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On the Algebraic Structure of Fuzzy Sets of Type 2

JUHANI NIEMINEN

This paper developes the further properties of fuzzy sets of type 2 (fuzzy grades) recently considered by Mizumoto and Tanaka. A class of modular fuzzy grades is constructed and the structure of distributive fuzzy grades is illustrated by means of a spectrum theorem of Plonka. Finally an open problem proposed by Mizumoto and Tanaka on convex fuzzy grades is solved.

1. INTRODUCTION

Fuzzy sets, introduced and considered by Zadeh in [5], offer a formal way of handling ill-defined objects and have many applications to automata, languages, pattern recognition, decision making, logic and control. After Zadeh had introduced the concept of fuzzy sets of type 2 (fuzzy grades), an extension for fuzzy sets, Mizumoto and Tanaka [2] showed that the algebra of fuzzy grades is a quasilattice (without using this name) and derived a lot of properties of the algebra of fuzzy grades. The purpose of this paper is to apply some properties of quasilattices for finding some further properties of the algebra of fuzzy grades. We shall show that endmaximal fuzzy grades constitute a modular quasilattice and moreover, normal endmaximal fuzzy grades a distributive quasilattice. As the quasilattices are semilattices, also the structure of fuzzy grades as a semilattice is briefly considered. The spectrum theorem of Plonka for distributive quasilattices [4, Theorem 3] is applied for finding some structural properties of the subsystem of convex fuzzy grades.

Basic properties of distributive quasilattices are studied in [4] by Plonka and some further remarks are made in [3].

The definitions and notations concerning fuzzy grades are those introduced in [2] and recalled in the next section. The definitions concerning quasilattices are those of [4] and [3].

2. PRELIMINARIES

In this section we recall briefly the necessary definitions and notations.

An algebra $\mathbf{Q} = (X, \wedge, \vee)$ is called a quasilattice, briefly QL if its fundamental operations $\wedge$ and $\vee$ satisfy the axioms:

(1) $x \wedge x = x, \qquad x \vee x = x.$

(2) $x \wedge y = y \wedge x, \qquad x \vee y = y \vee x.$

(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z), \quad (x \vee y) \vee z = x \vee (y \vee z).$

If the operations $\wedge$ and $\vee$ satisfy further the axiom

(4) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$

the algebra $\mathbf{Q}$ is called a distributive quasilattice and denoted by DQL. A quasilattice QL is modular, denoted briefly by MQL, if it satisfies the axioms (1), (2), (3) and the law (5) given below:

(5)
$$x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z),$$
$$x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z).$$

Obviously each DQL is a MQL, too.

Each quasilattice can be ordered in two different ways:

(i) $a \leqq b \Leftrightarrow b = a \vee b$, and (ii) $a \precsim b \Leftrightarrow a = a \wedge b$. If the orders $\leqq$ and $\precsim$ are mutually reciprocal, i.e. $a \leqq b \Leftrightarrow a \precsim b$, then, as noted by Matsushita [1, Theorem 1], both of the absorption laws hold and the quasilattice in question is in fact a lattice.

A fuzzy set A in a set X is characterized by a membership function μ_A which takes the values in the interval [0, 1], i.e.

$$\mu_A : X \to [0, 1].$$

The value of μ_A at $x \in X$, denoted by $\mu_A(x)$, represents the grade of membership (briefly grade) of x in A and is a point of [0, 1]. A fuzzy set A can be represented as follows:

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \dots + \mu_A(x_n)/x_n$$
$$= \sum_i \mu_A(x_i)/x_i, \quad x_i \in X,$$

where the operation + stands for logical sum (or). As shown in [2], the grades for ordinary fuzzy sets constitute a distributive lattice with the greatest and least elements.

A fuzzy set A of type 2 in a set X is the fuzzy set characterized by a fuzzy membership function μ_A as follows:

$$\mu_A : X \to [0, 1]^J ,$$

with the value $\mu_A(x)$ being called a fuzzy grade and being a fuzzy set in $[0, 1]$ (or in the subset J of $[0, 1]$). Thus by considering the algebraic structure of fuzzy grades we obtain also the algebraic structure of fuzzy sets of type 2. In this paper it is assumed that J is a finite set, but the results of this paper are easily generalized for the case, where J is continuous.

Let $\mu_A(x)$ and $\mu_B(x)$ be two fuzzy grades (i.e. fuzzy sets in $J \subseteq [0, 1]$) of fuzzy sets A and B of type 2, respectively. They can be represented as follows:

$$\begin{aligned}\mu_A(x) &= f(u_1)/u_1 + f(u_2)/u_2 + \ldots + f(u_n)/u_n \\ &= \sum_i f(u_i)/u_i , \quad u_i \in J , \\ \mu_B(x) &= g(w_1)/w_1 + g(w_2)/w_2 + \ldots + g(w_m)/w_m \\ &= \sum_j g(w_j)/w_j , \quad w_j \in J ,\end{aligned}$$

where f and g are membership functions of fuzzy grades $\mu_A(x)$ and $\mu_B(x)$, respectively, and the values $f(u_i)$ and $g(w_j)$ in $[0, 1]$ represent the grades for u_i and w_j in J, respectively.

