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Nash and Stackelberg Solutions to General Linear-Quadratic Two-Player Difference Games Part II. Open-Closed Strategies

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This paper deals with difference games in which the information structures of the players are not equal. Nash and Stackelberg solutions to deterministic linear-quadratic two-player difference games are presented. The open-closed strategies are considered where one of the players can only use the initial state whereas the other has access to perfect memory information of the state. Stackelberg open-closed strategies are treated only in the case where the leader has access to open-loop information. A combined dynamic programming and augmentation techniques approach is developed. A computationally straightforward recursive algorithm is derived for the Nash open-closed solution.

1. INTRODUCTION

In the first part of this paper [1] we studied game problems in which the information structures of the players were always equal. However, games with unequal information structures are also quite important because corresponding situations can easily occur in practice. Interesting game problems arise when the information structures of the players are no more equal. One of the players may for example have access to a perfect memory information set while the other player may only use the initial state vector when making his decision. This is just one possibility, more generally, the players could have different periodic information structures [2]. However, in the following we shall be restricted to the former situation. In the Nash problem both players may have either of the information structures whereas the Stackelberg problem is solved only when the follower has access to perfect memory information and the leader to open-loop information. In the opposite case, algorithms for the Stackelberg strategies are not found because stagewise solution techniques remain no more applicable when the leader plays a closed-loop strategy. Previously corresponding difference games where the players have different information structures have only been treated by Basar [3]. Foley and Schmitendorf [4] have presented some results on the continuous-time Nash open-closed problem with a special type of quadratic cost functionals

Let the system be again given by the linear equation

$$x(k+1) = A(k)\,x(k) + B_1(k)\,u(k) + B_2(k)\,v(k) \tag{2.1}$$

where $x(k) \in R^n$, $u(k) \in R^p$ and $v(k) \in R^q$ for all $k \in K$, $K = \{0, 1, \ldots, N-1\}$. The cost function which player i tries to minimize is of the following general quadratic form

$$\begin{aligned} J_i = {} & \tfrac{1}{2}x^T(N)\,S_i\,x(N) + \tfrac{1}{2}\sum_{k=0}^{N-1}[x^T(k)\,Q_i(k)\,x(k) + 2x^T(k)\,M_{i1}(k)\,u(k) + \\ & + 2x^T(k)\,M_{i2}(k)\,v(k) + u^T(k)\,R_{i1}(k)\,u(k) + 2u^T(k)\,N_i(k)\,v(k) + \\ & + v^T(k)\,R_{i2}(k)\,v(k)]\,, \quad i = 1, 2\,, \end{aligned} \tag{2.2}$$

where S_i, $Q_i(k)$, $R_{ij}(k)$ $i \neq j$, $i, j = 1, 2$ are symmetric positive semidefinite matrices and $R_{ii}(k)$, $i = 1, 2$ are symmetric positive definite matrices for all $k \in K$. The system and weighting matrices are time-varying and of appropriate dimensions (see [1]).

Considering now the game (2.1), (2.2) where player 1 has access to the open-loop information set and player 2 to the perfect-memory information set. Solution of such a game may be initiated by first considering the strategy of player 2 when the control of player 1 is fixed to an arbitrary open-loop strategy $u^0(k)$. At this point it is not yet necessary to know to what solution concept the open-loop strategy $u^0(k)$ is related. Moreover, the form of the optimization problem that player 2 has to solve when choosing his closed-loop strategy will be the same even if he plays a Nash strategy or if he plays a Stackelberg follower's strategy. Thus it is possible to determine first the general solution to the optimization problem of player 2.

The control of player 2 that minimizes J_2 with $u(k) = u^0(k)$ can be shown (see Appendix) to be the following affine feedback strategy, provided the required inverses exist:

$$v^*(k) = -G_2^{-1}(k)\,F_2(k)\,x(k) + t(k)\,, \tag{2.3}$$

where $t(k)$ is a q-vector defined by

$$t(k) = -G_2^{-1}(k)\,[E_2(k)\,u^0(k) + B_2^T(k)\,s(k)] \tag{2.4}$$

which depends on the initial state vector through the open-loop strategy of player 1. The n-vector $s(k)$ is given by the recursive equation

$$s(k) = K_2(k+1)\,u^0(k+1) + (A^0(k+1))^T\,s(k+1)\,, \quad s(N-1) = 0 \tag{2.5}$$

where

$$K_2(k) = F_{21}^T(k) - F_2^T(k)\,G_2^{-1}(k)\,E_2(k)\,, \quad K_2(N) = 0 \tag{2.6}$$

and

(2.7) $$A^0(k) = A(k) - B_2(k)\, G_2^{-1}(k)\, F_2(k)\,, \quad A^0(N) = 0\,.$$

Matrices $G_2(k)$, $E_2(k)$, $E_2(k)$ and $F_{21}(k)$ appearing above are defined in terms of $P_2(k+1)$ by the following equations

