KYBERNETIKA — VOLUME 14 (1978), NUMBER 1

Nash and Stackelberg Solutions to General Linear-Quadratic Two Player Difference Games Part I. Open-Loop and Feedback Strategies

RAIMO P. HÄMÄLÄINEN

Concepts needed in the definition of difference game problems are first studied in detail. Solutions for a general class of deterministic linear quadratic two-player nonzero-sum difference games are then developed. Nash and Stackelberg solutions for open-loop and zero-memory information structure are considered. An augmentation techniques and a dynamic programming approach are applied to obtain the solutions. A recursive algorithm is developed for the Nash open-loop solution. Computational difficulties, which are caused by the augmented representations when the number of time stages is great, are thus avoided.

1. INTRODUCTION

Nonzero-sum differential games constitute a relatively new research topic in control theory and applied mathematics. Increasing interest has been taken in these dynamic games due to their potential applicability in modelling decision problems in engineering and economics. The studies of nonzero-sum differential games can be considered to have been initiated in the articles by Starr and Ho [1] and [2]. However, the roots of many person decision problems go further into the past. Marschak [3] and Radner [4] studied the team problem and established the importance of the information structures of the decision makers for the solution. A team decision problem is an optimization problem which has many decision makers but only one common cost criterion. This means that there are no conflicts of interest between the players. The study of information structures was later continued in dynamic team problems by Ho and Chu [5] and [6]. The introduction of dynamics was to lead into difficult situations where present information is influenced by what has been done in the past. Zero-sum differential games have also been subject to extensive research since the appearence of Isaacs's book [7] and the paper by Ho et al., [8] where the linear-quadratic pursuit-evasion game was discussed. In a zero-sum game a single performance criterion is minimized by one of the players and maximized by the other. However, most of these early studies on zero-sum games were close to optimal control problems and did not have much in common with the problems appearing in nonzero-sum games.

Nonzero-sum differential games also resemble classical optimal control problems in some respects but because of multiple cost criteria it must be further specified what is damanded of an optimal solution. The aims of the players are no more completely antagonist as in the zero-sum case. Starr and Ho [1] and [2] discussed three solution concepts: Nash equilibrium, minimax and noninferior or Pareto optimal. Open-loop and closed-loop solutions were given to a two-player game with linear system and quadratic criteria. The Nash problem has also been treated by Case [9] and Lukes and Russel [10]. Krikelis and Rekasius [11] studied the steady-state problem and proposed an iterative method to obtain the solution to the pair of algebraic matrix Riccati equations related to the problem. In the above studies the uniqueness of the obtained affine Nash strategies could only be shown in the linear-quadratic open-loop problems. Uniqueness results of the closed-loop strategies were valid only when the solutions were assumed to be linear or in pure feedback form. It was not until some years later that a counterexample [12] was presented, in which the existence of nonlinear and nonunique Nash closed-loop strategies was established. Basar [13] considered the linear-quadratic discrete-time games with different information structures and showed that the nonuniqueness of Nash solutions could be overcome by including additive random perturbations in the state dynamics. This also yields the equivalence of the global and stagewise, i.e. closed-loop and feedback, Nash solutions in discrete-time linear-quadratic games.

The Stackelberg solution is another strategy, originally used in problems of static economic competition [14], which has been extended to dynamic games. Chen and Cruz [15] and Simaan and Cruz [16] and [17] illustrated widely the properties of Stackelberg open-loop, closed-loop and feedback strategies in multistage games and obtained the feedback solutions for continuous-time and discrete-time linear-quadratic games.

In addition to these papers on Nash and Stackelberg strategies a large number of more specific differential game problems has been treated in the literature only some of which can be mentioned here. Simaan and Cruz solved the sampled-data controls for a Nash problem [18] and the Stackelberg game between two groups of players where each player uses a Nash strategy within his group [19]. Among the few approaches to find approximative solution to nonzero-sum games with nonlinear dynamics are the works by Nishikawa et al. [20] and by Pau [21]. Mukundan and Elsner [22] presented numerical techniques to solve nonlinear game problems by restricting the solutions to the class of linear feedback strategies.

In this paper we shall first consider in detail the general definition of a difference game problem. A unique approach is then presented for the development of solution algorithms for deterministic discrete-time games with different information structures.

Nash and Stackelberg solutions are investigated in the cases of two-player open-loop and feedback strategies using general quadratic performance criteria, which include all the quadratic forms and cross-terms of the state and the players' controls. Corresponding problems have not before been dealt with in the literature. The recursive algorithm derived for the Nash open-loop solution is of special interest because it decreases significantly the computational difficulties in games with long intervals of play.

Currently these nonzero-sum difference games are subject to wide interest. This is manifested in the fact that the problems considered in the present paper were independently studied also by Doležal in [23] and [24]*), using only somewhat simpler quadratic criteria. He employed an entirely different method to solve the open-loop problems. The techniques is based on a discrete maximum principle approach. For the Nash open-loop solutions two recursive algorithms are found one of which corresponds to a special case of the one presented in this paper. The Stackelberg open-loop solution leads into a matrix two-point boundary-value problem, the analytic solution of which is generally not available. Not even the numerical solution of this problem is easily obtained. The corresponding boundary value problem for the continuous-time game was presented by Simaan abd Cruz [17]. Doležal [23, 24] did not consider games where the players' information structures are unequal (see part II of this paper [25]) and it is not clear whether his approach could be applied to these problems.

2. FORMULATION OF THE PROBLEM

In the following the general deterministic two-player linear-quadratic nonzero-sum discrete-time game is defined. This includes the specification of system dynamics, performance measures, the structure of the control strategies and the information available to the players.

Evolution of the system state $x(k)$ is described by the linear difference equation

$$x(k+1) = A(k)\,x(k) + B_1(k)\,u(k) + B_2(k)\,v(k)\,, \tag{2.1}$$

where $x(k) \in R^n$, $u(k) \in R^p$ and $v(k) \in R^q$ for all k belonging to the set of time-points $K = \{0, 1, \ldots, N-1\}$. The control vector $u(k)$ represents the decision of player 1 and the control vector $v(k)$ the decision of player 2 at stage k. The system matrix functions A, B_1 and B_2 are assumed time-varying and of dimensions $n \times n$, $n \times p$ and $n \times q$ respectively.

*) The author is indebted to an anonymous referee for pointing out these articles which were, however, published only after the submission of the present paper.

The performance measure for player i is a general quadratic cost function given by

$$(2.2)\qquad J_i = \tfrac{1}{2}x^T(N)\,S_i\,x(N) + \tfrac{1}{2}\sum_{k=0}^{N-1}\big[x^T(k)\,Q_i(k)\,x(k) + 2x^T(k)\,M_{i1}(k)\,u(k) + 2x^T(k)\,M_{i2}(k)\,v(k) + u^T(k)\,R_{i1}(k)\,u(k) + 2u^T(k)\,N_i(k)\,v(k) + v^T(k)\,R_{i2}(k)\,v(k)\big]\,,\quad i = 1, 2\,,$$

where the weighting matrices S_i, $Q_i(k)$, $M_{i1}(k)$, $M_{i2}(k)$, $R_{i1}(k)$, $N_i(k)$ and $R_{i2}(k)$ are of dimensions $n \times n$, $n \times n$, $n \times p$, $n \times q$, $p \times p$, $p \times q$ and $q \times q$ respectively for $i = 1, 2$ and $k \in K$. Moreover, the cost functions are assumed to be convex with S_1, S_2, $Q_1(k)$, $Q_2(k)$, $R_{12}(k)$ and $R_{21}(k)$ being symmetric positive semidefinite matrices and $R_{11}(k)$, $R_{22}(k)$ symmetric positive definite matrices for all $k \in K$.