Let $\circ$ and $\square$ denote max and min, respectively. The operations for fuzzy sets of type 2 are defined as follows:

Union:

$$\begin{aligned}A \cup B \Leftrightarrow \mu_{A \cup B}(x) &= \mu_A(x) \mathbin{\dot\cup} \mu_B(x) \\ &= (\sum_i f(u_i)/u_i) \mathbin{\dot\cup} (\sum_j g(w_j)/w_j) \\ &= \sum_{i,j} (f(u_i) \circ g(w_j))/(u_i \,\square\, w_j) .\end{aligned}$$

Intersection:

$$\begin{aligned}A \cap B \Leftrightarrow \mu_{A \cap B}(x) &= \mu_A(x) \mathbin{\dot\cap} \mu_B(x) \\ &= (\sum_i f(u_i)/u_i) \mathbin{\dot\cap} (\sum_j g(w_j)/w_j) \\ &= \sum_{i,j} (f(u_i) \circ g(w_j))/(u_i \circ w_j) .\end{aligned}$$

As shown in [2, Theorem 1], arbitrary fuzzy grades in J satisfy the laws (1), (2) and (3) with respect to the operations $\dot\cup$ and $\dot\cap$, i.e. they constitute a quasilattice QL(μ). Moreover,

$$\mu_A(x) \mathbin{\dot\cap} (\mu_A(x) \mathbin{\dot\cup} \mu_B(x)) = \mu_A(x) \mathbin{\dot\cup} (\mu_A(x) \mathbin{\dot\cap} \mu_B(x)) \tag{6}$$

holds for each two fuzzy grades $\mu_A(x), \mu_B(x) \in$ QL(μ).

As in [2], we substitute the notation $\mu_A(x)$ with the brief notation μ_A which means in what follows fuzzy grade.

Let $J = \{u_1, u_2, \ldots, u_n\}$ be a subset of $[0, 1]$ which satisfies $u_1 < u_2 < \ldots \ldots < u_n$. A fuzzy grade $\mu_A = \sum_i f(u_i)/u_i$ in J is said to be convex if for any integers i, k with $i \leqq k$ the relation $f(u_j) \geqq \min\{f(u_i), f(u_k)\}$ holds for each j, $i \leqq j \leqq k$. μ_A is said to be normal if $\max_i f(u_i) = 1$. Furthermore, a fuzzy grade which is normal and convex is referred to as a normal convex fuzzy grade.

Mizumoto and Tanaka have shown [2] that if μ_A and μ_B are convex (normal), then also $\mu_A \mathbin{\bar{\cap}} \mu_B$ and $\mu_A \mathbin{\bar{\cup}} \mu_B$ are convex (normal). Moreover, the convex fuzzy grades satisfy (4) with respect to $\bar{\cup}$ and $\bar{\cap}$, i.e. convex fuzzy grades constitute a DQL, and for normal convex fuzzy grades the absorption laws hold with respect to $\bar{\cup}$ and $\bar{\cap}$, i.e. the normal convex fuzzy grades constitute a lattice.

3. ON THE STRUCTURE OF QL(μ)

First, we prove a lemma about (6).

Lemma 1. Let QL be a modular quasilattice, then QL satisfies the law (6) with respect to the operations $\wedge$ and $\vee$.

Proof. By applying the modularity of QL we obtain: $x \wedge (x \vee y) = x \wedge ((x \wedge x) \vee y) = (x \wedge x) \vee (x \wedge y) = x \vee (x \wedge y)$. Thus (6) holds in QL.

In the following we construct a class of fuzzy grades which constitute a modular quasilattice. A fuzzy grade $\mu_A = \sum_i f(u_i)/u_i$ is called endmaximal, if for each value of j it holds: $f(u_1) = f(u_n) \geqq f(u_j)$, where $1 \leqq j \leqq n$ and $J = \{u_1, \ldots, u_n\}$, $u_1 < u_2 < \ldots < u_n$. Endmaximal fuzzy grades defined for a fixed J constitute a modular quasilattice as it will be shown. We need first a lemma.

Lemma 2. Let μ_A and μ_B be two endmaximal fuzzy grades in J. Then $\mu_A \mathbin{\bar{\cap}} \mu_B$ and $\mu_A \mathbin{\bar{\cup}} \mu_B$ are endmaximal, too.

Proof. Let $\mu_A = \sum_u f(u)/u$ and $\mu_B = \sum_w g(w)/w$, where $u, w \in J$. Let $p(a)$ be the grade of $a \in J$ in $\mu_A \mathbin{\bar{\cup}} \mu_B$. Then $p(a)$ can be represented as $p(a) = \sum_{u \Box w = a} f(u) \circ g(w)$, where the logical addition is performed over all the combinations u and w for which $u \Box w = a$.

When $a = u_1 = w_1$, $p(a) = \min\{f(u_1), g(w_1)\}$. In general, if $a = u \Box w$, then $a = u$ and $w \leqq a$, or $u \leqq a$ and $w = a$. Thus $p(a) = f(a) \circ \sum_{w \leqq a} g(w) + g(a) \circ \circ \sum_{u \leqq a} f(u)$, where $f(a) \leqq f(u_1)$, $\sum_{w \leqq a} g(w) = g(w_1)$, $g(a) \leqq g(w_1)$ and $\sum_{u \leqq a} f(u) = f(u_1)$ as μ_A and μ_B are endmaximal. So $f(a) \circ \sum_{w \leqq a} g(w) \leqq \min\{f(u_1), g(w_1)\}$ and the same holds for $g(a) \circ \sum_{u \leqq a} f(u)$, too. Accordingly, $p(a) \leqq p(u_1)$ for each $a \in J$, $a \neq u_1, u_n$.

When $a = u_n = w_n$, then $f(a) \circ \sum_{w \leqq a} g(w) = f(u_n) \circ g(w_n) = \min \{f(u_1), g(w_1)\}$ as μ_A and μ_B are endmaximal, and so $p(u_n) = p(u_1)$. Hence, $\mu_A \mathbin{\dot\cup} \mu_B$ is endmaximal. The proof is similar for $\mu_A \mathbin{\dot\cap} \mu_B$.