(2.8) $$G_2(k) = R_{22}(k) + B_2^T(k)\, P_2(k+1)\, B_2(k)$$

(2.9) $$F_2(k) = M_{22}^T(k) + B_2^T(k)\, P_2(k+1)\, A(k)$$

(2.10) $$E_2(k) = N_2^T(k) + B_2^T(k)\, P_2(k+1)\, B_1(k)$$

(2.11) $$F_{21}(k) = M_{21}^T(k) + B_1^T(k)\, P_2(k+1)\, A(k)\,.$$

The matrix $P_2(k)$ is obtained from the solution of the symmetric matrix Riccati difference equation

(2.12) $$P_2(k) = Q_2(k) + A^T(k)\, P_2(k+1)\, A(k) -$$
$$- \left[M_{22}^T(k) + B_2^T(k)\, P_2(k+1)\, A(k)\right]^T \left[R_{22}(k) + B_2^T(k)\, P_2(k+1)\, B_2(k+1)\right]^{-1} .$$
$$. \left[M_{22}^T(k) + B_2^T(k)\, P_2(k+1)\, A(k)\right]$$

with $P_2(N) = S_2$.

This solution gives a closed-loop strategy because the $t(k)$ term in the feedback equation (2.3) is a function of the initial state vector. Hence player 2 is assumed to have access to the perfect memory information set.

Sufficient conditions for the existence of the solution are that $B_{22}(k)$ is positive definite and that S_2, $Q_2(k)$ and $[Q_2(k) - M_{22}(k)\, R_{22}^{-1}(k)\, M_{22}^T(k)]$ are positive semidefinite matrices for all $k \in K$. Now $G_2(k)$ will clearly be nonsingular for all $k \in K$ since $R_{22}(k) > 0$, for all $k \in K$, and $S_2 \geqq 0$ implies that $P_2(N-1) \geqq 0$ which again guarantees that $P_2(N-2)$, $P_2(N-3), \ldots, P_2(0)$ will be positive semidefinite as can be concluded by the following equivalent expression for the Riccati-equations (2.12):

(2.13) $$P_2(k) = Q_2(k) - M_{22}(k)\, R_{22}^{-1}(k)\, M_{22}^T(k) +$$
$$+ \left[A(k) - B_2(k)\, R_{22}^{-1}(k)\, M_{22}^T(k)\right]^T .$$
$$. \left[P_2(k+1) - P_2(k+1)\, B_2(k) \left(R_{22}(k) + B_2^T(k)\, P_2(k+1)\, B_2(k)\right)^{-1} B_2(k)\, P_2(k+1)\right]$$
$$. \left[A(k) - B_2(k)\, R_{22}^{-1}(k)\, M_{22}^T(k)\right] .$$

The above defined matrices $P_2(k)$, $K_2(k)$ and $A^0(k)$ appear as weightings in the optimal cost-to-go of player 2 when it is expressed in the following form

(2.14) $$J_2^*(k) = \tfrac{1}{2}\left[x^T(k)\, P_2(k)\, x(k) + 2x^T(k)\, K_2(k)\, u^0(k) + 2s^T(k)\, A^0(k)\, x(k)\right] +$$
$$+ l_2^{(k)}(x_0)\,,$$

where $l_2^{(k)}(x_0)$ is a scalar function of the initial state. One may note that for a zero-control of player 1, when $u(k) = 0$ for all $k \in K$, (2.3) and (2.14) results in the solution of the remaining optimal regulator problem. This is naturally something that one would have expected to occur.