In addition to the usual quadratic terms of the system state vector and of the players' control vectors, the above cost functions also include all the cross-terms of these variables. The role of these cross-terms depends on the specific game problem in question and they cannot be given any general physical interpretation.

These general quadratic criteria are encountered when for example the performance measures of the players are defined in terms of the output of the controlled system. General nonzero-sum games have so far received minor interest in the literature although the corresponding output regulator problem plays an important role in optimal control theory. Analogously to the regulator case the solution of an "output"-game can be returned to the solution of a game with general performance criteria. To illustrate this let us consider system (2.1) with the following output-equation:

$$(2.3)\qquad y(k) = C(k)\,x(k) + D_1(k)\,u(k) + D_2(k)\,v(k)$$

where $y(k) \in R^m$ is the system output at stage k and $C(k)$, $D_1(k)$ and $D_2(k)$ are matrices of appropriate dimensions. The cost functions for player 1 and player 2 are now assumed to be of the simple quadratic form in $y(k)$, $u(k)$ and $v(k)$:

$$(2.4)\qquad J_i = \tfrac{1}{2}y^T(N)\,S_i'\,y(N) + \tfrac{1}{2}\sum_{k=0}^{N-1}\big[y^T(k)\,Q_i'(k)\,y(k) + u^T(k)\,R_{i1}'(k)\,u(k) + v^T(k)\,R_{i2}'(k)\,v(k)\big]\,,\quad i = 1, 2\,,$$

where S_1', S_2', $Q_1'(k)$, $Q_2'(k)$, $R_{12}'(k)$, $R_{21}'(k)$ are symmetric positive semidefinite and $R_{11}'(k)$, $R_{22}'(k)$, are symmetric positive definite matrices for all $k \in K$.

The above formulated game problem is solved by returning the cost functions (2.4) to the standard general form (2.2) by substituting the system output $y(k)$ in (2.4) by expression (2.3). The weighting matrices then become

$$(2.5)\qquad \begin{aligned} S_i &= C^T(N)\,S_i'\,C(N) \\ Q_i(k) &= C^T(k)\,Q_i'(k)\,C(k) \end{aligned}$$

$$M_{ij}(k) = C^T(k)\, Q_i'(k)\, D_j(k)$$
$$N_i(k) \;= D_1^T(k)\, Q_i'(k)\, D_2(k)$$
$$R_{ij}(k) \;= R_{ij}'(k) + D_j^T(k)\, Q_i'(k)\, D_j(k)\,, \quad i, j = 1, 2,\, k \in K\,.$$

It is interesting to note that in this case, when the original cost was given by (2.4), the cross-terms in (2.2), whose weighting matrices are $M_{ij}(k)$ and $N_i(k)$, $i, j = 1, 2$, appear only when $D_1(k) \neq 0$ or $D_2(k) \neq 0$ for some $k \in K$, that is, when the output at a stage is an explicit function of the control of one of the players at the same stage.

2.1. Information Structures and Strategies

A principal difference between game and optimal control problems is that the solution of a nonzero-sum game is strongly dependent on the information available to the players and on the structure of the permissible control laws. Thus it becomes necessary to define these explicitly before the game can be solved. The information of the past and present values of the state vector that player i has access to at stage k will be denoted by $z_i^{(k)}$. Since we are dealing with deterministic problems this information is perfect without any stochastic noise components. The information structure is called *perfect memory* when at time k the decision maker knows the present state vector and remembers perfectly all the past states i.e.:

$$z_i^{(k)} = \{x(0), x(1), \ldots, x(k)\}\,, \quad k \in K\,. \tag{2.6}$$

In the *zero-memory* case only the current time value of the state is available, that is

$$z_i^{(k)} = \{x(k)\}\,, \quad k \in K\,. \tag{2.7}$$

The third kind of information structure that is of importance is the *open-loop* information structure, where the player can only use the initial value of the state vector in his control law, in this case

$$z_i^{(k)} = \{x(0)\} \quad \text{for all} \quad k \in K\,. \tag{2.8}$$

In addition to the information $z_i^{(k)}$ player i is assumed to have exact knowledge of the system dynamics and of each others' cost functions.

The decisions of the players are based on the available information $z_i^{(k)}$ at different stages and the control laws are picked from a given class of admissible strategies. The strategy for player i at stage k is denoted by $\gamma_i^{(k)}$ and it is a function of the information $z_i^{(k)}$:

$$u(k) = \gamma_1^{(k)}(z_1^{(k)})\,, \quad \gamma_1^{(k)} \in \Gamma_1^{(k)} \quad \text{and} \quad z_1^{(k)} \in Z_1^{(k)} \tag{2.9}$$

$$v(k) = \gamma_2^{(k)}(z_2^{(k)})\,, \quad \gamma_2^{(k)} \in \Gamma_2^{(k)} \quad \text{and} \quad z_2^{(k)} \in Z_2^{(k)}\,, \tag{2.10}$$

where $\Gamma_1^{(k)}$, $\Gamma_2^{(k)}$ are the strategy spaces and $Z_1^{(k)}$, $Z_2^{(k)}$ are the information spaces in question at stage k. Generally, $\Gamma_1^{(k)}$ and $\Gamma_2^{(k)}$ are spaces of functions from $Z_1^{(k)}$ into R^p and correspondingly $Z_2^{(k)}$ into R^q.

2.2. Solution Concepts

The solution to a game problem is determined by the strategies and the information available to the players together with the properties that are required of the optimum. In the following Nash and Stackelberg solution concepts will be studied.

A pair of sets $((\gamma_1^{(0)*}, \ldots, \gamma_1^{(N-1)*}), (\gamma_2^{(0)*}, \ldots, \gamma_2^{(N-1)*}))$ with $\gamma_1^{(k)*} \in \Gamma_1^{(k)}$ and $\gamma_2^{(k)*} \in \Gamma_2^{(k)}$ for each $k \in K$ is called a *Nash solution* to the game if the following inequalities hold

$$(2.11) \quad J_1[(\gamma_1^{(0)*}, \ldots, \gamma_1^{(N-1)*}), (\gamma_2^{(0)*}, \ldots, \gamma_2^{(N-1)*})] \leqq \leqq J_1[(\gamma_1^{(0)}, \ldots, \gamma_1^{(N-1)}), (\gamma_2^{(0)*}, \ldots, \gamma_2^{(N-1)*}],$$

$$(2.12) \quad J_2[(\gamma_1^{(0)*}, \ldots, \gamma_1^{(N-1)*}), (\gamma_2^{(0)*}, \ldots, \gamma_2^{(N-1)*})] \leqq \leqq J_2[(\gamma_1^{(0)*}, \ldots, \gamma_1^{(N-1)*}), (\gamma_2^{(0)}, \ldots, \gamma_2^{(N-1)})],$$

for all $\gamma_1^{(k)} \in \Gamma_1^{(k)}$, $\gamma_2^{(k)} \in \Gamma_2^{(k)}$ and $k \in K$. This is an equilibrium strategy in the sense that neither of the players can lower his cost by unilaterally deviating from his Nash solution provided that the other player uses his Nash strategy.

