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Continuous Stochastic Approximation Procedure for Evaluating the Point at which the Regression Function Stops to Be Non-Positive

EL SAYED SOROUR

A continuous version of the stochastic approximation algorithm proposed in [2] is considered. A function $r(x)$ is observed continuously with Gaussian white noise. We want to estimate the point θ such that $r(x) \leq 0$ for $x \leq \theta$ while $r(x) > 0$ for $x > \theta$. The proving methods developed in the book by Nevelson and Hasminskij [4] are utilized to prove the convergence with probability one and in the mean square and the asymptotic normality of the procedure.

1. INTRODUCTION

Consider the following stochastic differential equation

$$\text{(1.1)} \qquad \mathrm{d}X(t) = -a(t)\left(r(X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}\zeta(t)\right), \quad X(t_0) = x,$$

where $\zeta(t)$ is the standard Wiener process.

Nevelson and Hasminskij [4] have proved the convergence of the procedure (1.1) with probability one to the set of roots of $r(x)$ under general conditions.

They have also proved that the procedure can converge to a narrower set. Namely, let Θ be the set of roots of $r(x)$, define $B \subset \Theta$ such that $\theta \in B$ if in some of its η-neighbourhood $U_\eta(\theta)$ there exists a continuously differentiable function $V(x)$ such that

$$V(\theta) = 0, \quad V(x) > 0 \quad \text{for} \quad x \neq \theta, \quad r(x)\frac{\mathrm{d}V}{\mathrm{d}x} \leqq 0 \quad \text{for} \quad x \in U_\eta(\theta).$$

Then, they have proved that the procedure (1.1) cannot converge with positive probability to a point $\theta \in B$. Our goal is to estimate the point θ such that $r(x) \leqq 0$ for $x \leqq \theta$ while $r(x) > 0$ for $x > \theta$, when $r(x)$ is observed continuously with Gaussian white noise. If the procedure (1.1) is used, the only thing we can deduce is that the procedure converges with probability one to the set of roots of $r(x)$, which does not help in our case (because $B = \emptyset$ (empty set)).

The problem was attacked before in the discrete time case by Guttman [3] and Friedman [2], to estimate the point θ at which the regression function stops to be a constant.

Using the continuous analogy of the procedure proposed in [2] and exploiting the proving methods in [4], we obtain results concerning the convergence of the procedure with probability one, in the mean square and the asymptotic normality of the procedure.

2. BASIC ASSUMPTIONS AND NOTATIONS

All random variables are supposed to be defined on a complete probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. Relations between random variables are meant with probability one. E denotes the expectation. The real line is denoted by $\boldsymbol{R}$ and the indicator function of a set A by I_A.

The following assumptions will be assumed to hold in the sequel.

(i) The function $r(x)$ is real-valued and continuous;

$$r(x) \leqq 0 \quad \text{for} \quad x \leqq \theta\,, \quad \text{while if} \quad x > \theta \quad \text{then} \quad r(x) > 0\,, \quad x \in \boldsymbol{R}\,. \tag{2.1}$$

(ii) The function $\sigma(t, x)$ is real-valued and continuous function of its arguments for $t \in [t_0, \infty)$, $x \in \boldsymbol{R}$.

(iii) For each N, there exists L_N for which

$$|r(x) - r(y)| + |\sigma(t, x) - \sigma(t, y)| \leqq L_N|x - y| \tag{2.2}$$

for $|x| \leqq N$, $|y| \leqq N$; $t_0 \leqq t \leqq N$.

(iv) $\zeta(t)$ is independent (standard) Wiener process, consistent with a non-decreasing family $[\mathfrak{F}_t, t \geqq t_0]$ of σ-fields of events.

(v) $X^{s,\xi}(t)$ is the regular solution, which is continuous with probability one, of the stochastic differential equation of the form

$$\mathrm{d}X(t) = b(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}\zeta(t) \tag{2.3}$$

with $X(s) = \xi$, ξ is $\mathfrak{F}_s$-measurable.

(vi) The differential operator

$$\frac{\partial}{\partial t} + b(t, x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2}{\partial x^2}$$

of (2.3) is denoted by L.

3. CONDITIONS AND PRELIMINARIES

The following conditions will be needed as referred to.

Conditions on the regression function $r(x)$

R1: For some $K_1 > 0$, $r(x) \geqq K_1(x - \theta)$ for $x > \theta$.

R2: For some positive constants K_2 and K_3, there exist positive constants ϱ and $h \geqq \varrho$ such that

$$r(x) \leqq K_2(x - \theta) \quad \text{for} \quad x \in [\theta - \varrho, \theta]\,,$$

$$r(x) \leqq K_3(x - \theta) \quad \text{for} \quad x \in (-\infty, -h + \theta)\,.$$

R3: For some constant $K_4 > 0$, $(x - \theta)\, r(x) \geqq K_4(x - \theta)^2$, $x \in \boldsymbol{R}$.