As the strategy of player 2 will always be given by (2.3) when player 1 uses any open-loop strategy $u^0(k)$, we can proceed in the problem solution by substituting (2.3) in the system equation and by expressing the cost function of player 1 in terms of the initial state vector. After the substitution we have

$$x(k+1) = A^0(k)\,x(k) + B_1(k)\,u(k) + B_2(k)\,\iota(k)\,. \tag{2.15}$$

Introducing now the augmented vectors $\bar{x}$, $\bar{x}_0$ and $\bar{u}$ defined in Chapter 3 of the first part of this paper [1] together with the qN-vector $\bar{\iota}$

$$\bar{\iota} \triangleq [\iota^T(0) \mid \iota^T(1) \mid \ldots \mid \iota^T(N-1)]^T\,, \tag{2.16}$$

the system equation (2.1) can be written with the aid of the auxiliary control variable $\bar{\iota}$ in the form

$$\bar{x} = \bar{A}^0\bar{x}_0 + \bar{B}_1^0\bar{u} + \bar{B}_2^0\bar{\iota}\,, \tag{2.17}$$

where $\bar{A}^0$ is a block diagonal $nN \times nN$-matrix. It is defined analogously to $\bar{A}$ by eq. (3.6) in [1] when $\Phi(k, l)$ is replaced by $\Phi^0(k, l)$ which is the fundamental matrix associated with (2.15) i.e. we have now

$$\Phi^0(k, l) = A^0(k-1)\,A^0(k-2)\cdot\ldots\cdot A^0(l)\,,\quad k > l \tag{2.18}$$

and

$$\Phi^0(k, k) = I\,.$$

Similarly, $\bar{B}_1^0$ and $\bar{B}_2^0$ are defined by eq. (3.8) in [1] when the $\Phi(k, l)$ matrices are replaced by $\Phi^0(k, l)$ matrices respectively for all k and l.

The cost function of player 1 when $v(k) = v^*(k)$ can readily be written by employing these augmented vectors:

$$\begin{aligned} J_1^0 = \tfrac{1}{2}[&\bar{x}^T\bar{Q}_1^0\bar{x} + 2\bar{x}^T\bar{M}_{11}^0\bar{u} + 2\bar{x}^T\bar{M}_{12}^0\bar{\iota} + \bar{u}^T\bar{R}_{11}^0\bar{u} + 2\bar{u}^T\bar{N}_1^0\bar{\iota} + \\ &+ \bar{\iota}^T\bar{R}_{12}^0\bar{\iota} + \bar{x}_0^T\bar{Q}_{01}^0\bar{x}_0 + 2\bar{x}_0^T\bar{M}_{011}^0\bar{u} + 2\bar{x}_0^T\bar{M}_{012}^0\bar{\iota}] \end{aligned} \tag{2.19}$$

The augmented weighting matrices $\bar{Q}_1^0, \bar{M}_{11}^0, \bar{M}_{12}^0, \bar{R}_{11}^0, \bar{N}_1^0, \bar{R}_{12}^0, \bar{Q}_{01}^0, \bar{M}_{011}^0$ and $\bar{M}_{012}^0$ are again defined analogously to $\bar{Q}_1, \bar{M}_{11}, \bar{M}_{12}, \bar{R}_{11}, \bar{N}_2, \bar{R}_{12}, \bar{Q}_{01}, \bar{M}_{011}$ and $\bar{M}_{012}$ using (3.10)–(3.15) in [1]. Particularly some of the matrices remain unchanged:

$$\bar{R}_{11}^0 \equiv \bar{R}_{11}\,,\quad \bar{N}_1^0 \equiv \bar{N}_1\,,\quad \bar{R}_{12}^0 \equiv \bar{R}_{12}\,.$$

The other weighting matrices are obtained by modifying the definitions of the corresponding original weightings so that the block matrices $Q_1(k)$, $M_{11}(k)$ and $M_{12}(k)$

are replaced by $Q_1^0(k)$, $M_{11}^0(k)$ and $M_{12}^0(k)$ respectively for each $k \in K$ when

$$Q_1^0(k) = Q_1(k) - M_{12}(k)\,G_2^{-1}(k)\,F_2(k) - F_2^T(k)\,G_2^{-1}(k)\,M_{12}^T(k) + \\ + F_2^T(k)\,G_2^{-1}(k)\,R_{12}(k)\,G_2^{-1}(k)\,F_2(k) \tag{2.20}$$

$$M_{11}^0(k) = M_{11}(k) - F_2^T(k)\,G_2^{-1}(k)\,N_1^T(k) \tag{2.21}$$

$$M_{12}^0(k) = M_{12}(k) - F_2^T(k)\,G_2^{-1}(k)\,R_{12}(k)\,. \tag{2.22}$$

Thus the block structure and the dimensions of the matrices are not changed.