In practice the stagewise definition of the Nash solution is often more important. Instead of the two inequalities (2.11) and (2.12) we now have two inequalities at each stage. The pair $(\gamma_1^{(k)*}, \gamma_2^{(k)*})$ with $\gamma_1^{(k)*} \in \Gamma_1^{(k)}$ and $\gamma_2^{(k)*} \in \Gamma_2^{(k)}$ is a *stagewise Nash solution* at stage $k \in K$ if it satisfies the inequalities

$$(2.13) \quad J_1[(\gamma_1^{(0)*}, \ldots, \gamma_1^{(k-1)*}, \gamma_1^{(k)*}, \gamma_1^{(k+1)*}, \ldots, \gamma_1^{(N-1)*}), (\gamma_2^{(0)*}, \ldots, \gamma_2^{(N-1)*})] \leqq \leqq J_1[(\gamma_1^{(0)*}, \ldots, \gamma_1^{(k-1)*}, \gamma_1^{(k)}, \gamma_1^{(k+1)*}, \ldots, \gamma_1^{(N-1)*}), (\gamma_2^{(0)*}, \ldots, \gamma_2^{(N-1)*})]$$

$$(2.14) \quad J_2[(\gamma_1^{(0)*}, \ldots, \gamma_1^{(N-1)*}), (\gamma_2^{(0)*}, \ldots, \gamma_2^{(k-1)*}, \gamma_2^{(k)*}, \gamma_2^{(k+1)*}, \ldots, \gamma_2^{(N-1)*})] \leqq \leqq J_2[(\gamma_1^{(0)*}, \ldots, \gamma_1^{(N-1)*}), (\gamma_2^{(0)*}, \ldots, \gamma_2^{(k-1)*}, \gamma_2^{(k)}, \gamma_2^{(k+1)*}, \ldots, \gamma_2^{(N-1)*})].$$

As the equilibrium conditions are here satisfied at each stage, when controls at the other stages are fixed to their optimal values, recursive solution algorithms derived by dynamic programming techniques become applicable.

Another solution concept frequently studied is the Stackelberg strategy. It can be described by a situation where one of the players knows only his own cost function but the other player knows both cost functions. It may also be so that one of the players is forced to wait until the other player announces his decision, before making his own. A general solution principle is to first optimize the response of the folllower to any fixed strategy of the leader. Then the leader's cost is minimized with the knowledge of the follower's response. Suppose that player 1, the leader, and player 2, the follower, both have the zero-memory information structure. The definition of a *stagewise Stackelberg solution* can then be given in the following way. Consider first the follower's strategy at stage k when the leader's control is fixed. The follower's optimal response to a fixed $\gamma_1^{(k)}$ at stage $k \in K$ is $\gamma_2^{(k)\circ}$ if

$$
\begin{aligned}
(2.15) \qquad & J_2[(\gamma_1^{(0)*}, \ldots, \gamma_1^{(k-1)*}, \gamma_1^{(k)}, \gamma_1^{(k+1)*}, \ldots, \gamma_1^{(N-1)*}), \\
& (\gamma_2^{(0)*}, \ldots, \gamma_2^{(k-1)*}, \gamma_2^{(k)\circ}, \gamma_2^{(k+1)*}, \ldots, \gamma_2^{(N-1)*})] \leqq \\
& \leqq J_2[(\gamma_1^{(0)*}, \ldots, \gamma_1^{(k-1)*}, \gamma_1^{(k)}, \gamma_1^{(k+1)*}, \ldots, \gamma_1^{(N-1)*}), \\
& (\gamma_2^{(0)*}, \ldots, \gamma_2^{(k-1)*}, \gamma_2^{(k)}, \gamma_2^{(k+1)*}, \ldots, \gamma_2^{(N-1)*})]
\end{aligned}
$$

for all $\gamma_2^{(k)} \in \Gamma_2^{(k)}$.

Thus the optimal response becomes a function of the leader's control i.e. $\gamma_2^{(k)\circ} = \gamma_2^{(k)\circ}(u(k))$. The pair $(\gamma_1^{(k)*}, \gamma_2^{(k)*})$ is then a stagewise Stackelberg solution at stage k with player 1 leading if

$$
\begin{aligned}
& J_1[(\gamma_1^{(0)*}, \ldots, \gamma_1^{(k-1)*}, \gamma_1^{(k)*}, \gamma_1^{(k-1)*}, \ldots, \gamma_1^{(N-1)*}), \\
& (\gamma_2^{(0)*}, \ldots, \gamma_2^{(k-1)*}, \gamma_2^{(k)*}, \gamma_2^{(k+1)*}, \ldots, \gamma_2^{(N-1)*})] \leqq \\
(2.16) \qquad & \leqq J_1[(\gamma_1^{(0)*}, \ldots, \gamma_1^{(k-1)*}, \gamma_1^{(k)}, \gamma_1^{(k+1)*}, \ldots, \gamma_1^{(N-1)*}), \\
& (\gamma_2^{(0)*}, \ldots, \gamma_2^{(k-1)*}, \gamma_2^{(k)\circ}, \gamma_2^{(k+1)*}, \ldots, \gamma_2^{(N-1)*})]
\end{aligned}
$$

for all $\gamma_1^{(k)} \in \Gamma_1^{(k)}$, where $\gamma_2^{(k)*} = \gamma_2^{(k)\circ}(\gamma_1^{(k)*})$. It is seen that the assumed information structures enter implicitly the definition of this solution concept. The Stackelberg solution with player 2 leading would be defined in an analogous manner.

The Stackelberg solution to a game, where more general than the zero-memory information structures are allowed, may become quite complicated. The follower's response at a certain stage can then become dependent of values of the state of the leader's policy even at other time points besides the current time. This can make the determining of the leader's optimal strategy very difficult in general.

3. OPEN-LOOP SOLUTIONS

Consider the game problem of chapter 2 with the cost functions defined by (2.2) and the system dynamics by (2.1). The concept of open-loop solution implies that both players have access to the open-loop information set. This means that they can use only the initial state vector when determining their optimal policies. The admissible strategies for each stage $k \in K$ are functions mapping $z_1 = \{x_0\}$ into R^p for player 1 and $z_2 = \{x_0\}$ into R^q for player 2 when $x_0 = x(0)$.