If μ_A and μ_B are normal endmaximal, then the endmaximality implies that $f(u_1) = f(u_n) = g(w_1) = g(w_n) = 1$. According to the proof above, $\mu_A \mathbin{\dot\cup} \mu_B$ and $\mu_A \mathbin{\dot\cap} \mu_B$ are normal endmaximal, too.

Theorem 1. The endmaximal fuzzy grades in J constitute a modular quasilattice.

Proof. We must show that $\mu_A \mathbin{\dot\cup} (\mu_B \mathbin{\dot\cap} (\mu_A \mathbin{\dot\cup} \mu_C)) = (\mu_A \mathbin{\dot\cup} \mu_B) \mathbin{\dot\cap} (\mu_A \mathbin{\dot\cup} \mu_C)$ and $\mu_A \mathbin{\dot\cap} (\mu_B \mathbin{\dot\cup} (\mu_A \mathbin{\dot\cap} \mu_C)) = (\mu_A \mathbin{\dot\cap} \mu_B) \mathbin{\dot\cup} (\mu_A \mathbin{\dot\cap} \mu_C)$ for each three endmaximal fuzzy grades μ_A, μ_B and μ_C in J. In fact we shall prove the validity of the first law; the proof is analogous for the second.

Let the fuzzy grades μ_A, μ_B and μ_C be: $\mu_A = \sum_{u \in J} f(u)/u$, $\mu_B = \sum_{w \in J} g(w)/w$, and $\mu_C = \sum_{z \in J} h(z)/z$. Let $p(a)$ be the grade of $a \in J$ in the fuzzy grade $\mu_A \mathbin{\dot\cup} (\mu_B \mathbin{\dot\cap} (\mu_A \mathbin{\dot\cup} \mu_C))$. We shall show that the grade $q(a)$ of $a \in J$ in the fuzzy grade $(\mu_A \mathbin{\dot\cup} \mu_B) \mathbin{\dot\cap} (\mu_A \mathbin{\dot\cup} \mu_C)$ is equal to $p(a)$ for each $a \in J$, from which the desired equality of fuzzy grades follows. $p(a)$ can be represented as

$$p(a) = \sum_{u \square (w \circ (z \square u')) = a} f(u) \circ g(w) \circ h(z) \circ f(u') .$$

The solutions of $u \square (w \circ (z \square u')) = a$ divide u, w, z and u' into the following six classes:

$$\begin{Bmatrix} u = a \\ w \leqq a \\ z \text{ free} \\ u' \text{ free} \end{Bmatrix}; \quad \begin{Bmatrix} u = a \\ w \text{ free} \\ z \leqq a \\ u' \leqq a \end{Bmatrix}; \quad \begin{Bmatrix} u \leqq a \\ w = a \\ z \text{ free} \\ u' \geqq a \end{Bmatrix}; \quad \begin{Bmatrix} u \leqq a \\ w = a \\ z \geqq a \\ u' \text{ free} \end{Bmatrix}; \quad \begin{Bmatrix} u \leqq a \\ w \geqq a \\ z = a \\ u' \leqq a \end{Bmatrix}; \quad \begin{Bmatrix} u \leqq a \\ w \geqq a \\ z \leqq a \\ u' = a \end{Bmatrix}.$$

According to these six classes $p(a)$ can be divided into six partial results $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$, $\boxed{5}$ and $\boxed{6}$, where $p(a) = \boxed{1} + \boxed{2} + \boxed{3} + \boxed{4} + \boxed{5} + \boxed{6}$. The partial results are given below.

$$\begin{aligned} \boxed{1} &= f(a) \circ \sum_{w \leqq a} g(w) \circ \sum_{z} h(z) \circ \sum_{u'} f(u') \\ &= f(a) \circ \sum_{w \leqq a} g(w) \circ \sum_{z} h(z), \quad \text{as} \quad f(a) \leqq \sum_{u'} f(u') . \end{aligned}$$

$$\begin{aligned} \boxed{2} &= f(a) \circ \sum_{w} g(w) \circ \sum_{z \leqq a} h(z) \circ \sum_{u' \leqq a} f(u') \\ &= f(a) \circ \sum_{w} g(w) \circ \sum_{z \leqq a} h(z), \quad \text{as} \quad f(a) \leqq \sum_{u' \leqq a} f(u') . \end{aligned}$$

$$\boxed{3} = \sum_{u \leqq a} f(u) \circ g(a) \circ \sum_{z} h(z) \circ \sum_{u' \geqq a} f(u') .$$

$$\boxed{4} = \sum_{u \leqq a} f(u) \circ g(a) \circ \sum_{z \geqq a} h(z) \circ \sum_{u'} f(u')$$
$$= \sum_{u \leqq a} f(u) \circ g(a) \circ \sum_{z \geqq a} h(z) .$$

$$\boxed{5} = \sum_{u \leqq a} f(u) \circ \sum_{w \geqq a} g(w) \circ h(a) \circ \sum_{u' \leqq a} f(u')$$
$$= \sum_{u \leqq a} f(u) \circ \sum_{w \geqq a} g(w) \circ h(a) .$$

$$\boxed{6} = \sum_{u \leqq a} f(u) \circ \sum_{w \geqq a} g(w) \circ \sum_{z \leqq a} h(z) \circ f(a)$$
$$= f(a) \circ \sum_{w \geqq a} g(w) \circ \sum_{z \leqq a} h(z) .$$

Moreover, $\boxed{2} + \boxed{6} = \boxed{2}$, as $\sum_{w} g(w) \geqq \sum_{w \geqq a} g(w)$.