The dynamic game problem faced by player 1 has now been brought into a form (2.19) and (2.15) which allows the derivation of the open-loop strategies of player 1 subject to the closed-loop strategy of player 2 both in the Nash and Stackelberg cases.

2.1. The Nash Open-Closed Solution

The game problem of player 1 is now to solve the static minimization problem with the cost given in terms of the initial state vector by (2.19) subject to the constraint (2.15) Due to the assumed Nash solution concept the minimization of J_1^0 with respect to $\bar{u}$ is not affected by the terms that only depend on $\bar{t}$ although $\bar{t}$ becomes a function of the Nash strategy of player 1.

Direct derivation of J_1^0 in (2.19) yields then

$$\bar{u}^* = (\bar{G}_1^0)^{-1}\,[\bar{F}_1^0\bar{x}_0 + \bar{E}_1^0\bar{t}]\,, \tag{2.23}$$

where

$$\bar{G}_1^0 = \bar{R}_{11}^0 + (\bar{B}_1^0)^T\,\bar{M}_{11}^0 + [(\bar{M}_{11}^0)^T + (\bar{B}_1^0)^T\,\bar{Q}_1^0]\,\bar{B}_1^0\,, \tag{2.24}$$

$$\bar{F}_1^0 = (\bar{M}_{011}^0)^T + [(M_{11}^0)^T + (\bar{B}_1^0)^T\,\bar{Q}_1^0]\,\bar{A}^0\,, \tag{2.25}$$

$$\bar{E}_1^0 = \bar{N}_1^0 + (\bar{B}_1^0)^T\,\bar{M}_{12}^0 + [(\bar{M}_{11}^0)^T + (\bar{B}_1^0)^T\,\bar{Q}_1^0]\,\bar{B}_2^0\,. \tag{2.26}$$

The open-loop solution $\bar{u}^*$ in (2.22) is still in an implicit form as the elements of $\bar{t}$ are $t(k)$ vectors which were defined in terms of the open-loop strategy of player 1. Hence we have to identify these strategies with each other and solve the resulting linear matrix equation. The recursion (5.5) yields the linear relationship between $\bar{s}$ and $\bar{u}^0$

$$\bar{s} = \bar{K}_2\bar{u}^0\,, \tag{2.27}$$

where

$$\bar{s}^0 \triangleq [(s^0\,(0))^T \mid (s^0\,(1))^T \mid \ldots \mid (s^0\,(N-1))^T]^T\,, \tag{2.28}$$

$$\bar{u}^0 \triangleq [(u^0\,(0))^T \mid (u^0\,(1))^T \mid \ldots \mid (u^0\,(N-1))^T]^T \tag{2.29}$$

and the $n \times p$ block elements of $\bar{K}_2$, an $nN \times pN$ matrix, are given by

$$[\bar{K}_2]_{kl} \triangleq \begin{cases} (\Phi^0(l-1, k))^T K_2(l-1) & \text{for} \quad k < l \\ 0 & \text{else} \end{cases} \tag{2.30}$$

when $k, l = 1, 2, \ldots, N$.

Further by the definition of $t(k)$ we have

$$\bar{t} = -\bar{L}_u \bar{u}^0 - \bar{L}_s \bar{s}\,, \tag{2.31}$$

where $\bar{L}_u$ and $\bar{L}_s$ are $qN \times pN$ and $qN \times nN$ block diagonal matrices, respectively, given by the $q \times p$ blocks

$$[\bar{L}_u]_{kk} = G_2^{-1}(k-1)\, E_2(k-1) \tag{2.32}$$

and by the $q \times n$ blocks

$$[\bar{L}_s]_{kk} = G_2^{-1}(k-1)\, B_2^T(k-1) \tag{2.33}$$

for $k = 1, 2, \ldots, N$.
Combining these two equations (2.27) and (2.31) yields the linear dependence of $\bar{t}$ on $\bar{u}^0$

$$\bar{t} = -[\bar{L}_u + \bar{L}_s \bar{K}]\, \bar{u}^0 = -\bar{L}_t \bar{u}^0 \tag{2.34}$$

where $\bar{L}_t$ becomes an upper block triangular matrix.