In order to obtain the optimal strategies the problem will be rewritten with the aid of augmented state, initial state and control vectors $\bar{x}$, $\bar{x}_0$, $\bar{u}$ and $\bar{v}$

(3.1) $$\bar{x} \triangleq [x^T(1) \,\vdots\, x^T(2) \,\vdots\, \dots \,\vdots\, x^T(N)]^T$$

(3.2) $$\bar{x}_0 \triangleq [x^T(0) \,\vdots\, x^T(0) \,\vdots\, \dots \,\vdots\, x^T(0)]^T$$

(3.3) $$\bar{u} \triangleq [u^T(0) \,\vdots\, u^T(1) \,\vdots\, \dots \,\vdots\, u^T(N-1)]^T$$

(3.4) $$\bar{v} \triangleq [v^T(0) \,\vdots\, v^T(1) \,\vdots\, \dots \,\vdots\, v^T(N-1)]^T$$

which have nN, nN, pN and qN components respectively. The system dynamics (2.1) can now be replaced by the static equation

(3.5) $$\bar{x} = \bar{A}\bar{x}_0 + \bar{B}_1\bar{u} + \bar{B}_2\bar{v}$$

where the $nN \times nN$-matrix $\bar{A}$ is block-diagonal and its k-th $n \times n$-block is

(3.6) $$[\bar{A}]_{kk} \triangleq \Phi(k, 0)\,, \quad k = 1, 2, \dots, N\,,$$

with $\Phi(k, l)$ being the fundamental matrix associated with (2.1) given by

(3.7) $$\Phi(k, l) = A(k-1)\,A(k-2)\cdot \dots \cdot A(l)\,, \quad k > l$$

and

$$\Phi(k, k) = I\,.$$

$\bar{B}_1$ and $\bar{B}_2$ are lower block triangular matrices of dimensions $nN \times pN$ and $nN \times qN$ respectively with the $n \times p$ and $n \times q$ blocks defined by

(3.8) $$[\bar{B}_i]_{kl} \triangleq \begin{cases} \Phi(k, l)\,B_i(l-1) & \text{for} \quad k \geqq l\,, \\ 0 & \text{else} \end{cases}$$

when $k, l = 1, 2, \dots, N$ and $i = 1, 2$.

By using these augmented vectors and matrices the general quadratic cost functions (2.2) are transformed into the following equivalent form

(3.9) $$J_i = \tfrac{1}{2}[\bar{x}^T\bar{Q}_i\bar{x} + 2\bar{x}^T\bar{M}_{i1}\bar{u} + 2\bar{x}^T\bar{M}_{i2}\bar{v} + \bar{u}^T\bar{R}_{i1}\bar{u} + 2\bar{u}^T\bar{N}_i\bar{v} + \\ + \bar{v}^T\bar{R}_{i2}\bar{v} + \bar{x}_0^T\bar{Q}_{0i}\bar{x}_0 + 2\bar{x}_0^T\bar{M}_{0i1}\bar{u} + 2\bar{x}_0^T\bar{M}_{0i2}\bar{v}]\,, \quad i = 1, 2,$$

where $\bar{Q}_i$ and $\bar{Q}_{0i}$ are symmetric block-diagonal $nN \times nN$ matrices whose $n \times n$ blocks are defined for $i = 1, 2$ by

(3.10) $$[\bar{Q}_i]_{kk} \triangleq \begin{cases} Q_i(k) & \text{for} \quad k = 1, 2, \dots, N-1\,, \\ S_i & \text{for} \quad k = N \end{cases}$$

and

$$[\bar{Q}_{0i}]_{kk} \triangleq \begin{cases} Q_i(0) & \text{for} \quad k = 1\,, \\ 0 & \text{for} \quad k = 2, 3, \ldots, N\,. \end{cases} \tag{3.11}$$

The block-matrices $\bar{M}_{11}$, $\bar{M}_{21}$, $\bar{M}_{011}$, $\bar{M}_{021}$ are $nN \times pN$ dimensional and correspondingly $\bar{M}_{12}$, $\bar{M}_{22}$, $\bar{M}_{012}$, $\bar{M}_{022}$ are $nN \times qN$ dimensional defined for $i, j = 1, 2$ by

$$[\bar{M}_{ij}]_{kl} \triangleq \begin{cases} M_{ij}(k) & \text{for} \quad l = k + 1\,, \\ 0 & \text{else} \end{cases} \tag{3.12}$$

when $k = 1, 2, \ldots, N$ and

$$[\bar{M}_{0ij}]_{kl} \triangleq \begin{cases} M_{ij}(0) & \text{for} \quad k = l = 1\,, \\ 0 & \text{else}\,, \end{cases} \tag{3.13}$$

when $k, l = 1, 2, \ldots, N$ and $i, j = 1, 2$.

The symmetric matrices $\bar{R}_{1j}$ and $\bar{R}_{2j}$ are block-diagonal of dimensions $pN \times pN$ for $j = 1$ and $qN \times qN$ for $j = 2$ respectively:

$$[\bar{R}_{ij}]_{kk} \triangleq R_{ij}(k-1) \quad \text{for} \quad k = 1, 2, \ldots, N \tag{3.14}$$

Finally we have the $pN \times qN$ block-diagonal matrices $\bar{N}_1$ and $\bar{N}_2$:

$$[\bar{N}_i]_{kk} \triangleq N_i(k-1) \quad \text{for} \quad k = 1, 2, \ldots, N\,. \tag{3.15}$$

The original dynamic game problem has thus become a static game where the quadratic cost criterion is expressed in terms of the initial state vector and the augmented control vectors. Different solutions based on the open-loop information structure can now be determined in a straightforward manner. It is clear that the constant terms depending solely on x_0 like $x_0^T \bar{Q}_{0i} \bar{x}$ effect only the values of the of optimal costs of the players without entering the solution procedure.

3.1. The Nash Open-Loop Solution

The Nash open-loop equilibrium solution is obtained by first deriving the N decisions of each player with fixed strategies for the other player and finally requiring these to be satisfied simultaneously. We define the augmented strategy vectors by

$$\bar{\gamma}_1(\bar{x}_0) \triangleq [(\gamma_1^{(1)}(x_0))^T \mid (\gamma_1^{(2)}(x_0))^T \mid \ldots \mid (\gamma_1^{(N-1)}(x_0))^T]^T \tag{3.16}$$

$$\bar{\gamma}_2(\bar{x}_0) \triangleq [(\gamma_2^{(1)}(x_0))^T \mid (\gamma_2^{(2)}(x_0))^T \mid \ldots \mid (\gamma_2^{(N-1)}(x_0))^T]^T\,. \tag{3.17}$$