By combining the results above we obtain the following expression for $p(a)$:

$$p(a) = f(a) \circ \sum_{w \leqq a} g(w) \circ \sum_{z} h(z) + f(a) \circ \sum_{w} g(w) \circ \sum_{z \leqq a} h(z) +$$
$$+ \sum_{u \leqq a} f(u) \circ g(a) \circ \sum_{z} h(z) \circ \sum_{u' \geqq a} f(u') + \sum_{u \leqq a} f(u) \circ g(a) \circ \sum_{z \geqq a} h(z) +$$
$$+ \sum_{u \leqq a} f(u) \circ \sum_{w \geqq a} g(w) \circ h(a) .$$

$q(a)$ has the expression

$$q(a) = \sum_{(u \square w) \circ (z \square u') = a} f(u) \circ g(w) \circ h(z) \circ f(u') .$$

The solutions of $(u \square w) \circ (z \square u') = a$ divide u, w, z, and u' into the following eight classes.

$$\begin{Bmatrix} u = a \\ w \leqq a \\ z \geqq a \\ u' \text{ free} \end{Bmatrix}; \quad \begin{Bmatrix} u = a \\ w \leqq a \\ z \text{ free} \\ u' \geqq a \end{Bmatrix}; \quad \begin{Bmatrix} u \leqq a \\ w = a \\ z \geqq a \\ u' \text{ free} \end{Bmatrix}; \quad \begin{Bmatrix} u \leqq a \\ w = a \\ z \text{ free} \\ u' \geqq a \end{Bmatrix};$$

$$\begin{Bmatrix} u \geqq a \\ w \text{ free} \\ z = a \\ u' \leqq a \end{Bmatrix}; \quad \begin{Bmatrix} u \text{ free} \\ w \geqq a \\ z = a \\ u' \leqq a \end{Bmatrix}; \quad \begin{Bmatrix} u \geqq a \\ w \text{ free} \\ z \leqq a \\ u' = a \end{Bmatrix}; \quad \begin{Bmatrix} u \text{ free} \\ w \geqq a \\ z \leqq a \\ u' = a \end{Bmatrix}.$$

According to these eight classes $p(a)$ can be represented as a logical sum of eight partial results $\boxed{1}$, $\boxed{2}$, ..., $\boxed{8}$ given below.

$$\boxed{1} = f(a) \circ \sum_{w \leqq a} g(w) \circ \sum_{z \geqq a} h(z) \circ \sum_{u'} f(u')$$
$$= f(a) \circ \sum_{w \leqq a} g(w) \circ \sum_{z \geqq a} h(z), \quad \text{as} \quad f(a) \leqq \sum_{u'} f(u') .$$

$$\boxed{2} = f(a) \circ \sum_{w \leqq a} g(w) \circ \sum_{z} h(z) \circ \sum_{u' \geqq a} f(u')$$
$$= f(a) \circ \sum_{w \leqq a} g(w) \circ \sum_{z} h(z), \quad \text{as} \quad f(a) \leqq \sum_{u' \geqq a} f(u').$$

Moreover, $\boxed{1} + \boxed{2} = \boxed{2}$, as $\sum_{z} h(z) \geqq \sum_{z \geqq a} h(z)$.

$$\boxed{3} = \sum_{u \leqq a} f(u) \circ g(a) \circ \sum_{z \geqq a} h(z) \circ \sum_{u'} f(u')$$
$$= \sum_{u \leqq a} f(u) \circ g(a) \circ \sum_{z \geqq a} h(z).$$

$$\boxed{4} = \sum_{u \leqq a} f(u) \circ g(a) \circ \sum_{z} h(z) \circ \sum_{u' \geqq a} f(u')$$

$$\boxed{5} = \sum_{u \geqq a} f(u) \circ \sum_{w} g(w) \circ h(a) \circ \sum_{u' \leqq a} f(u')$$

$$\boxed{6} = \sum_{u} f(u) \circ \sum_{w \geqq a} g(w) \circ h(a) \circ \sum_{u' \leqq a} f(u')$$
$$= \sum_{w} g(w) \circ h(a) \circ \sum_{u' \leqq a} f(u').$$

Moreover,

$$\boxed{5} + \boxed{6} = h(a) \circ \sum_{u' \leqq a} f(u') \circ \left[\sum_{w \geqq a} g(w) + \sum_{w} g(w) \circ \sum_{u \geqq a} f(u) \right]$$
$$= h(a) \circ \sum_{u' \leqq a} f(u') \circ \sum_{w \geqq a} g(w).$$

It follows from the endmaximality of μ_B that $\sum_{w \geqq a} g(w) = \sum_{w} g(w)$, and so $\sum_{w \geqq a} g(w) \geqq$ $\geqq \sum_{w} g(w) \circ \sum_{u \geqq a} f(u)$. Further, $\sum_{u' \leqq a} f(u') = \sum_{u \leqq a} f(u)$.

$$\boxed{7} = \sum_{u \geqq a} f(u) \circ \sum_{w} g(w) \circ \sum_{z \leqq a} h(z) \circ f(a)$$
$$= \sum_{w} g(w) \circ \sum_{z \leqq a} h(z) \circ f(a).$$

$$\boxed{8} = \sum_{u} f(u) \circ \sum_{w \geqq a} g(w) \circ \sum_{z \leqq a} h(z) \circ f(a)$$
$$= \sum_{w \geqq a} g(w) \circ \sum_{z \leqq a} h(z) \circ f(a).$$

Moreover, $\boxed{7} + \boxed{8} = \boxed{7}$, as $\sum_{w \geqq a} g(w) \leqq \sum_{w} g(w)$. By collecting the partial results given above, we obtain for $q(a)$ the expression:

$$q(a) = f(a) \circ \sum_{w \leqq a} g(w) \circ \sum_{z} h(z) + \sum_{u \leqq a} f(u) \circ g(a) \circ \sum_{z \geqq a} h(z) +$$
$$+ \sum_{u \leqq a} f(u) \circ g(a) \circ \sum_{z} h(z) \circ \sum_{u' \geqq a} f(u') + h(a) \circ \sum_{u \leqq a} f(u) \circ \sum_{w \geqq a} g(w) +$$
$$+ \sum_{w} g(w) \circ \sum_{z \leqq a} h(z) \circ f(a).$$

Thus $p(a) = q(a)$ for each $a \in J$, and the first modularity law holds for endmaximal fuzzy grades.