R4: There exists $B > 0$, such that

$$r(x) = B(x - \theta) + f(x, \theta)\,, \quad x \in \boldsymbol{R}\,,$$

where

$$|f(x, \theta)| = o(|x - \theta|) \quad \text{as} \quad x \to \theta\,.$$

Conditions on the functions $a(t)$ and $\delta(t)$

A1:
$$\int_{t_0}^{\infty} a(t)\, \delta(t)\, \mathrm{d}t = \infty\,; \quad \int_{t_0}^{\infty} a^2(t)\, \mathrm{d}t < \infty\,; \quad \lim_{t \to \infty} a(t) = 0\,;$$

$$\int_{t_0}^{\infty} a(t)\, \delta^2(t)\, \mathrm{d}t < \infty \quad \text{and} \quad \lim_{t \to \infty} \delta(t) = 0\,.$$

A2:
$$a(t) = \frac{a}{t^\alpha}\,; \quad a > 0\,; \quad \frac{1}{2} < \alpha < \frac{2}{3}\,; \quad t \geqq t_0\,.$$

A3:
$$\delta(t) = \frac{\delta}{t^\gamma}\,; \quad \delta > 0\,; \quad \frac{\alpha}{2} < \gamma < 1 - \alpha\,; \quad t \geqq t_0\,.$$

Conditions on $\sigma(t, x)$

C1:
$$|\sigma(t, x)|^2 \leqq K^2(1 + x^2) \quad \text{for all} \quad t \geqq t_0\,, \quad x \in \boldsymbol{R}\,.$$

C2:
$$\lim_{\substack{t \to \infty \\ x \to \theta}} \sigma(t, x) = \sigma_0\,.$$

We shall also need the following theorem due to M. B. Nevelson, R. Z. Hasminskij [4].

Theorem 3.1. Let us have a nonnegative real-valued function $V(t, x)$, which is continuously differentiable with respect to t, and twice continuously differentiable with respect to x, and a set A for which

(3.1) $$\inf_{t \geqq t_0} V(t, x) \to \infty \quad \text{for} \quad |x| \to \infty ;$$

let us assume that

(3.2) $$LV \leqq -\alpha(t)\, \varphi(t, x) + g(t)\,(1 + V),$$

where

(3.3) $$g(t) > 0 ; \quad \int_{t_0}^{\infty} g(t)\, \mathrm{d}t < \infty ; \quad \alpha(t) > 0 ; \quad \int_{t_0}^{\infty} \alpha(t)\, \mathrm{d}t = \infty ;$$

(3.4) $$\varphi(t, x) \geqq 0 \quad \text{for all} \quad t \geqq t_0 ; \quad x \in \boldsymbol{R}$$

and for all $M > \varrho > 0$

$$\inf_{t \geqq T(\varrho), x \in U_{\varrho, M}(A)} \varphi(t, x) > 0 ,$$

where $U_{\varrho,M}(A) = v_\varrho(A) \cap \{x : |x| < M\}$, $v_\varrho(A)$ is the complement of the ϱ-neighbourhood of the set A; L is the differential operator of (2.3). Further let the conditions

(3.5) $$\inf_{t \geqq t_0} V(t, x) > 0 \quad \text{for} \quad x \notin A ; \quad V(t, x) = 0 \quad \text{for} \quad x \in A$$

and

(3.6) $$\lim_{x \to A} \sup_{t \geqq t_0} V(t, x) = 0$$

be valid. Moreover let $b(t, x)$ satisfy (2.2) ($r(x)$ is replaced by $b(t, x)$). Then the solution of (2.3) converges with probability one to the set A for all $x \in \boldsymbol{R}$.

(This is one-dimensional version of the Theorem 3.8.1 of M. B. Nevelson and R. Z. Hasminskij [4].)

4. CONVERGENCE THEOREMS

Let $a(t)$ and $\delta(t)$ be positive real-valued continuous functions. Let $X^x(t)$ be the regular solution of the stochastic differential equation

(4.1) $$\mathrm{d}X(t) = -a(t)\left[(r(X(t)) - \delta(t))\, \mathrm{d}t + \sigma(t, X(t))\, \mathrm{d}\zeta(t)\right],$$

with $X(t_0) = x$; $x \in \boldsymbol{R}$, $t \geqq t_0$.