The control of player 1 that minimizes $J_1(\bar{u}, \bar{\gamma}_2)$ for a fixed $\bar{\gamma}_2$ is then

$$\bar{u}^\circ = -\bar{G}_1^{-1}[\bar{F}_1\bar{x}_0 + \bar{E}_1\bar{\gamma}_2] \tag{3.18}$$

and the control of player 2 that minimizes $J_2(\bar{\gamma}_1, \bar{v})$ for a fixed $\bar{\gamma}_1$ is

$$\bar{v}^\circ = -\bar{G}_2^{-1}[\bar{F}_2\bar{x}_0 + \bar{E}_2\bar{\gamma}_1], \tag{3.19}$$

where $\bar{G}_1$ and $\bar{G}_2$ are symmetric matrices defined by

$$\bar{G}_i = \bar{R}_{ii} + \bar{B}_i^T\bar{M}_{ii} + [\bar{M}_{ii}^T + \bar{B}_i^T\bar{Q}_i]\,\bar{B}_i, \quad i = 1, 2, \tag{3.20}$$

and

$$\bar{F}_i = \bar{M}_{0ii}^T + [\bar{M}_{ii}^T + \bar{B}_i^T\bar{Q}_i]\,\bar{A}, \quad i = 1, 2, \tag{3.21}$$

$$\bar{E}_1 = \bar{N}_1 + \bar{B}_i^T\bar{M}_{12} + [\bar{M}_{11}^T + \bar{B}_1^T\bar{Q}_1]\,\bar{B}_2 \tag{3.22}$$

$$\bar{E}_2 = \bar{N}_2^T + \bar{B}_2^T\bar{M}_{21} + [\bar{M}_{22}^T + \bar{B}_2^T\bar{Q}_2]\,\bar{B}_1. \tag{3.23}$$

The Nash open-loop strategy pair $(\bar{\gamma}_1^*, \bar{\gamma}_2^*)$ results when (3.18) and (3.19) are satisfied simultaneously. Solving the pair of equations

$$\bar{\gamma}_1^* = \bar{u}^\circ(\bar{x}_0, \bar{\gamma}_2^*), \tag{3.24}$$

$$\bar{\gamma}_2^* = \bar{v}^\circ(\bar{x}_0, \bar{\gamma}_1^*) \tag{3.25}$$

yields

$$\bar{u}^* = \bar{\gamma}_1^*(x_0) = -\bar{H}_u\bar{x}_0, \tag{3.26}$$

$$\bar{v}^* = \bar{v}_2^*(x_0) = -\bar{H}_v\bar{x}_0, \tag{3.27}$$

where

$$\bar{H}_u = [\bar{G}_1 - \bar{E}_1\bar{G}_2^{-1}\bar{E}_2]^{-1}\,[\bar{F}_1 - \bar{E}_1\bar{G}_2^{-1}\bar{F}_2], \tag{3.28}$$

$$\bar{H}_v = [\bar{G}_2 - \bar{E}_2\bar{G}_1^{-1}\bar{E}_1]^{-1}\,[\bar{F}_2 - \bar{E}_2\bar{G}_2^{-1}\bar{F}_1]. \tag{3.29}$$

The solution (3.26) and (3.27) thus obtained is at each stage a linear function of the initial state for both players. Necessary condition for the existence of this equilibrium solution is invertibility of $\bar{G}_1, \bar{G}_2$ and of either the matrix $[\bar{G}_1 - \bar{E}_1\bar{G}_2^{-1}\bar{E}_2]$ or the matrix $[\bar{G}_2 - \bar{E}_2\bar{G}_1^{-1}\bar{E}_1]$. The equivalence of these latter conditions is revealed by the following general matrix identity

$$[\bar{G}_1 - \bar{E}_1\bar{G}_1^{-1}\bar{E}_2]^{-1} = \bar{G}_1^{-1} + \bar{G}_1^{-1}\bar{E}_1[\bar{G}_2 - \bar{E}_2\bar{G}_1^{-1}\bar{E}_1]^{-1}\,\bar{E}_2\bar{G}_1^{-1}. \tag{3.30}$$

Thus it is sufficient to check the nonsingularity of the lower dimensional matrix only.

3.2. The Stackelberg Open-Loop Solution

The Stackelberg open-loop solution is determined at the start of the game in such a way that the leader first computes the follower's optimal reaction to his set of decisions on the whole interval of the play. After this the leader minimizes his own cost function assuming that the follower will response rationally at all stages in the future. The follower's optimal strategy is thus determined by the values of the leader's control.

Suppose the leader's control is $\bar{u}$. Then the followers response minimizing $\bar{J}_2(\bar{u}, \bar{\gamma}_2)$ is

$$\bar{\gamma}_2^{\circ}(\bar{u}) = -\bar{G}_2^{-1}[\bar{F}_2\bar{x}_0 + \bar{E}_2\bar{u}], \tag{3.31}$$

where matrices $\bar{G}_2$, $\bar{F}_2$ and $\bar{E}_2$ are those defined above by (3.20), (3.21) and (3.23). The leader then seeks a solution that minimizes $\bar{J}_1(\bar{\gamma}_1, \bar{\gamma}_2^{\circ}(\bar{\gamma}_1))$. This results in the linear strategy

$$\bar{u}^* = \bar{\gamma}_1^*(\bar{x}) = -\bar{H}_u\bar{x}_0, \tag{3.32}$$

where

$$\bar{H}_u = [\bar{G}_{1L} - \bar{E}_{1L}\bar{G}_2^{-1}\bar{E}_2]^{-1}[\bar{F}_{1L} - \bar{E}_{1L}\bar{G}_2^{-1}\bar{F}_2] \tag{3.33}$$

and

$$\bar{G}_{1L} = \bar{G}_1 - \bar{E}_2^T\bar{G}_2^{-1}\bar{E}_1^T, \tag{3.34}$$

$$\bar{F}_{1L} = \bar{F}_1 - \bar{E}_2^T\bar{G}_2^{-1}\bar{F}_{12}, \tag{3.35}$$

$$\bar{E}_{1L} = \bar{E}_1 - \bar{E}_2^T\bar{G}_2^{-1}\bar{G}_{12} \tag{3.36}$$

with

$$\bar{G}_{12} = \bar{R}_{12} + \bar{B}_2\bar{M}_{12} + [\bar{M}_{12}^T + \bar{B}_2^T\bar{Q}_1]\bar{B}_2 \tag{3.37}$$

$$\bar{E}_{12} = \bar{M}_{012}^T + [\bar{M}_{12}^T + \bar{B}_2^T\bar{Q}_1]\bar{A}. \tag{3.38}$$

Matrices $\bar{G}_1$, $\bar{F}_1$ and $\bar{E}_1$ are defined by equations (3.20), (3.21) and (3.22). The follower's Stackelberg strategy $\bar{\gamma}_2^*$ is then his optimal response to the leader's Stackelberg control:

$$\bar{v}^* = \bar{\gamma}_2^*(\bar{x}_0) = \bar{\gamma}_2^{\circ}(\bar{\gamma}_1^*(\bar{x}_0)) = -\bar{H}_v\bar{x}_0 \tag{3.39}$$

where

$$\bar{H}_v = \bar{G}_2^{-1}[\bar{F}_2 - \bar{E}_2\bar{H}_u] \tag{3.40}$$

or it can also be written as

$$\bar{H}_v = [\bar{G}_2 - \bar{E}_2\bar{G}_1^{-1}\bar{E}_{1L}]^{-1}[\bar{F}_2 - \bar{E}_2\bar{G}_1^{-1}\bar{F}_{1L}]. \tag{3.41}$$

Here it is required that the matrices $\bar{G}_{1L}$, $\bar{G}_2$ and either $[\bar{G}_{1L} - \bar{E}_{1L}\bar{G}_2^{-1}\bar{E}_2]$ or $[\bar{G}_2 - \bar{E}_2\bar{G}_1^{-1}\bar{E}_{1L}]$ are invertible. The obtained Stackelberg open-loop solutions (3.32) and (3.39) yield at each stage a pair of strategies $(\gamma_1^{(k)*}(x_0), \gamma_2^{(k)*}(x_0))$ which are linear functions of the initial state x_0.