The Nash open-loop strategy $\bar{\gamma}_1^*(x_0)$ for player 1 results from (2.23) after the substitution of $\bar{t}$ by $-\bar{L}_t \bar{u}^*$ provided that the required inverses exist:

$$\bar{u}^* = \bar{\gamma}_1^*(x_0) = -[\bar{G}_1^0 - \bar{E}_1^0 \bar{L}_t]^{-1}\, \bar{F}_1^0 \bar{x}_0\,. \tag{2.35}$$

The obtained solution $\bar{u}^*$ yields a linear strategy $\gamma_1^{(k)*}(x_0)$ for player 1 at each stage. Correspondingly the closed-loop strategy of player 2 in this Nash open-closed game will be a linear function of the current time and initial state vectors at each stage. This closed-loop strategy $v^*(k) = \gamma_2^{(k)*}(x(k), x_0)$ is naturally obtained from (2.3) when we set $u^0(k) = \gamma_1^{(k)*}(x_0)$ in (2.4) and (2.5).

It is clear that the Nash open-closed game, where player 1 plays closed-loop and player 2 open-loop, is solved in an analogous manner only by changing the roles of u and v.

2.2. The Stackelberg Open-Closed Solution

As was pointed out before the general form of the closed-loop strategy of player 2, the follower, is the affine strategy (2.3) for all open-loop strategies of player 1, the leader. The Stackelberg solution concept implies that the leader knows what the strategy of the follower will be after the leader has decided his own strategy. This means that the leader takes into account the dependence of the $t(k)$ term appearing in the follower's strategy on his own strategy $u(k)$.

The open-loop strategy for the leader is thus obtained by minimizing J_1^0 in (2.19) with respect to $\bar{u}$ subject to the system equation (2.15) and to the equation

$$\bar{t} = -\bar{L}_t \bar{u} \,. \tag{2.36}$$

These constraints mean that, unlike in the Nash case, all the terms in J_1^0 except for $\bar{x}_0^T \bar{Q}_{01}^0 \bar{x}_0$ enter the optimization problem.

The solution is again first determined in the implicit form

$$\bar{u}^* = -[\bar{G}_1^0 - \bar{L}_t^T (\bar{E}_1^0)^T]^{-1} [(\bar{F}_1^0 - \bar{L}_t^T \bar{F}_{12}^0)\,\bar{x}_0 + (\bar{E}_1^0 - \bar{L}_t^T \bar{G}_{12}^0)\,\bar{t}] \tag{2.37}$$

where

$$\bar{G}_{12}^0 = \bar{R}_{12}^0 + (\bar{B}_2^0)^T \bar{M}_{12}^0 + [(\bar{M}_{12}^0)^T + (B_2^0)^T \bar{Q}_1^0]\, B_2^0 \,, \tag{2.38}$$

$$\bar{F}_{12}^0 = (\bar{M}_{012}^0)^T + [(\bar{M}_{12}^0)^T + (B_2^0)^T \bar{Q}_1^0]\, \bar{A}^0 \,. \tag{2.39}$$

The open-loop strategy can then be directly solved after setting $\bar{t} = -\bar{L}_t \bar{u}^*$ which yields

$$\bar{u}^* = \bar{\gamma}^*(x_0) = -[\bar{G}_1^0 + \bar{L}_t^T \bar{G}_{12}^0 \bar{L}_t - \bar{L}_t^T (\bar{E}_1^0)^T \bar{E}_1^0 \bar{L}_t]^{-1} [\bar{F}_1^0 - \bar{L}_t \bar{F}_{12}^0]\, \bar{x}_0 \,. \tag{2.40}$$

It is again assumed that the required inverses exist. Thus the Stackelberg open-closed solution becomes a linear strategy of the initial state for the leader and a linear strategy of the current time state and the initial state for the follower.

The above presented solution techniques for the Nash and Stackelberg open-closed strategies may at first sight seem very complicated. However, the procedures are in principle quite straightforward. Firstly the Riccati-equation for $P_2(k)$ is solved backwards in time starting from the last stage with $P_2(N) = S_2$. This can be done independently since the equation (2.12) for $P_2(k)$ is not coupled with any equations related to the strategies of player 1. Secondly the solution to $P_2(k)$ determines the matrices needed in (2.35) and (2.40) and the open-loop strategies of player 1 are obtained after performing the matrix inversions and multiplications. Thirdly the closed-loop strategies of player 2 are given by equation (5.3) when $t(k)$ is taken from (5.29) with $u^0(k)$ replaced by $\gamma_1^{(k)*}(x_0)$.

The major problems are again the computational difficulties which are due to the size of the matrices involved since their dimensions are proportional to the number of time points included.