Note that only the endmaximality of μ_B was used in the proof, and so the modular law is valid for fairly wide class of fuzzy grades. Further, the proof of the law $\mu_A \mathbin{\bar{\cup}} (\mu_B \mathbin{\bar{\cap}} (\mu_A \mathbin{\bar{\cup}} \mu_C))$ can be performed if $g(w_n) \geqq g(w_j)$ for each j, $w_j \in J$, or if $g(u_i) \geqq f(u_i)$ for each $u_i = w_i \in J$. The second modularity law needs naturally dual properties for μ_B.

The following theorem illustrates weakened absorption laws that are valid in the modular case. Accordingly, these laws hold for fuzzy grades when the fuzzy grade corresponding to μ_B in the modular law has suitable properties illustrated above.

Theorem 2. In a MQL the following weakened laws of absorption are valid:

(i) $x \vee y \vee (x \wedge y) = x \vee y\,,\quad x \wedge y \wedge (x \vee y) = x \wedge y.$

(ii) $x \vee (x \wedge y) \vee (y \wedge z) = (x \vee y) \wedge (x \vee (y \wedge z))\,,$

$x \wedge (x \vee y) \wedge (y \vee z) = (x \wedge y) \vee (x \wedge (y \vee z)).$

Proof. The laws of (i) are proved in [3]. On the other hand, by putting $z = y$ in (ii) we obtain the laws of (i).

(ii) : 1 $x \vee (x \wedge y) \vee (y \wedge z) = x \vee [(x \wedge y) \vee (y \wedge z)] =$

$= x \vee [y \wedge (x \vee (y \wedge z))] = (x \vee y) \wedge (x \vee (y \wedge z)).$

(ii) : 2 $x \wedge (x \vee y) \wedge (y \vee z) = x \wedge [y \vee (x \wedge (y \vee z))] =$

$= (x \wedge y) \vee (x \wedge (y \vee z)).$

By applying the proof scheme of Theorem 4 in [2], one can easily see that the normal endmaximal fuzzy grades constitute a distributive quasilattice.

Theorem 3. The normal endmaximal fuzzy grades constitute a distributive quasilattice.

Surprisingly, there are no greatest elements among normal endmaximal fuzzy grades, and so it seems to be so that this subset of fuzzy grades does not constitute a lattice. $\sum_{u \in J} 1/u$ satisfies the law $\mu_A \mathbin{\bar{\cup}} \sum_{u \in J} 1/u = \sum_{u \in J} 1/u$ for each normal endmaximal fuzzy grade μ_A, but also $\mu_A \mathbin{\bar{\cap}} \sum_{u \in J} 1/u = \sum_{u \in J} 1/u$ holds.

As shown in [2, Theorem 9], normal convex fuzzy grades constitute a distributive lattice. On the other hand, the set of fuzzy grades is always a semilattice with respect to $\bar{\cup}$ (and to $\bar{\cap}$, too). In the following we shall make a few observations on the $\bar{\cup}$-semilattice structure of QL(μ).

We call a fuzzy grade $\mu_A = \sum_{u \in J} f(u)/u$ *b-maximal*, if $f(u_1) \geqq f(u_i)$ for each $u_i \in J$.

Lemma 3. If μ_A and μ_B are two b-maximal fuzzy grades in J, then $\mu_A \mathbin{\bar{\cup}} \mu_B$ has the same property in J.

Proof. Let $p(a)$ be the grade of $a \in J$ in $\mu_A \mathbin{\bar{\cup}} \mu_B$, i.e. $p(a) = f(a) \circ \sum_{w \leqq a} g(w) + g(a) \circ \sum_{u \leqq a} f(u)$, where $\mu_A = \sum_{u \in J} f(u)/u$ and $\mu_B = \sum_{w \in J} g(w)/w$. Let us assume that $f(u_1) \geqq g(w_1)$ for $u_1 = w_1 \in J$. According to the b-maximality, $\sum_{w \leqq a} g(w)/w = g(w_1)$ and $\sum_{u \leqq a} f(u)/u = f(u_1)$. Thus $p(a) = f(a) \circ g(w_1) + g(a) \circ f(u_1)$, and this expression for $p(a)$ implies that $p(a) \leqq \min\{g(w_1), g(a)\} \leqq g(w_1)$, whence the fuzzy grade $\mu_A \mathbin{\bar{\cup}} \mu_B$ is b-maximal.

Theorem 4. The b-maximal fuzzy grades constitute a distributive $\bar{\cup}$-semilattice.

Proof. A $\vee$-semilattice is distributive with respect to its operation $\vee$, if the relation $a \vee b \geqq c$ implies the existence of two elements $a_1 \leqq a$ and $b_1 \leqq b$ such that $a_1 \vee b_1 = c$.

As we consider fuzzy grades as a $\bar{\cup}$-semilattice, the order relation $\sqsubseteq$ is given by the rule: $\mu_A \sqsubseteq \mu_B \Leftrightarrow \mu_A \mathbin{\bar{\cup}} \mu_B = \mu_B$.