One observes that irrespective of the entirely different solution concepts the Stackelberg open-loop strategies are determined by the same kind of equations as the Nash open-loop strategies. The only deviations are the additional terms that enter the leader's $\bar{G}_{1L}$, $\bar{F}_{1L}$ and $\bar{E}_{1L}$ matrices.

A game with player 1 being the follower and player 2 the leader would be solved in the same way, merely a change in the indexing in the above equations would be needed.

3.3. Recursive Algorithms

The practical applicability of the preceding forms for the solutions are likely to decrease due to dimensionality problems when N becomes large, because the solutions are given in terms of the augmented matrices. Therefore one would rather have the solutions in a form where high dimensional matrix equations would be transformed into a series of low dimensional equations.

It is, indeed, possible to express the Nash open-loop solution in a much more convenient way by replacing the augmented vector representation by two recursive matrix equations. An equally simple recursive procedure is, however, not obtained for the Stackelberg open-loop solution. Thus we shall only consider the Nash open-loop solution in this context. Derivation of the recursive equations is based on an implicit version of the pair of equations (3.24) and (3.25), where the augmented state vector $\bar{x}$ has not been eliminated. Then the matrices involved become of suitable block triangular and block diagonal forms allowing the derivation of a stagewise solution procedure beginning from the lowest row of block matrices. We omit the details but it can be verified by performing some algebraic manipulations that the Nash open-loop solution (3.26) and (3.27) is given in the following feedback form

$$u^*(k) = -H_u(k)\,x(k) \tag{3.42}$$

$$v^*(k) = -H_v(k)\,x(k) \tag{3.43}$$

The feedback gains $H_u(k)$ and $H_v(k)$ are

$$H_u(k) = [G_1(k) - E_1(k)\,G_2^{-1}(k)\,E_2(k)]^{-1}\,[F_1(k) - E_1(k)\,G_2^{-1}(k)\,F_2(k)] \tag{3.44}$$

$$H_v(k) = [G_2(k) - E_2(k)\,G_1^{-1}(k)\,E_1(k)]^{-1}\,[F_2(k) - E_2(k)\,G_1^{-1}(k)\,F_1(k)], \tag{3.45}$$

 where the matrices $G_i(k)$, $F_i(k)$ and $E_i(k)$ are defined by

$$G_i(k) = R_{ii}(k) + B_i^T(k)\,P_i(k+1)\,B_i(k) \quad i = 1, 2 \tag{3.46}$$

$$F_i(k) = M_{ii}^T(k) + B_i^T(k)\,P_i(k+1)\,A(k) \quad i = 1, 2 \tag{3.47}$$

and

$$E_1(k) = N_1(k) + B_1^T(k)\,P_1(k+1)\,B_2(k) \tag{3.48}$$

$$E_2(k) = N_2^T(k) + B_2^T(k)\,P_2(k+1)\,B_1(k)\,. \tag{3.49}$$

The Nash control laws are then obtained by solving recursively the coupled asymmetric Riccati-type matrix difference equations for $P_1(k)$ and $P_2(k)$

$$\begin{aligned} P_1(k) = {} & Q_1(k) - M_{11}(k)\,H_u(k) - M_{12}(k)\,H_v(k) + \\ & + A^T(k)\,P_1(k+1)\,[A(k) - B_1(k)\,H_u(k) - B_2(k)\,H_v(k)] \end{aligned} \tag{3.50}$$

$$\begin{aligned} P_2(k) = {} & Q_2(k) - M_{22}(k)\,H_v(k) - M_{21}(k)\,H_u(k) + \\ & + A^T(k)\,P_2(k+1)\,[A(k) - B_1(k)\,H_u(k) - B_2(k)\,H_v(k)] \end{aligned} \tag{3.51}$$

with

$$P_1(N) = S_1\,, \quad P_2(N) = S_2\,.$$

Finally the system equation must be employed repetitively in order to eliminate the current time state vectors from the solution and to obtain the open-loop strategies. This yields

$$u^*(k) = \gamma_1^{(k)*}(x_0) = -H_u(k)\,\Psi(k)\,x_0 \tag{3.52}$$

and

$$v^*(k) = \gamma_2^{(k)*}(x_0) = -H_v(k)\,\Psi(k)\,x_0\,, \tag{3.53}$$

where $\Psi(k)$ satisfies the difference equation

(3.54)

$$\Psi(k) = [A(k-1) - B_1(k-1)\,H_u(k-1) - B_2(k-1)\,H_v(k-1)]\,\Psi(k-1)\,,$$
$$\Psi(0) = I\,.$$

The necessary condition for the existence of the solution involving the augmented matrices is now replaced by corresponding conditions at each stage i.e. invertibility of $G_1(k)$, $G_2(k)$ and of either $[G_1(k) - E_1(k)\,G_2^{-1}(k)\,E_2(k)]$ or $[G_2(k) - E_2(k)\,G_1^{-1}(k)\,E_1(k)]$ for all $k \in K$.

The solution algorithm for this Nash open-loop game proceeds stagewise backwards so that at a stage k the $G_i(k)$, $F_i(k)$ and $E_i(k)$ matrices are first determined by the aid of $P_i(k+1)$ which is known from the preceding step. The feedback gains $H_u(k)$ and $H_v(k)$ can then be evaluated from (3.44) and (3.45) and used to obtain $P_i(k)$ from (3.50) and (3.51). When the feedback gains have been computed in this manner for the whole interval, equation (3.54) is solved recursively in the forward direction to yield the open-loop strategies (3.52) and (3.53).

4. FEEDBACK SOLUTIONS

The feedback strategies for a game are characterized by the stagewise definition of the conditions for the solution and by the zero-memory information structure for both players. This definition of the solution is probably the most important one from the practical point of view. For example, if a game theoretic approach is used in designing regulators or decentralized control systems, it is natural to assume that optimization of control at each stage is solely based on current-time values of the state vector. The stagewise definition of solutions is also convenient from the computational point of view since recursive solution algorithms can be derived inductively by straightforward dynamic programming techniques.

4.1. The Nash Feedback Solution

Consider again the game with cost functions (2.2) and system governed by (2.1). Let $J_i^*(k)$ denote the optimal cost-to-go from stage k to the final stage i.e. $J_i^*(k)$ is the cost for player i over the interval $[k, k+1, \ldots, N]$ with the players' controls being fixed to the optimal strategies on that interval.

Starting from the final stage one readily sees that

$$J_i(N) = J_i^*(N) = \tfrac{1}{2}x^T(N)\, S_i\, x(N)\,. \tag{4.1}$$

At stage $N-1$ the game problem faced by the players becomes a static nonzero-sum game when the system equation is used to eliminate $x(N)$ from the cost functions. The Nash solution to this game is obtained in the same way as in the open-loop case. This yields a linear strategy in $x(N-1)$ for both players and the resulting costs are again quadratic in the current time state vector $x(N-1)$. By an inductive argumentation this can be shown to be true for all stages.