It is evident that the differential operator L of (4.1) is

$$L = \frac{\partial}{\partial t} - a(t)\,(r(x) - \delta(t))\,\frac{\partial}{\partial x} + \frac{1}{2}\,\sigma^2(t, x)\, a^2(t)\,\frac{\partial^2}{\partial x^2}\,. \tag{4.2}$$

Theorem 4.1. If **R1**, **A1** and **C1** hold, then $X^x(t) \to \theta$ for $t \to \infty$ with probability one.

Proof. Without loss of generality we can take $\theta = 0$. Defining

$$V(t, x) = x^2\,, \tag{4.3}$$

it is evident that $V(t, x)$ fulfils the conditions (3.1), (3.5) and (3.6) of Theorem 3.1 with the set $A = \{0\}$.

From (4.1), (4.2) and (4.3) we have

$$LV \leqq -2x\, a(t)\,(r(x) - \delta(t)) + K^2 a^2(t)\,(1 + V)\,. \tag{4.4}$$

For $x > (2/K_1)\,\delta(t)$, we have by using **R1** $r(x) > 2\,\delta(t)$. Thus from (4.4) we get

$$LV \leqq -2\, a(t)\,\delta(t)\, x + K^2 a^2(t)\,(1 + V)\,. \tag{4.5}$$

For $0 \leqq x \leqq (2/K_1)\,\delta(t)$ the inequality (4.4) can be written as

$$LV \leqq -2\, a(t)\, x\, r(x) + \frac{4}{K_1}\, a(t)\,\delta^2(t) + K^2 a^2(t)\,(1 + V)\,. \tag{4.6}$$

For $x < 0$ the inequality (4.4) can be written as

$$LV \leqq -2\, a(t)\, x\, r(x) + 2\, a(t)\,\delta(t)\, x + K^2 a^2(t)\,(1 + V)\,.$$

Using (2.1) we get

$$LV \leqq -2\, a(t)\,\delta(t)\,|x| + K^2\, a^2(t)\,(1 + V)\,. \tag{4.7}$$

Defining

$$\varphi(t, x) = \begin{cases} x\, r(x) & \text{for} \quad 0 \leqq x < \dfrac{2}{K_1}\,\delta(t)\,; \\ |x| & \text{otherwise}\,; \end{cases}$$

$$\alpha(t) = 2\, a(t)\,\delta(t)\,,$$

$$g(t) = \frac{4}{K_1}\, a(t)\,\delta^2(t) + K^2 a^2(t)\,,$$

the inequalities (4.5), (4.6) and (4.7) can be written as

$$LV \leqq -\alpha(t)\,\varphi(t, x) + g(t)\,(1 + V)\,, \quad \text{for} \quad t \geqq T_1\,. \tag{4.8}$$

From (2.1) and **A1** it is easy to see that $\varphi(t, x)$ satisfies (3.4) and from **A1** it is evident that $g(t)$ satisfies (3.3), thus by (4.8) the condition (3.2) is also satisfied completing the proof of the theorem.

Remark 4.2. The condition **R1** can be somehow weakened to extend the class of the regression functions for which Theorem 4.1 is still valid. Let **A1** and **R1** in Theorem 4.1 be replaced by **A1'** and **R1'**.

A1': $$\int_{t_0}^{\infty} a(t)\,\delta(t)\,\mathrm{d}t = \infty\,, \quad \int_{t_0}^{\infty} a^2(t)\,\mathrm{d}t < \infty\,, \quad \lim_{t\to\infty} \delta(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} a(t) = 0\,.$$

Define $\tau(t) > 0$, $\tau(t) \to 0$ for $t \to \infty$ such that $\inf_{x>\tau(t)} r(x) > 2\,\delta(t)$ (this is possible by virtue of **A1'**).

R1': $$\int_{t_0}^{\infty} a(t)\,\delta(t)\,\tau(t)\,\mathrm{d}t < \infty\,.$$

Still we can conclude that Theorem 4.1 is valid.

The proof can be carried out in steps as in the proof of Theorem 4.1.

In fact for $x > \tau(t)$ the inequality (4.4) can be written as (4.5). For $0 \leqq x \leqq \tau(t)$ we have

$$\text{(4.6')} \qquad LV \leqq -2\,a(t)\,x\,r(x) + 2\,a(t)\,\delta(t)\,\tau(t) + K^2 a^2(t)\,(1 + V)\,.$$

For $x < 0$ as in Theorem 4.1, we have (4.7).

Defining

$$\varphi'(t, x) = \begin{cases} x\,r(x)\,, & 0 \leqq x \leqq \tau(t)\,, \\ |x| & \text{otherwise}\,; \end{cases}$$

$$\alpha(t) = 2\,a(t)\,\delta(t)\,;$$

$$g'(t) = 2\,a(t)\,\delta(t)\,\tau(t) + K^2 a^2(t)$$

then (4.5), (4.6'), (4.7) can be written as

$$\text{(4.8')} \qquad LV \leqq -\alpha(t)\,\varphi'(t, x) + g'(t) \quad \text{for} \quad t \geqq T_1$$

and the proof can be completed so as in Theorem 4.1.