Although the preceding open-closed solution algorithms for the Nash and Stackelberg games are partly given by low dimensional recursive equations, there remains, however, some operations which have to be performed using the augmented matrices. As it was already noted in [1] these will make the computations increasingly cumbersome when the time interval becomes long. In this section we shall present a recursive algorithm for the solution of the Nash open-closed game, where the equations with augmented matrices have been replaced by low dimensional matrix difference equations. The Stackelberg open-loop solution can not, however, be put into an equally simple recursive form.

We omit the rather complicated derivation of the following equations but it can be shown that the Nash open-closed solution (2.36), (2.3) is given in feedback form by

$$u^*(k) = -H_u(k)\,x(k) \tag{3.1}$$

$$t^*(k) = -H_t(k)\,x(k)\,, \tag{3.2}$$

$$\begin{aligned} v^*(k) &= -G_2^{-1}(k)\,F_2(k)\,x(k) + t^*(k) = \\ &= -[G_2^{-1}(k)\,F_2(k) + H_t(k)]\,x(k)\,, \end{aligned} \tag{3.3}$$

where the feedback gains are

$$\begin{aligned} H_u(k) &= [G_1(k) - E_1(k)\,G_3^{-1}(k)\,E_3(k)]^{-1}\,. \\ &\cdot [F_1(k) - E_1(k)\,G_2^{-1}(k)\,F_2(k) - E_1(k)\,G_3^{-1}(k)\,F_3(k)]\,, \end{aligned} \tag{3.4}$$

$$\begin{aligned} H_t(k) &= [G_3(k) - E_3(k)\,G_1^{-1}(k)\,E_1(k)]^{-1}\,. \\ &\cdot [F_3(k) - E_3(k)\,G_1^{-1}(k)\,F_1(k) + E_3(k)\,G_1^{-1}(k)\,E_1(k)\,G_2^{-1}(k)\,F_2(k)] \end{aligned} \tag{3.5}$$

with

$$G_1(k) = R_{11}(k) + B_1^T(k)\,P_1(k+1)\,B_1(k)\,, \tag{3.6}$$

$$F_1(k) = M_{11}(k) + B_1^T(k)\,P_1(k+1)\,A(k)\,, \tag{3.7}$$

$$E_1(k) = N_1(k) + B_1^T(k)\,P_1(k+1)\,B_2(k)\,, \tag{3.8}$$

$$G_3(k) = G_2(k) - B_2^T(k)\,P_3(k+1)\,B_2(k)\,, \tag{3.9}$$

$$F_3(k) = B_2^T(k)\,P_3(k+1)\,[B_2(k)\,G_2^{-1}(k)\,F_2(k) - A(k)]\,, \tag{3.10}$$

$$E_3(k) = E_2(k) - B_2^T(k)\,P_3(k+1)\,B_1(k)\,. \tag{3.11}$$

Matrices $G_2(k)$, $F_2(k)$ and $E_2(k)$ in the above equations are defined in terms of $P_2(k+1)$ by (2.8)–(2.10). All the three unknown matrices $P_1(k+1)$, $P_2(k+1)$ and $P_3(k+1)$ are obtained from the solutions of matrix difference equations. The symmetric Riccati equation for $P_2(k+1)$ was given in (2.12) and it does not depend on the pair of coupled asymmetric equations for $P_1(k+1)$ and $P_3(k+1)$:

(3.12)
$$P_1(k) = Q_1^0(k) - M_{11}^0(k)\,H_u(k) - M_{12}^0(k)\,H_t(k) + \\ + (A^0(k))^T P_1(k+1)\left[A^0(k) - B_1(k)\,H_u(k) - B_2(k)\,H_t(k)\right],$$

(3.13)
$$P_3(k) = K_2(k)\,H_u(k) + (A^0(k))^T P_3(k+1)\,. \\ .\left[A^0(k) - B_1(k)\,H_u(k) - B_2(k)\,H_t(k)\right]$$

with

$$P_1(N) = S_1 \quad \text{and} \quad P_3(N) = 0\,.$$

This pair of equations is moreover coupled with the solution of (5.7) since $P_2(k+1)$ appears in the expressions for $K_2(k)$, $H_u(k)$ and $H_t(k)$.