Assume first that at a stage $k+1$ the optimal costs are quadratic functions of the form

$$J_i^*(k+1) = \tfrac{1}{2}x^T(k+1)\, P_i(k+1)\, x(k+1)\,, \quad i = 1, 2\,. \tag{4.2}$$

Consider then the game prpblem that remains to be solved at stage k. The costs become

$$(4.3)\qquad \begin{aligned} J_i(k) = \tfrac{1}{2}[x^T(k)\,Q_i(k)\,x(k) + 2x^T(k)\,M_{i1}(k)\,u(k) + \\ 2x^T(k)\,M_{i2}(k)\,v(k) + u^T(k)\,R_{i1}(k)\,u(k) + 2u^T(k)\,N_i(k)\,v(k) + \\ + v^T(k)\,R_{i2}(k)\,v(k)] + \tfrac{1}{2}x^T(k+1)\,P_i(k+1)\,x(k+1)\,, \quad i = 1,2\,. \end{aligned}$$

Elimination of $x(k + 1)$ from $J_i(k)$ results in a static game problem whose Nash solution is again easily obtained and it yields a pair of linear strategies

$$(4.4)\qquad u^*(k) = \gamma_1^{(k)*}(x(k)) = -H_u(k)\,x(k)\,,$$

$$(4.5)\qquad v^*(k) = \gamma_1^{(k)*}(x(k)) = -H_v(k)\,x(k)\,,$$

where $H_u(k)$, $H_v(k)$ and the related matrices in their definitions are those given in terms of $P_i(k + 1)$ by equations (3.44)–(3.49). Existence of the solution requires invertibility of matrices $G_1(k)$, $G_2(k)$ and $[G_1(k) - E_1(k)\,G_2^{-1}(k)\,E_2(k)]$. The optimal cost-to-go is now easily observed to be quadratic in $x(k)$ for both players:

$$(4.6)\qquad J_i^*(k) = \tfrac{1}{2}x^T(k)\,P_i(k)\,x(k)\,, \quad i = 1, 2\,.$$

with $P_1(k)$ and $P_2(k)$ given by

$$(4.7)\qquad \begin{aligned} P_1(k) = Q_1 - M_{11}H_u - H_u^TM_{11}^T - M_{12}H_v - H_v^TM_{12}^T + \\ + H_u^TR_{11}H_u + H_u^TN_1H_v + H_v^TN_1^TH_u + H_v^TR_{12}H_v + \\ + [A - B_1H_u - B_2H_v]^T\,P_1(k+1)\,[A - B_1H_u - B_2H_v] \end{aligned}$$

and

$$(4.8)\qquad \begin{aligned} P_2(k) = Q_2 - M_{21}H_u - H_u^TM_{21}^T - M_{22}H_v - H_v^TM_{22}^T + \\ + H_u^TR_{21}H_u + H_u^TN_2H_v + H_v^TN_2^TH_u + H_v^TR_{22}H_v + \\ + [A - B_1H_u - B_2H_v]^T\,P_2(k+1)\,[A - B_1H_u - B_2H_v]\,, \end{aligned}$$

where the argument k of the time-varying matrices on the right hand sides has been dropped for convenience.

The above results now imply by backward induction that $J_1^*(k)$ and $J_2^*(k)$ are quadratic of the form (4.6) for all $k \in K$. The solution on the whole interval can thus be a recurcive algorithm where a static game is solved at each stage starting from the last stage and proceeding backwards to the initial stage. This is possible because of the assumed zero-memory information structure since the solution at a certain stage will not depend functionally on controls at the earlier stages.

The Nash feedback solution is now given by (4.4) and (4.5) for all $k \in K$. The feedback gains $H_u(k)$ and $H_v(k)$ are determined in terms of the matrices $G_i(k)$, $F_i(k)$ and $E_i(k)$ which depend on $P_i(k + 1)$, $i = 1, 2$. Here matrices $P_1(k)$ and $P_2(k)$ determine the values of the cost-to-go and they are obtained by solving the pair of symmetric coupled Riccati-type matrix difference equations defined by (4.7) and (4.8) with the boundary conditions

(4.9) $$P_i(N) = S_i, \quad i = 1, 2.$$

It is interesting to compare these equations to the corresponding pair of matrix difference equations (3.50), (3 51) obtained in the Nash open-loop case. The structure of the equations is much the same, only symmetry together with a number of terms are missing from the open-loop equations. This is analogous to the differences between Nash open-loop and Nash feedback solutions in continuous time linear quadratic games [12].

Further, one might wish to compare these values of the cost functions (4.6) with those obtained when both players use arbitrary linear feedback strategies of the type $u(k) = -H_u(k)\,x(k)$ and $v(k) = -H_v(k)\,x(k)$. The cost-to-go would again be obtained from (4.6) by solving the difference equations (4.7) and (4.8) for $P_1(k)$ and $P_2(k)$ where $H_u(k)$ and $H_v(k)$ are substituated by the arbitrary feedback gains in question. The remaining problem would become linear without coupling between the equations. For example the optimal costs-to-go in the Nash open-loop game could be calculated in this way.

4.2. The Stackelberg Feedback Solution

The solution procedure for the Stackelberg feedback strategy is quite similar to that in the Nash case. A series of static games is solved recursively proceeding backwards from the final stage, $k = N - 1$, to the initial stage, $k = 0$. The same technique remains applicable also in this Stackelberg problem because of the assumed zero-memory information structure and due to the fact that the optimal costs-to-go become again quadratic in the current-time state vector.

The game (2.2) and system (2.1) is considered with player 1 being the leader and player 2 the follower. Omitting further details of the derivation of this algorithm the recursive equations defining the Stackelberg feedback solutions are given. At a stage $k \in K$ the Stackelberg optimal cost-to-go for player i is a quadratic function of $x(k)$

(4.10) $$J_i^*(k) = \tfrac{1}{2}x^T(k)\,P_i(k)\,x(k), \quad i = 1, 2$$

and the feedback solutions are linear strategies

(4.11) $$u^*(k) = \gamma_1^{(k)*}(x(k)) = -H_u(k)\,x(k),$$

(4.12) $$v^*(k) = \gamma_2^{(k)*}(x(k)) = -H_v(k)\,x(k).$$

In the above equations $P_1(k)$ and $P_2(k)$ satisfy the pair of symmetric coupled Riccati type matrix equations (4.7) and (4.8) with the final conditions (4.9). The feedback gain matrices, which also appear in the equations for $P_1(k)$ and $P_2(k)$, are given by

$$(4.13)\quad H_u(k) = [G_{1L}(k) - E_{1L}(k)\, G_2^{-1}(k)\, E_2(k)]^{-1}\, [F_{1L}(k) - E_{1L}(k)\, G_2^{-1}(k)\, F_2(k)]$$

$$(4.14)\quad H_v(k) = [G_2(k) - E_2(k)\, G_{1L}^{-1}(k)\, E_{1L}(k)]^{-1}\, [F_2(k) - E_2(k)\, G_{1L}^{-1}(k)\, F_{1L}(k)]$$