To show that Remark 4.1 extends the class of the regression functions, the following example is used.

Example 4.1. Put

$$\begin{aligned} r(x) &= 0\,, && -\infty < x \leqq 0\,; \\ &= x^2\,, && 0 < x \leqq 1\,; \\ &= 1\,, && x > 1\,. \end{aligned}$$

 Take $a(t) = 1/t^{2/3}$ and $\delta(t) = 1/t^{1/3}$; then from the definition of $\tau(t)$, we deduce that

$$\tau(t) = O\left(\frac{1}{t^{1/6}}\right).$$

It is evident that $r(x)$ does not satisfy **R1** while **A1'** and **R1'** are satisfied.

Theorem 4.2. If **R1**, **R2**, **A1** and **C1** hold, then

$$\lim_{t\to\infty} \mathsf{E}|X^x(t) - \theta|^2 = 0 .$$

Proof. As before we can take $\theta = 0$. Let $V(t, x)$ be chosen as in proving Theorem 4.1. Then (4.4) can be written as

$$LV \leqq -2\, a(t)\, x\, r(x) + 2\, a(t)\, \delta(t)\, x + K^2\, a^2(t)\, (1 + x^2)\,. \tag{4.9}$$

For $x > 0$ and by using the inequality $|x| \leqq 1 + x^2$, we can write (4.9) as

$$LV \leqq -2\, a(t)\, x\, r(x) + 2\, a(t)\, \delta(t) + 2\, a(t)\, \delta(t)\, x^2 + K^2\, a^2(t)\, (1 + x^2)\,.$$

Using **R1** we get

$$\begin{aligned} LV &\leqq -2K_1\, a(t)\, x^2 + 2\, a(t)\, \delta(t) + K^2\, a^2(t) + x^2(2\, a(t)\, \delta(t) + K^2\, a^2(t)) = \\ &= -K_1\, a(t)\, x^2 \left(2 - \frac{2}{K_1}\, \delta(t) - \frac{K^2}{K_1}\, a(t)\right) + 2\, a(t)\, \delta(t) + K^2\, a^2(t)\,. \end{aligned}$$

From **A1** it follows that there exists T_1 such that

$$\begin{aligned} LV &\leqq -K_1\, a(t)\, x^2 + 2\, a(t)\, \delta(t) + K^2\, a^2(t) = \\ &= -K_1\, a(t)\, V + 2\, a(t)\, \delta(t) + K^2\, a^2(t) \end{aligned} \tag{4.10}$$

for $t \geqq T_1$.

For $-\varrho \leqq x < 0$ the inequality (4.9), by using **R2**, can be written as

$$\begin{aligned} LV &\leqq -2K_2\, a(t)\, x^2 + 2\, a(t)\, \delta(t)\, x + K^2\, a^2(t)\, (1 + x^2) \leqq \\ &\leqq -2K_2\, a(t)\, x^2 + K^2\, a^2(t)\, (1 + x^2)\,; \end{aligned}$$

then as before there exists T_2 such that

$$LV \leqq -K_2\, a(t)\, V + K^2\, a^2(t)\,, \quad \text{for} \quad t > T_2\,. \tag{4.11}$$

For $-h \leqq x \leqq -\varrho$ the inequality (4.9) can be written as

$$\begin{aligned} LV &\leqq 2\, a(t)\, \delta(t)\, x + K^2\, a^2(t)\, (1 + x^2) \leqq K^2\, a^2(t)\, (1 + x^2) \leqq \\ &\leqq K^2\, a^2(t)\, (1 + h^2)\,. \end{aligned} \tag{4.12}$$

Finally for $x < -h$ the inequality (4.9) can be written as

$$LV \leqq -2\, a(t)\, x\, r(x) + K^2\, a^2(t)\,(1 + x^2)\,.$$

Using **R2** we get

$$LV \leqq -2K_3\, a(t)\, x^2 + K^2\, a^2(t)\,(1 + x^2)\,,$$

and as before

$$LV \leqq -K_3\, a(t)\, V + K^2\, a^2(t) \quad \text{for} \quad t > T_3\,. \tag{4.13}$$

