}^{\infty} = zd/a$, where

$$d = [bz^{\varepsilon}]_a\,. \tag{31}$$

Proof. By Theorem 9

$$f((n + \varepsilon)\,T) = \left\langle \frac{z}{a}\,[bz^{\varepsilon}z^{n}]_a \right\rangle_z = \left\langle \frac{z}{a}\,[[bz^{\varepsilon}]_a z^{n}]_a \right\rangle_z\,, \quad n = 0, 1, 2, \dots.$$

Putting

$$d = [bz^{\varepsilon}]_a$$

the proof is completed.

Consider $b(s)/a(s)$, $\partial b \leqq \partial a$ as the $\mathscr{L}$-transfer function of a given continuous plant, and a sampling period T.

As it is known, the $\mathscr{Z}$-transfer function of $b(s)/a(s)$ for $H(s) = (1 - e^{-sT})/s$ (zero-order holder) has the form $d(z)/c(z)$, $\partial d \leqq \partial c$, where

$$\frac{z\, d(z)}{(z-1)\, c(z)} = \mathscr{Z}(\{f(nT)\}_{n=0}^{\infty})$$

and

$$\mathscr{L}(f(t)) = \frac{b(s)}{s\, a(s)}\,.$$

If

$$H(s) = \frac{Ts+1}{Ts^2}(1 - e^{-sT})^2$$

(first-order holder), then the $\mathscr{Z}$-transfer function of $b(s)/a(s)$ has the form $d(z)/c(z)$, $\partial d \leqq \partial c$, where

$$\frac{z^2\, d(z)}{(z-1)^2\, c(z)} = \mathscr{Z}(\{f(nT)\}_{n=0}^{\infty})$$

and

$$\mathscr{L}(f(t)) = \frac{(Ts+1)\, b(s)}{Ts^2\, a(s)}\,.$$

All theorems for conversions between the $\mathscr{L}$ and $\mathscr{Z}$ transforms are applicable in this way for transfer functions.

Example 1. Consider the $\mathscr{L}$-transfer function $b(s)/a(s)$, $b = 1$, $a = 6 + 11s + 6s^2 + s^3$, the sampling period $T = 1$, time delay between the input and output $\varepsilon = 0{\cdot}5$, and the sampling and holding element

$$H(s) = \frac{1 - e^{-sT}}{s}\,.$$

Compute the $\mathscr{Z}$-transfer function of $b(s)/a(s)$ for the given condition.

Solution. First, compute the $\mathscr{Z}$-transform of $b(s)/(s\, a(s))$ in the form

$$\frac{z\, d(z)}{(z-1)\, c(z)}$$

by using Theorem 7. Then the $\mathscr{Z}$-transfer function of $b(s)/a(s)$ for $\varepsilon = 0$ is given as

$$d(z) = 2{\cdot}095873420\ \mathrm{E} - 03 + 4{\cdot}236740183\ \mathrm{E} - 02z + 4{\cdot}209674297\ \mathrm{E} - 02z^2\,,$$

$$c(z) = -2{\cdot}478752177\ \mathrm{E} - 03 + 7{\cdot}484065426\ \mathrm{E} - 02z - 5{\cdot}530017928\ \mathrm{E} - \\ - 01z^2 + z^3\,.$$

The $\mathscr{Z}$-transfer function of $b(s)/a(s)$ for $\varepsilon = 0{\cdot}5$ can be obtained in the form $e(z)/c(z)$, where

$$e(z) = [d(z)\,.\,z^{\varepsilon}]_{(z-1)c(z)}$$

on using Theorem 12:

$$e(z) = 1{\cdot}127862800\ \mathrm{E} - 04 + 1{\cdot}391807043\ \mathrm{E} - 02z + 6{\cdot}237646413\ \mathrm{E} - 02z^2 + \\ + 1{\cdot}015269737\ \mathrm{E} - 02z^3\,.$$

After the inverse procedure, from $e(z)/c(z)$ back to the given $\mathscr{L}$-transfer function $b(s)/a(s)$, the polynomials b, a are computed with the accuracy of fifteen significant digits.

Example 2. Find the properties of the conversion between the $\mathscr{L}$ and $\mathscr{Z}$-transforms for extremely small period T.

For $T \to 0^+$ we can write $e^{pT} = 1 + pT$. Consider $b(p)/a(p) = \mathscr{L}(f(t))$. Using Theorem 5

$$f(nT) = \left\langle \frac{p}{a(p)}\,[b(p)\,(1 + pT)^n]_{a(p)} \right\rangle_p\,. \tag{32}$$

The $\mathscr{Z}$-transform of $\{f(nT)\}_{n=0}^{\infty}$ is given as

$$\sum_{n=0}^{\infty} f(nT)\, z^{-n} = \left\langle \frac{p}{a(p)} \left[b(p) \sum_{n=0}^{\infty} \left(\frac{1 + pT}{z} \right)^n \right]_{a(p)} \right\rangle_p\,.$$

For $|(1 + pT)/z| < 1$ we can write

$$\sum_{n=0}^{\infty} f(nT)\, z^{-n} = \left\langle \frac{p}{a(p)} \left[\frac{b(p)\, z\,.\,\dfrac{1}{T}}{\dfrac{z-1}{T} - p} \right]_{a(p)} \right\rangle_p\,.$$

From the proof of Theorem 3

$$\left\langle \frac{p}{a(p)} \left[\frac{b}{s - p} \right]_{a(p)} \right\rangle_p = \frac{b(s)}{a(s)}$$

follows. Hence

$$\sum_{n=0}^{\infty} f(nT)\, z^{-n} = \frac{1}{T} \frac{z\, b\left(\dfrac{z-1}{T}\right)}{a\left(\dfrac{z-1}{T}\right)} = \frac{z\, \hat{b}(z)}{\hat{a}(z)}\,.$$

Put

$$\hat{a}(z) = T^{\partial a}\, a\left(\frac{z-1}{T}\right)$$

(33)
$$\hat{b}(z) = T^{\partial a - 1}\, b\left(\frac{z-1}{T}\right)$$

then, as it is evident,

$$\lim_{T \to 0^+} \hat{a}(z) = (z-1)^{\partial a}\, a_{\partial a}\,, \quad \lim_{T \to 0^+} \hat{b} = (z-1)^{\partial a - 1}\, b_{\partial a - 1}$$

and

(34)
$$\lim_{T \to 0^+} \frac{z \hat{b}}{\hat{a}} = \frac{z b_{\partial a - 1}}{(z-1)\, a_{\partial a - 1}}\,.$$

Hence for extremely small sampling period T, the $\mathscr{Z}$-transform can be converted back into the $\mathscr{L}$-transform with some numerical error only, see (34).

CONCLUSION

Procedures for conversions between the $\mathscr{L}$ and $\mathscr{Z}$ transforms developed in this paper are suitable, due to small number of operations and high accuracy, for numerical and theoretical studies of the sampling process in the DDC problems.

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