Let μ_A, μ_B and μ_C be three b-maximal fuzzy grades such that $\mu_C \sqsubseteq \mu_A \mathbin{\bar{\cup}} \mu_B$, where $\mu_A = \sum_{u \in J} f(u)/u$, $\mu_B = \sum_{w \in J} g(w)/w$ and $\mu_C = \sum_{z \in J} h(z)/z$. We shall use also the notation $\mu_A \mathbin{\bar{\cup}} \mu_B = \sum_{x \in J} t(x)/x$.

The relation $\mu_C \sqsubseteq \mu_A \mathbin{\bar{\cup}} \mu_B$, i.e. $\mu_C \mathbin{\bar{\cup}} \mu_A \mathbin{\bar{\cup}} \mu_B = \mu_A \mathbin{\bar{\cup}} \mu_B$, implies that $t(a) = h(a) \circ \sum_{x \leqq a} t(x) + t(a) \circ \sum_{z \leqq a} h(z)$, and in particular, $t(u_1) = h(u_1) \circ t(u_1)$ when $a = u_1 = w_1 = z_1 = x_1$. Thus $t(u_1) \leqq h(u_1)$, and as we consider b-maximal fuzzy grades only, $t(a) = h(a) \circ t(u_1) + t(a) \circ h(u_1)$. But $t(a) \leqq t(u_1) \leqq h(u_1)$, and so $t(a) = h(a) \circ t(u_1) + t(a)$, whence $t(a) \geqq h(a) \circ t(u_1)$. Further, $t(u_1) \geqq t(a)$ and as $h(a) \circ t(u_1) \leqq t(a)$, we can conclude that $h(a) \leqq t(a)$ when $a \neq u_1$.

On the other hand, $t(a) = f(a) \circ \sum_{w \leqq a} g(w) + g(a) \circ \sum_{u \leqq a} f(u)$, and for $a = u_1 = w_1$, $t(u_1) = f(u_1) \circ g(w_1)$. We assume that $f(u_1) \geqq g(u_1)$, and so $g(u_1) = t(u_1)$; the proof is similar in the opposite case. Thus when $a \neq u_1$, $t(a) = f(a) \circ g(u_1) + g(a) \circ f(u_1) \geqq h(a)$, as shown above. Hence, $h(a) = h(a) \circ t(a) = h(a) \circ f(a) \circ g(u_1) + h(a) \circ g(a) \circ f(u_1)$ when $a \neq u_1$. The expression for $h(a) \circ t(a)$ as a model we define two new fuzzy grades μ'_A and $\mu'_B : \mu'_B = h(u_1) + \sum_{w > u_1} g(w) \circ h(w)$ and $\mu'_A = [f(u_1) \sqcap h(u_1)] + \sum_{u > u_1} f(u) \circ h(u)$. As $h(u_1) \geqq t(u_1) = g(u_1)$, μ'_B is b-maximal, and obviously also μ'_A is b-maximal. Let $q(a)$ denote the grade of $a \in J$ in $\mu_B \mathbin{\bar{\cup}} \mu'_B$. $q(u_1) = g(u_1) \circ h(u_1) = g(u_1)$, and when $a > u_1$, $q(a) = g(a) \circ h(u_1) + g(a) \circ h(a) \circ g(u_1) \doteq g(a) + g(a) \circ h(a) = g(a)$. Thus $\mu_B \mathbin{\bar{\cup}} \mu'_B = \mu_B$. Similarly we see that $\mu_A \mathbin{\bar{\cup}} \mu'_A = \mu_A$.

Let $r(a)$ denote the grade of $a \in J$ in $\mu'_A \mathbin{\bar{\cup}} \mu'_B$. First, $r(u_1) = h(u_1) \circ (f(u_1) \,\square\, h(u_1)) = h(u_1)$. Further when $a > u_1$, $r(a) = h(a) \circ f(a) \circ h(u_1) + h(a) \circ g(a) \circ (f(u_1) \,\square\, h(u_1)) = h(a) \circ f(a) + h(a) \circ g(a) = h(a) \circ (f(a) + g(a))$. On the other hand $h(a) = h(a) \circ t(a) = h(a) \circ (f(a) \circ g(u_1) + g(a) \circ f(u_1))$, whence $h(a) \leqq f(a) \circ g(u_1) + g(a) \circ f(u_1)$. Moreover, $f(a) + f(a) \circ g(u_1) + g(a) + g(a) \circ f(u_1) = f(a) + g(a)$, and so $h(a) \leqq f(a) + g(a)$ for each $a > u_1$. Hence $r(a) = h(a) \circ (f(a) + g(a)) = h(a)$ for each $a > u_1$, and so $\mu_C = \mu'_A \mathbin{\bar{\cup}} \mu'_B$.

According to [4, Theorem 3] a DQL is of the form $\{\bigcup(T \mid T \in \mathcal{T});\ \bar{\cup},\ \bar{\cap}\}$, where $T \cap T' = \emptyset$ for $T \neq T'$, $T, T' \in \mathcal{T}$. $\mathcal{T}$ is partially ordered with a l.u.b. determined by a certain relation $\leqq^*$. Each of the sets $T \in \mathcal{T}$ is a distributive lattice with respect to $\bar{\cap}$ and $\bar{\cup}$. For each pair T_1 and T_2 (and T_2, T_3), where $T_1 \leqq^* T_2 \leqq^* T_3$ there exists a homomorphism $\varphi_{T_1,T_2} : T_1 \to T_2$ such that

$$\varphi_{T,T}(x) = x\,, \quad \varphi_{T_2,T_3}(\varphi_{T_1,T_2}(x)) = \varphi_{T_1,T_3}(x)\,,$$

and if $a \in T_4$, $b \in T_5$,

$$a \mathbin{\bar{\cup}} b = \varphi_{T_4,T_{45}}(a) \mathbin{\bar{\cup}} \varphi_{T_5,T_{45}}(b) \quad \text{and} \quad a \mathbin{\bar{\cap}} b = \varphi_{T_4,T_{45}}(a) \mathbin{\bar{\cap}} \varphi_{T_5,T_{45}}(b)$$

where l.u.b. $(T_1, T_2) = T_{12}$. We only will see the characteristic property of the lattices T, the direct spectrum of which the distributive subquasilattice of convex fuzzy grades of $\mathrm{QL}(\mu)$ is.