Defining

$$\begin{aligned} \beta(t) &= 2\, a(t)\, \delta(t) + K^2\, a^2(t)\,,\\ T &= \max(T_1, T_2, T_3)\,,\\ K_5 &= \min(K_1, K_2, K_3)\,,\\ a'(t) &= K_5\, a(t)\,,\\ W(t, x) &= V(t, x) \exp\left(\int_T^t a'(u)\, \mathrm{d}u\right), \end{aligned} \tag{4.14}$$

then (4.10), (4.11), (4.12) and (4.13) can be written as

$$LV \leqq \begin{cases} K^2\, a^2(t)\,(1 + h^2)\,, & -h \leqq x < -\varrho\,;\\ -a'(t)\, V + \beta(t) & \text{otherwise}\,. \end{cases} \tag{4.15}$$

From (4.2) and (4.14) we get

$$LW = \exp\left(\int_T^t a'(u)\, \mathrm{d}u\right)(LV + a'(t)\, V)\,. \tag{4.16}$$

From Lemma 3.5.1 in [4], Fatou lemma and the regularity of the procedure we get

$$\mathsf{E}(W(t, X(t)) - W(s, X(s))) \leqq \mathsf{E} \int_s^t LW(u, X(u))\, \mathrm{d}u\,. \tag{4.17}$$

Let us denote

$$A = [-\varrho, \infty) \cup (-\infty, -h)$$

and

$$p(t) = \mathsf{P}(X(t) \in A)\,.$$

Using (4.15) and (4.16), the inequality (4.17) can be written as

$$\mathsf{E}W(t, X^x(t)) - \mathsf{E}W(s, X^x(s)) \leqq \mathsf{E}\Big[I_A \int_s^t \beta(u) \exp\left(\int_T^u a'(v)\, \mathrm{d}v\right) \mathrm{d}u +$$

$$+ I_{A^c} \int_s^t (K^2 a^2(u)(1 + h^2) + a'(u) h^2) \exp\left(\int_T^u a'(v)\,\mathrm{d}v\right) \mathrm{d}u \Bigg] =$$

$$= p(t) \int_s^t \beta(u) \exp\left(\int_T^u a'(v)\,\mathrm{d}v\right) \mathrm{d}u + q(t) \int_s^t (K^2 a^2(u)(1 + h^2) +$$

$$+ a'(u) h^2) \exp\left(\int_T^u a'(v)\,\mathrm{d}v\right) \mathrm{d}u\,, \quad \text{for} \quad t > s \geqq T \geqq t_0\,.$$

Thus

(4.18)

$$\mathsf{E}V(X(t)) \leqq \mathsf{E}W(s, X^x(s)) \exp\left(-\int_T^t a'(v)\,\mathrm{d}v\right) + p(t) \int_s^t \beta(u) \exp\left(-\int_u^t a'(v)\,\mathrm{d}v\right) \mathrm{d}u +$$

$$+ K^2 q(t) \int_s^t a^2(u)(1 + h^2) \exp\left(-\int_u^t a'(v)\,\mathrm{d}v\right) \mathrm{d}u +$$

$$+ q(t) h^2 \int_s^t a'(u) \exp\left(-\int_u^t a'(v)\,\mathrm{d}v\right) \mathrm{d}u\,.$$

Consider the right-hand side of (4.18). The first term tends to zero for $t \to \infty$ since $\mathsf{E}W(s, X^x(s))$ is bounded. The second and the third terms tend to zero for $t \to \infty$ by Problem 4.4.1 in [4].

Let us denote the last term by $g(t)$, i.e.

$$g(t) = q(t) h^2 \int_s^t a'(u) \exp\left(-\int_u^t a'(v)\,\mathrm{d}v\right) \mathrm{d}u\,.$$

By integration we get

$$g(t) = q(t) h^2 \left(1 - \exp\left(-\int_s^t a'(v)\,\mathrm{d}v\right)\right).$$

Using Theorem 4.1, $q(t) \to 0$ for $t \to \infty$, which implies with **A1** that $g(t) \to 0$, for $t \to \infty$. Thus

$$\lim_{t\to\infty} \mathsf{E}V(X^x(t)) = \lim_{t\to\infty} \mathsf{E}(X^x(t))^2 = 0\,,$$

completing the proof of the theorem.

5. THE ASYMPTOTIC NORMALITY OF THE PROCEDURE

To establish the asymptotic normality of the procedure, we give the following lemma. Its proof can be carried out as that of Lemma 6.2.1 in [4].

Lemma 5.1. If **R3**, **A2**, **A3** and **C1** hold, then

$$\mathsf{E}|X^{s,\xi}(t) - \theta|^2 = O(t^{-\alpha}), \quad \text{for} \quad t \to \infty .$$

Here $X^{s,\xi}(t)$ is the solution of the stochastic differential equation (4.1) and $\mathsf{E}|\xi|^2 < \infty$.