These equations provide a means for obtaining the Nash open-closed solution in a feedback form. The solution can be further developed so that its representation will correspond to the assumed information structures of the players. The control of player 1 and the auxiliary control variable $t^*(k)$ are given as open-loop strategies by

(3.14)
$$u^*(k) = \gamma_1^{(k)*}(x_0) = -H_u(k)\,\Psi^0(k)\,x_0$$

and

(3.15)
$$t^*(k) = \gamma_3^{(k)*}(x_0) = -H_t(k)\,\Psi^0(k)\,x_0$$

where $\Psi^0(k)$ satisfies the matrix difference equation

(3.16)
$$\Psi^0(k) = \left[A^0(k-1) - B_1(k-1)\,H_u(k-1) - B_2(k-1)\,H_t(k-1)\right]\Psi^0(k-1)$$

with the initial condition

$$\Psi^0(0) = I\,.$$

Substituting (3.15) into (3.3) yields then the closed-loop strategy of player 2

(3.17)
$$v^*(k) = \gamma_2^{(k)*}(x(k), x_0) = \\ = -G_2^{-1}(k)\,F_2(k)\,x(k) - H_t(k)\,\Psi^0(k)\,x_0\,.$$

Thus we have obtained a recursive algorithm to solve the Nash open-closed game. In the first phase the feedback gains $H_u(k)$ and $H_t(k)$ are solved. The recursion proceeds backwards in time starting from the last stage in such a manner that at stage k we first evaluate $P_1(k)$, $P_2(k)$ and $P_3(k)$ from equations (3.12). (2.12) and (3.13), whose right hand sides include only matrices known from the previous stage $k+1$. These matrices then determine the values of all the matrices needed at the next stage, $k-1$. The other phase of the algorithm can be started when we have reached

$k = 0$. Then, $\Psi^0(k)$ is evaluated recursively in the forward direction from (3.16) and finally the solution strategies are obtained at each stage from (3.14) and (3.17).

A comparison of this with the recursive algorithm for the Nash open-open game in [1] shows that here we have to solve three matrix difference equations instead of those two in the open-open game. However, in both cases there are only two coupled asymmetric matrix difference equations. It is interesting to note that the control problem of player 2, who plays a closed-loop strategy in the open-closed game, has certain characteristic features similar to those of the discrete-time linear-quadratic tracking problem. The affine feedback law (2.3) has a noncausal property in the sense that the "tracking" term $t(k)$ is a function of $s(k)$, which again depends on $s(k + 1)$. Further the independent Riccati difference equation (2.12) for $P_2(k)$ which determines the feedback gain in (2.3) is the same symmetric equation that is obtained in the linear-quadratic regulator and tracking problems.

The form of this discrete-time Nash open-closed solution differs from the corresponding continuous time solution presented by Foley and Schmitendorf [4] for a game with simpler quadratic cost functions. In their solution the closed-loop strategy of the other player is a pure feedback control-law, which does not depend explicitly on the initial state, unlike in this discrete-time case. On the basis of the results of [3] it can be concluded that the affine strategies for these open-closed games become unique when a random perturbation term is included in the system dynamics.

4. CONCLUSION

An approach using combined augmentation and dynamic programming techniques was developed for the solution of open-closed strategies in discrete-time linear-quadratic differential games. The Stackelberg open-closed game is solved only in the case when the leader plays open-loop and the follower closed-loop. In the opposite case the solution is not available by the present techniques. The solution procedures for the Nash and Stackelberg solutions become in general much alike and their connections can be easily seen due to the unified notations used. A recursive computational algorithm, where the augmented matrix representations are replaced by recursive difference equations, is derived for the Nash open-closed strategies. In this case we arrive at three matrix equations the symmetric one of which is the standard Riccati equation of optimal regulator problems and it becomes independent of the two other equations. The discrete-time Nash open-closed strategy becomes a function of the current time state and also of the initial state. This differs in form from the pure feedback strategy obtained in continuous-time open-closed games. Sufficient conditions guaranteeing the existence of the presented solutions have not been developed. These remain as open question which are in general difficult to approach and especially the general forms of the cost functions make the analysis even more complicated.

It will be shown by backward induction that the control of player 2, who plays closed-loop in the present open-closed game problem, is given by (2.3) when player 1 uses an open-loop strategy $u^0(k)$. The optimization problem of player 2 can be solved recursively backwards by dynamic programming techniques because the current time state, which includes knowledge of all previous decisions, is available to player 2.