As shown in [4], $\varphi_{T_4,T_{45}} = a \odot b = a \wedge (a \vee b) = a \vee (a \wedge b)$ and $b \odot a = \varphi_{T_5,T_{45}}(b) = b \vee (b \wedge a) = b \wedge (b \vee a)$. Let $\mu_A = \sum_{u \in J} f(u)/u$ and $\mu_B = \sum_{w \in J} g(w)/w$ be two convex fuzzy grades. As J is a finite set, we can conclude that there exists at least one $u_r \in J$ such that $f_{max} = f(u_r) \geqq f(u)$ for each $u \in J$, and similarly a point $w_s : g_{max} = g(w_s) \geqq g(w)$ for each $w \in J$. Let $\mu_A \odot \mu_B = \sum_{z \in J} t(z)/z$ and $\mu_A \mathbin{\bar{\cup}} \mu_B = \sum_{x \in J} h(x)/x$. Now $h(a) = f(a) \circ \sum_{w \leqq a} g(w) + g(a) \circ \sum_{u \leqq a} f(u)$, and further $t(a) = h(a) \circ \sum_{u \geqq a} f(u)/u + f(a) \circ \sum_{x \geqq a} h(x)/x$. By combining these partial results, we obtain

$$\begin{aligned} t(a) &= f(a) \circ \sum_{w \leqq a} g(w) \circ \sum_{u \geqq a} f(u) + f(a) \circ \sum_{x \geqq a} \Big[f(x) \circ \sum_{w \leqq x} g(w) + g(x) \circ \sum_{u \leqq x} f(u)\Big] \\ &= f(a) \circ \sum_{w \leqq a} g(w) + f(a) \circ \sum_{x \geqq a} f(x) \circ g_{\max} + f(a) \circ \sum_{x \geqq a} g(x) \circ f_{\max} \\ &= f(a) \circ \sum_{w \leqq a} g(w) + f(a) \circ g_{\max} + f(a) \circ \sum_{x \geqq a} g(x) \\ &= f(a) \circ g_{\max}. \end{aligned}$$

Similarly we obtain for $\mu_B \odot \mu_A = \sum_{y \in J} d(y)/y : d(a) = g(a) \circ f_{\max}$ for each $a \in J$.

On the other hand, $f_{\max} \leqq g_{\max}$ or $g_{\max} \leqq f_{\max}$. In the first case $\mu_A \odot \mu_B$ does not alter μ_A and in $\mu_B \odot \mu_A$ the operation $\odot$ "cuts" μ_B into the level of μ_A; if $g_{\max} \leqq f_{\max}$, we obtain a dual result. In fact, after the operation $\odot$ we have two convex fuzzy grades with the same maximum grade. Let $T(b)$ be the class of fuzzy grades with the maximum grade b. In this class the fuzzy grade $0/u_1 + 0/u_2 + \ldots + 0/u_{m-1} + b/u_m$ ($J = \{u_1, \ldots, u_m\}$ and $u_1 < u_2 < \ldots < u_m$) is the greatest element such that $b/u_m \,\dot{\cup}\, \mu = b/u_m$ and $b/u_m \,\dot{\cap}\, \mu = \mu$ for each $\mu \in T(b)$. According to [4, Theorem 2], $T(b)$ is a lattice with respect to the operations $\dot{\cup}$ and $\dot{\cap}$. So the distributive quasilattice of convex fuzzy grades is a direct spectrum in the meaning of [4, Theorem 3] of distributive lattices $T(b)$, where each two fuzzy grades have the same maximum grade b. Moreover $T(a) \leqq^* T(b) \Leftrightarrow b \leqq a$, and l.u.b. $(T(a), T(b)) = T(c)$, $c = \min\{a, b\}$.

In the case of normal endmaximal fuzzy grades the following structure is found. If $\mu_A \odot \mu_B = \sum_{z\in J} t(z)/z$, then $t(a) = g(a) + f(a)$ for each $a \in J$, and similarly, if $\mu_B \odot \mu_A = \sum_{x\in J} d(x)/x$, then $d(a) = g(a) + f(a)$ for each $a \in J$. Each $T \in \mathcal{T}$ consists of a single fuzzy grade only and trivially such a T is a lattice. If $T_1 = \{\mu_A\}$ and $T_2 = \{\mu_B\}$, then $T_1 \leqq^* T_2 \Leftrightarrow g(a) \geqq f(a)$ for each $a \in J$, and l.u.b. $(T_1, T_2) = \{\mu_C = \sum_{v\in J} h(v)/v\}$, where $h(a) = g(a) + f(a)$ for each $a \in J$.

4. A PROBLEM ON FUZZY GRADES UNDER $\uplus$ AND $\Cap$

In [2] Mizumoto and Tanaka defined except the operations $\dot{\cup}$ and $\dot{\cap}$ also two other operations on fuzzy grades: $\uplus$ and $\Cap$. Let $\mu_A = \sum_{u\in J} f(u)/u$ and $\mu_B = \sum_{w\in J} g(w)/w$, then

$$\mu_A \uplus \mu_B = \left(\sum_{u\in J} f(u)/u\right) \uplus \left(\sum_{w\in J} g(w)/w\right) = \sum_{u,w\in J} f(u)\, g(w)/u \,\Box\, w\,,$$

and

$$\mu_A \Cap \mu_B = \sum_{u,w\in J} f(u)\, g(w)/u \circ w\,,$$

where $f(u)\, g(w)$ stands for the algebraic product of $f(u)$ and $g(w)$. It is left to the reader in [2] as an unsolved problem to prove or to disprove that $\mu_A \uplus \mu_B$ and $\mu_A \Cap \mu_B$ are convex fuzzy grades if μ_A and μ_B are convex.