Proof. As before, we can take $\theta = 0$. Take $V(t, x)$ as in proving Theorem 4.1. By using **R3**, **A2**, **A3** and **C1** in (4.4) we get

$$LV \leqq -2K_4 \frac{a}{t^\alpha} V + \frac{2a\delta}{t^{\alpha+\gamma}} x + \frac{K^2a^2}{t^{2\alpha}} (1 + V) . \tag{5.1}$$

Using the inequality

$$|x| \leqq \eta^{-1}t^{-\gamma} + \eta t^\gamma |x|^2 , \quad \eta > 0 , \tag{5.2}$$

we get

$$LV \leqq -2K_4at^{-\alpha}V + K^2a^2t^{-2\alpha}V + 2a\delta\eta t^{-\alpha}V + K^2a^2t^{-2\alpha} + 2a\delta\eta^{-1}t^{-\alpha-2\gamma} . \tag{5.3}$$

By chosing η sufficiently small, since **A3** implies that $-\alpha - 2\gamma < -2\alpha$, we get

$$LV \leqq -K_4at^{-\alpha}V + K^2a^2t^{-2\alpha} , \quad \text{for} \quad t > T . \tag{5.4}$$

From Theorem 3.1, and (5.4), we can easily deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{E}V(X^{s,\xi}(t)) = \mathsf{E}LV(X^{s,\xi}(t)) \leqq -K_4at^{-\alpha}\mathsf{E}V(X^{s,\xi}(t)) + K^2a^2t^{-2\alpha} ;$$

its solution can be written as

$$\mathsf{E}V(X^{s,\xi}(t)) \exp\left(\frac{aK_4}{1-\alpha} t^{1-\alpha}\right) \leqq \int_T^t K^2a^2u^{-2\alpha} \exp\left(\frac{aK_4}{1-\alpha} u^{1-\alpha}\right) \mathrm{d}u + $$
$$+ \mathsf{E}V(X^{s,\xi}(T)) \exp\left(\frac{aK_4}{1-\alpha} T^{1-\alpha}\right).$$

Then

$$\mathsf{E}V(X^{s,\xi}(t)) \leqq m_1 \exp(-m_2t^{1-\alpha}) \int_t^T u^{-2\alpha} \exp(m_2u^{1-\alpha}) \,\mathrm{d}u + m_3 \exp(-m_2t^{1-\alpha}) .$$

(Here, as well as in the sequel, m with subscript will denote positive constants, possibly of different values in different formulas.)

$$\mathsf{E}V(X^{s,\xi}(t)) \leqq m_1 \exp(-m_2t^{1-\alpha}) \Big[t^{-\alpha} \exp(m_2t^{1-\alpha}) - T^{-\alpha} \exp(m_2T^{1-\alpha}) + $$
$$+ m_2 \int_T^t u^{-\alpha-1} \exp(m_2u^{1-\alpha}) \,\mathrm{d}u \Big] + m_3 \exp(-m_2t^{1-\alpha}) =$$
$$= m_1t^{-\alpha} + m_1 \exp(-m_2t^{1-\alpha}) + m_1 I(t) ,$$

where

$$I(t) = \int_T^t u^{-\alpha-1} \exp\left(-m_2(t^{1-\alpha} - u^{1-\alpha})\right) du .$$

It is sufficient to prove that $I(t) = O(t^{-\alpha})$, for $t \to \infty$. Substituting $z = t^{1-\alpha} - u^{1-\alpha}$, we obtain

$$t^\alpha I(t) = \frac{1}{1-\alpha} \int_0^{t^{1-\alpha} - T^{1-\alpha}} \frac{t^{\alpha-1}}{\left(1 - \frac{z}{t^{1-\alpha}}\right)^{1/(1-\alpha)}} \exp\left(-m_2 z\right) dz .$$

Then $\lim_{t\to\infty} t^\alpha I(t) = 0$, which completes the proof of the lemma.

Theorem 5.1. If **R1**, **R4**, **A2**, **A3**, **C1** and **C2** hold, then the asymptotic distribution of $t^{\alpha/2}(X^x(t) - \theta)$ is normal with

$$\text{mean} = 0 \tag{5.5}$$

and

$$\text{varience} = \frac{a\sigma_0^2}{2B} . \tag{5.6}$$

Proof. From **R4**, it follows that there exists $\eta > 0$ such that

$$|f(x, \theta)| \leqq \frac{B}{2} |x - \theta| \quad \text{for} \quad |x - \theta| < \eta . \tag{5.7}$$

From **C1** it follows that

$$|\sigma(t, x)| \leqq K' \quad \text{for} \quad |x - \theta| < \eta . \tag{5.8}$$