Consider first stage $N-1$. After the system equation (2.1) has been employed to eliminate $x(N)$ from the cost-to-go $J_2(N-1)$, it is easily seen that the optimal decision of player 2 minimizing $J_2(N-1)$ is given by

$$\text{(A 1)} \quad v^*(N-1) = -G_2^{-1}(N-1)\left[F_2(N-1)\,x(N-1) - E_2(N-1)\,u^0(N-1)\right].$$

where $G_2(N-1)$, $F_2(N-1)$, and $E_2(N-1)$ are defined by (2.8), (2.9) and (2.10) with $P_2(N) = S_2$. Substituting (A 1) into the cost-to-go and rearranging the terms results in the following expression

$$\text{(A 2)} \quad J_2^*(N-1) = \tfrac{1}{2}[x^T(N-1)\,P_2(N-1)\,x(N-1) + 2x^T(N-1)\,.$$
$$.\,K_2(N-1)\,u^0(N-1)] + \tfrac{1}{2}(u^0(N-1))^T\,T_2(N-1)\,u^0(N-1)$$

where

$$\text{(A 3)} \quad P_2(N-1) =$$
$$= Q_2(N-1) + A^T(N-1)\,S_2\,A(N-1) - F_2(N-1)\,G_2^{-1}(N-1)\,F_2(N-1),$$

$$\text{(A 4)} \quad K_2(N-1) =$$
$$= M_{21}(N-1) + A^T(N-1)\,S_2\,B_1(N-1) - F_2(N-1)\,G_2^{-1}(N-1)\,E_2(N-1)$$

and

$$\text{(A 5)} \quad T_2(N-1) =$$
$$= R_{21}(N-1) + B_1^T(N-1)\,S_2\,B_1(N-1) - E_2(N-1)\,G_2^{-1}(N-1)\,E_2(N-1)\,.$$

These results (A 1) and (A 2) clearly agree with equations (2.3) and (2.14) for $k = N-1$ since by (2.4) we have

$$t(N-1) = -G_2^{-1}(N-1)\,E_2(N-1)\,u^0(N-1)\,.$$

It is next assumed that (2.3) and (2.14) are true for $k = l+1$, $l \in K$. The optimization problem faced by player 2 at stage l is then to minimize the cost-to-go

$$\text{(A 6)} \quad J_2(l) = \tfrac{1}{2}[x^T(l)\,Q_2(l)\,x(l) + 2x^T(l)\,M_{21}(l)\,u^0(l) +$$
$$+ 2x^T(l)\,M_{22}(l)\,v(l) + (u^0(l))^T\,R_{21}(l)\,u^0(l) + 2(u^0(l))^T\,N_2(l)\,v(l) +$$
$$+ v^T(l)\,R_{22}(l)\,v(l) + x^T(l+1)\,P_2(l+1)\,x(l+1) +$$
$$+ 2x^T(l+1)\,K_2(l+1)\,u^0(l+1) + 2s^T(l+1)\,A^0(l+1)\,x(l+1)] + l_2^{(l)}(x_0)$$

 subject to

$$x(l+1) = A(l)\,x(l) + B_1(l)\,u^0(l) + B_2(l)\,v(l)\,. \tag{A 7}$$

In the cost function (A 6) there is a number of terms such as $(u^0(l))^T R_{21}(l)\,u^0(l)$ and $l_2^{(l)}(x_0)$ which do not enter the optimization problem since they depend solely on the initial state. In addition to $u^0(l)$, which was assumed to be an open-loop strategy, also $s(l+1)$ is a function of the initial state by definition, equation (2.5). The minimizing control $v^*(l)$ is readily seen to be the affine strategy

$$\begin{aligned} v^*(l) = \\ = -[R_{22}(l) + B_2^T(l)\,P_2(l+1)\,B_2(l)]^{-1}\,[M_{22}^T(l) + B_2^T(l)\,P_2(l+1)\,A^T(l)]\,x(l) - \\ - [R_{22}(l) + B_2^T(l)\,P_2(l+1)\,B_2(l)]^{-1}\,[(N_2^T(l) + B_2^T(l)\,P_2(l+1)\,B_1(l))\,u^0(l) + \\ + B_2^T(l)\,(K_2(l+1)\,u^0(l+1) + (A^0(l+1))^T s(l+1))]\,, \end{aligned} \tag{A 8}$$

which is the same as (2.3) with $k = l$. Inserting this optimal strategy (A 8) and $x(l+1)$ from (A 7) into (A 6) results after some algebra in the expression for $J^*(l)$ which is of the form (2.14) with $l = k$. Thus the proof of our assertion is completed.

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