where

$$(4.15)\qquad G_{1L} = G_1(k) - E_2^T(k)\, G_2^{-1}(k)\, E_1^T(k)\,,$$

$$(4.16)\qquad F_{1L} = F_1(k) - E_2^T(k)\, G_2^{-1}(k)\, F_{12}(k)\,,$$

$$(4.17)\qquad E_{1L} = E_1(k) - E_2^T(k)\, G_2^{-1}(k)\, G_{12}(k)$$

and

$$(4.18)\qquad G_{12}(k) = R_{12}(k) + B_2^T(k)\, P_1(k+1)\, B_2(k)\,,$$

$$(4.19)\qquad F_{12}(k) = M_{12}^T(k) + B_2^T(k)\, P_1(k+1)\, A(k)\,.$$

The matrices $G_i(k)$, $F_i(k)$ and $E_1(k)$ above are defined in terms of $P_i(k+1)$, $i = 1, 2$, by equations (3.46)–(3.49). The solution exists provided $G_{1L}(k)$, $G_2(k)$ and moreover either $[G_{1L}(k) - E_{1L}(k)\, G_2^{-1}(k)\, E_2(k)]$ or $[G_2(k) - E_2(k)\, G_{1L}^{-1}(k)\, E_{1L}(k)]$ are nonsingular for all $k \in K$.

The solution procedure for this Stackelberg feedback solution is algorithmically of the same kind as the preceding Nash feedback one. Similarly as in the open-loop case the corresponding equations defining the Nash feedback strategies yield the Stackelberg feedback strategies with player 1 being the leader when $G_1(k)$, $F_1(k)$ and $E_1(k)$ are replaced by $G_{1L}(k)$, $F_{1L}(k)$ and $E_{1L}(k)$ respectively. One might expect a more clear difference in the structure of the Nash and Stackelberg solution algorithms since the Nash problem is a real two player game compared to the Stackelberg problem, where the leader is in fact the only decision maker. A detailed inspection of the definitions of the auxiliary matrices, however, reveals that after performing the required substitutions the analytic representations of the corresponding matrix difference equations become quite different. Yet the practical numerical solution algorithms, which are based on the recursive equations derived above, remain similar.

5. CONCLUSION

In this paper we have developed algorithms for open-loop and feedback strategies in deterministic general linear-quadratic two-person nonzero-sum difference games with the Nash and the Stackelberg solution concepts. The feedback strategies are derived by a dynamic programming techniques and the solution is in both cases obtained by solving a pair of recursive matrix equations. The open-loop strategies are found by converting the dynamic game problem into a static one by an augmentation method. In these two cases both the Nash and the Stackelberg solution procedures are given algorithmically in the same formats. A computationally advantageous recursive algorithm is derived for the Nash open-loop solution. The augmented form representation is replaced by two nonsymmetric matrix-difference equations, which resemble the symmetric ones obtained in the feedback case.

6. ACKNOWLEDGEMENT

This work was financially supported by the Academy of Finland, the Finnish Cultural Foundation and the Walter Ahlström Foundation.

(Received October 20, 1976.)

REFERENCES

[1] A. W. Starr, Y. - C. Ho: Nonzero-sum differential games. J. Optim. Theory. Applic. *3* (1969), 3, 184—206.

[2] A. W. Starr, Y. - C. Ho: Further properties of nonzero-sum differential games. J. Optim. Theory. Applic. *3* (1969), 4, 207—219.

[3] J. Marschak: Elements for a theory of teams. Management Science *1* (1955), 127—137.

[4] R. Radner: Team decision problems. Ann. Math. Statistics *33* (1962), 857—881.

[5] Y. - C. Ho, K. - C. Chu: Team decision theory and information structures in optimal control problems — part I. IEEE Trans. Autom. Control *AC - 17* (1972), 1, 15—22.

[6] K. - C. Chu: Team decision theory and information structures in optimal control problems — part II. IEEE Trans. Autom. Control *AC-17* (1972), 1, 22—28.

[7] R. Isaacs: Differential Games. A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization. John Wiley and Sons, New York 1965.

[8] Y. - C. Ho, A. E. Bryson, Jr, S. Baron: Differential games and optimal pursuit-evasion strategies. IEEE Trans Autom. Control *AC-10* (1965), 4, 385—389.

[9] J. H. Case: Toward a theory of many player differential games. SIAM J. Control *7* (1969), 2, 179—197.

[10] D. L. Lukes, D. L. Russel: A global theory for linear-quadratic differential games. J. Math. Anal. Applic. *33* (1971), 96—123.

[11] N. J. Krikelis, Z. V. Rekasius: On the solution of the optimal linear control problems under conflict of interest. IEEE Trans. Autom. Control *AC-16* (1971), 2, 140—147.

[12] T. Basar: A counterexample in linear quadratic games: Existence of nonlinear Nash solutions. J. Optim. Theory. Applic. *14* (1974), 4, 425—430.

[13] T. Basar: Nash strategies for *M*-person differential games with mixed information structures. Automatica *11* (1975), 5, 547—551.

[14] H. Stackelberg: Marktform und Gleichgewicht. Springer Verlag, Wien—Berlin 1934.

[15] C. I. Chen, J. B. Cruz, Jr: Stackelberg solution for two-person games with biased information patterns. IEEE Trans. Autom. Control. *AC-17* (1972), 6, 791—798.

[16] M. Simaan, J. B. Cruz, Jr: On the Stackelberg strategy in nonzero-sum games. J. Optim. Theory. Applic. *11* (1973), 5, 533—555.

[17] M. Simaan, J. B. Cruz, Jr: Additional aspects of the Stackelberg strategy in nonzero-sum games. J. Optim. Theory. Applic. *11* (1973), 6, 613—627.

[18] M. Simaan, J. B. Cruz, Jr: Sampled-data Nash controls in non-zero-sum differential games. Int. J. Control *17* (1973), 6, 1201—1209.

[19] M. Simaan, J. B. Cruz, Jr: A Stackelberg solution for games with many players. IEEE Trans. Autom. Control. *AC-18* (1973), 3, 322—324.

[20] Y. Nishikawa, H. Itakura, H. Kitagawa: Nonzero-sum differential games with nonlinear dynamics. Institute of Electronics and Communication Engineers of Japan, Tokyo. Report NLP 71-18, 1972.

[21] L. F. Pau: Differential games and a Nash equilibrium searching algorithm. SIAM J. Control *13* (1975), 4, 835—852.

[22] R. Mukundan, W. B. Elsner: Linear feedback strategies in non-zero-sum differential games. Int. J. Systems Sci. *6* (1975), 6, 513—532.

[23] J. Doležal: Necessary optimality conditions for *N*-player nonzero-sum multistage games. Kybernetika *12* (1976), 4, 268—295.

[24] J. Doležal: Hierarchical solution concepts for static and multistage decision problems with two objectives. Kybernetika *12* (1976), 5, 363—385.

[25] R. P. Hämäläinen: Nash and Stackelberg solutions to general linear-quadratic two-player difference games. Part II. Open-closed strategies. Kybernetika, *14* (1978), to appear.

Dr. Raimo P. Hämäläinen, Systems Theory Laboratory, Helsinki University of Technology, 02150 Espoo 15. Finland.