Let us define

$$\begin{aligned} f^*(x, \theta) &= f(x, \theta) && \text{for} \quad |x - \theta| \leqq \eta , \\ &= f\left(\eta + \theta \frac{(x - \theta)}{|x - \theta|}, \theta\right) \frac{|x|}{\eta} && \text{for} \quad |x - \theta| > \eta \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} \sigma^*(t, x) &= \sigma(t, x) && \text{for} \quad |x - \theta| \leqq \eta , \\ &= \sigma\left(t, (\eta + \theta) \frac{(x - \theta)}{|x - \theta|}\right) && \text{for} \quad |x - \theta| > \eta . \end{aligned} \tag{5.10}$$

Without loss of generality, we can take $\theta = 0$. Let us consider the following auxiliary stochastic differential equation

$$dX^*(t) = -at^{-\alpha}[(BX^*(t) + f^*(X^*(t)) - \delta t^{-\gamma}) dt + \sigma^*(t, X^*(t)) d\zeta(t)] \tag{5.11}$$

with $X^*(s) = \xi$ where ξ is $\mathfrak{F}_s$-measurable and $\mathsf{E}|\xi|^2 < \infty$.

From (5.9) it follows that **R3** holds; then from Lemma 5.1

$$\mathsf{E}|X^{*s,\xi}(t)|^2 = O(t^{-\alpha}) \quad \text{for} \quad t \to \infty . \tag{5.12}$$

Denoting $Y^*(t) = t^{\alpha/2} X^{*\xi}(t)$, from (5.11) we get

$$\mathrm{d}Y^*(t) = \left[\left(\frac{\alpha}{2} t^{-1} - aBt^{-\alpha}\right) Y^*(t) - at^{-\alpha/2} f^*(X(t)) + a\delta t^{-\alpha/2-\gamma}\right] \mathrm{d}t - \tag{5.13}$$
$$- at^{-\alpha/2} \sigma^*(t, X^*(t)) \, \mathrm{d}\zeta(t) .$$

Then

$$Y^*(t) = a\delta \int_s^t \left(\frac{t}{u}\right)^{\alpha/2} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) u^{-\alpha/2-\gamma} \, \mathrm{d}u - \tag{5.14}$$
$$- a \int_s^t \left(\frac{t}{u}\right)^{\alpha/2} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) u^{-\alpha/2} f^*(X^*(u)) \, \mathrm{d}u -$$
$$- a \int_s^t \left(\frac{t}{u}\right)^{\alpha/2} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) u^{-\alpha/2} \sigma^*(u, X(u)) \, \mathrm{d}\zeta +$$
$$+ t^{\alpha/2} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - s^{1-\alpha})\right) \xi .$$

Let us consider the right-hand side of (5.14). Denote the 2-nd integral by $I_2(t)$. From (5.9) it follows that given $\eta > 0$, there exists $\varrho_1 > 0$, such that

$$|f^*(x)| \leqq \eta^2 |x| \quad \text{for} \quad |x| < \varrho_1 .$$

From Theorem 4.1 it follows

$$\mathsf{P}[\sup_{u \geqq T} |X^\xi(t)| < \varrho_1] > 1 - \eta ;$$

by this we can write

(5.15)

$$\mathsf{P}[|I_2(t)| > \eta] \leqq \eta + \mathsf{P}\left[\left|\int_T^t \left(\frac{t}{u}\right)^{\alpha/2} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) f^* u^{-\alpha/2} \, \mathrm{d}u\right| > \eta ;\right.$$
$$\left. \sup |X^\xi(t)| > \varrho_1 \right] \leqq$$
$$\leqq \eta + \mathsf{P}\left[\int_T^t \left(\frac{t}{u}\right)^{\alpha/2} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) |f^*| \, u^{-\alpha/2} \, \mathrm{d}u > \eta\right] \leqq$$
$$\leqq \eta + \eta \left[\int_T^t \left(\frac{t}{u}\right)^{\alpha/2} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) \mathsf{E}|X^*(u)| \, u^{-\alpha/2} \, \mathrm{d}u\right].$$

 By using the inequality $|X(t)| \leqq t^{-\alpha/2} + t^{\alpha/2}|X(t)|^2$ and (5.12), the right-hand side of (5.15) is less than or equal to

$$\eta + 2\eta \int_T^t \left(\frac{t}{u}\right)^{\alpha/2} u^{-\alpha} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) du .$$

Denoting

$$G(t) = \eta \int_T^t \left(\frac{t}{u}\right)^{\alpha/2} u^{-\alpha} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) du$$

and using the substitution (5.3) $z = t^{1-\alpha} - u^{1-\alpha}$, we have

$$(5.16) \qquad G(t) = m'\eta \int_0^{t^{1-\alpha}-T^{1-\alpha}} \frac{1}{\left(1 - \dfrac{z}{t^{1-\alpha}}\right)^{\alpha/2(1-\alpha)}} \exp\left(\frac{-aB}{1-\alpha} z\right) dz .$$

Thus

$$\lim_{t\to\infty} G(t) = \eta m' \int_0^\infty \exp\left(\frac{-aB}{1-\alpha} z\right) dz = \eta m'' ,$$

and then

$$\lim_{t\to\infty} \mathsf{P}[|I_2(t)| > \eta] = 0 .$$

Considering the 3-rd integral and defining

$$\eta_1(t) = -a \int_s^t \left(\frac{t}{u}\right)^{\alpha/2} u^{-\alpha/2} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right)(\sigma^* - \sigma_0)\, d\zeta(u) ,$$

$$\eta_2(t) = -a \int_s^t \left(\frac{t}{u}\right)^{\alpha/2} u^{-\alpha/2} \exp\left(\frac{-aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right)\sigma_0\, d\zeta(u)$$

then

$$\mathsf{E}(\eta_1(t))^2 = a^2 \int_s^t \left(\frac{t}{u}\right)^{\alpha} u^{-\alpha} \mathsf{E}|\sigma^* - \sigma_0|^2 \exp\left(\frac{-2aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) du .$$

From Theorem 4.1, (5.8), (5.10) and **C2**

$$\lim_{t\to\infty} \mathsf{E}|\sigma^*(t, X^*(t)) - \sigma_0|^2 = 0 .$$

Thus given $\eta > 0$, there exists $T(\eta)$, such that

$$(5.17) \quad \mathsf{E}(\eta_1(t))^2 \leqq a^2 \int_s^{T(\eta)} \left(\frac{t}{u}\right)^{\alpha} u^{-\alpha}\, \mathsf{E}|\sigma^* - \sigma_0|^2 \exp\left(\frac{-2aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) du +$$
$$+ a^2\eta \int_{T(\eta)}^t \left(\frac{t}{u}\right)^{\alpha} u^{-\alpha} \exp\left(\frac{-2aB}{1-\alpha}(t^{1-\alpha} - u^{1-\alpha})\right) du .$$

Using (5.8), the first integral in the right hand side of (5.17) tends to zero for $t \to \infty$, while the 2-nd integral tends to 0 for $t \to \infty$.

Then $\mathsf{E}(\eta_1(t))^2 \to 0$, for $t \to \infty$ which implies that

$\eta_1(t) \to 0$ in probability;

$\eta_2(t)$ is normal random variable with mean zero, and with

$$\text{variance} = \lim_{t\to\infty} a^2\sigma_0^2 \int_s^t \left(\frac{t}{u}\right)^\alpha u^{-\alpha} \exp\left(\frac{-2aB}{1-\alpha}\,(t^{1-\alpha} - u^{1-\alpha})\right) \mathrm{d}u\,.$$

Using the substitution (5.3) $z = t^{1-\alpha} - u^{1-\alpha}$, the

$$\text{variance} = -\frac{a^2\sigma_0^2}{2B'} \int_0^\infty \exp\left(-\frac{2aB}{1-\alpha}\,z\right) \mathrm{d}z\,, \quad B' = \frac{\alpha - 1}{2}\,.$$

$$= a\sigma_0^2/2B\,.$$

It is evident that the 4-th term tends to zero for $t \to \infty$. Then $Y^*(t)$ is normal with variance $= a\sigma_0^2/2B$, and with

$$\text{mean} = \lim_{t\to\infty} a\delta \int_s^t \left(\frac{t}{u}\right)^{\alpha/2} \exp\left(\frac{-aB}{1-\alpha}\,(t^{1-\alpha} - u^{1-\alpha})\right) u^{-\alpha/2-\gamma}\,\mathrm{d}u\,.$$

From **A3** it follows that $-\alpha/2 - \gamma < -\alpha$; then $u^{-\alpha/2-\gamma} = \varrho(u)\,u^{-\alpha}$, $\varrho(u) \to 0$ as $u \to \infty$, thus

$$\text{mean} = \lim_{t\to\infty} a\delta \int_s^t \left(\frac{t}{u}\right)^{\alpha/2} u^{-\alpha} \exp\left(\frac{-aB}{1-\alpha}\,(t^{1-\alpha} - u^{1-\alpha})\right) \varrho(u)\,\mathrm{d}u\,.$$

This integral tends to 0 for $t \to \infty$ as (5.16). Thus $t^{\alpha/2} X^*(t)$ is normal with

$$\text{mean} = 0\,,$$

$$\text{variance} = \frac{a\sigma_0^2}{2B}\,.$$

The proof can be completed as in Theorem 6.5.1 in [4].

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RNDr. Dipl. Eng. El Sayed Farag Sorour, CSc., Ebn-Marawan 8, Hammamat El Koppa, 16, Cairo. Egypt.