Theorem 5. If μ_A and μ_B are convex fuzzy grades in J, then $\mu_A \uplus \mu_B$ and $\mu_A \Cap \mu_B$ are convex fuzzy grades in J, too.

Proof. We shall use induction over the number $|J|$ of points in J. If $|J| = 1$ or 2, then $\mu_A \uplus \mu_B$ and $\mu_A \Cap \mu_B$ are trivially convex fuzzy grades in J; it is mentioned in [2] that this holds also when $|J| = 3$. We assume that $\mu_A \uplus \mu_B$ and $\mu_A \Cap \mu_B$ are convex fuzzy grades when $|J| \leqq m - 1$ and we shall show that this is the case also when $|J| = m$.

Let $J = \{u_1, \ldots, u_m\} = \{w_1, \ldots, w_m\} = \{x_1, \ldots, x_m\}$, $u_i = w_j = x_k$ if $i = j = k$, and let $u_1 < u_2 < u_3 < \ldots < u_{m-1} < u_m$. Further, we denote $\mu_A \uplus \mu_B = \sum_{x\in J} t(x)/x$, and we assume that there exists a point $x_q \in J$ such that

$$t(x_q) < \min\{t(x_r),\ t(x_s)\}\,,\quad r < q < s\,. \tag{7}$$

If $s < m$, then the expression (7) contradicts to the induction assumption, as μ_A and μ_B are convex also in $\{u_1, u_2, \ldots, u_s\}$ and $s \leqq m - 1$. So we can assume that $s = m$. Let $1 < r$, and let us consider the following expression for $t(a)$, $a \in J$:

$$t(a) = f(a) \sum_{w \leqq a} g(w) + g(a) \sum_{u \leqq a} f(u)\,.$$

If $f_{\max} = f(u_0)$ and $g_{\max} = g(w_0)$ are reached for each u_0 and w_0 when $u_0, w_0 \geqq x_r$, then $t(a) = t'(a)$ for all values $a \in J$ satisfying $a \geqq x_r$, where $\sum_{x \in J'} t'(x)/x = \mu'_A \cup \mu'_B$, $\mu'_A = \sum_{u \in J'} f(u)/u$, $\mu'_B = \sum_{w \in J'} g(w)/w$ and $J' = \{x_r, x_{r+1}, \ldots, x_m\}$. As μ_A and μ_B are convex in J, μ'_A and μ'_B are convex in J', and according to the induction assumption, the expression (7) does not hold in J', whence (7) is not valid in J.

If $f_{\max} = f(u_0)$ is reached by $u_0 < x_r$, and $g_{\max} = g(w_0)$ by $w_0 \geqq x_r$ for each w_0, then $t(a) = t''(a)$ for each $a \geqq x_r$, $a \in J$, where $\mu''_A \cup \mu''_B = \sum_{x \in J''} t''(x)/x$, $\mu''_A = f''(x_r) + \sum_{u \in J''} f(u)/u$, $f''(x_r) = f_{\max}$, $\mu''_B = \sum_{w \in J'} g(w)/w$, $J' = \{x_r, \ldots, x_m\}$, and $J'' = \{x_{r+1}, \ldots, x_m\}$. As μ_A and μ_B are convex in J, μ''_A and μ''_B are convex in J' and hence according to the induction assumption $t''(x_q) \geqq \min\{t''(x_r),\ t''(x_s)\}$. As $t''(x_r) \geqq t(x_r)$ and $t''(a) = t(a)$ for $a \in J''$, the expression (7) does not hold.

The proof is similar, if $g_{\max} = g(w_0)$ is changes with $f_{\max}$ above, or if $u_0, w_0 < x_r$. Hence we can assume that $x_r = x_1$.

By using similar considerations as above and the induction assumption, we can conclude that $t(x_q) \geqq \min\{t(x_1), t(x_{m-1})\}$ and $t(x_q) \geqq \min\{t(x_2), t(x_m)\}$. As $t(x_q) < \min\{t(x_1), t(x_m)\}$, $t(x_q) \geqq t(x_2), t(x_{m-1})$. Further, by similar arguments, the interval $\{x_q, \ldots, x_m\}$ implies that $t(x_{m-1}) \geqq \min\{t(x_q), t(x_m)\}$, and the interval $\{x_1, \ldots, x_q\}$ that $t(x_2) \geqq \min\{t(x_1), t(x_m)\}$. By combining the results above, $t(x_2) = t(x_{m-1}) = t(x_q)$. By applying now the induction assumption to the interval $\{x_1, \ldots, x_i\}$, $x_2 \leqq x_i \leqq x_{m-1}$, we obtain that $t(x_2) \geqq \min\{t(x_1), t(x_i)\}$, and as $t(x_2) = t(x_q) < t(x_1)$, the relation $t(x_2) \geqq t(x_i)$ holds for each i, $2 \leqq i \leqq m - 1$. On the other hand, $t(x_i) \leqq t(x_q) < \min\{t(x_1), t(x_m)\}$, and by using the same way as in the case of $t(x_q)$ before, we can prove that $t(x_i) \geqq t(x_2)$, for each i, $2 \leqq i \leqq m - 1$. Hence, $t(x_2) = t(x_3) = \ldots = t(x_{m-1})$, and $t(x_1)$, $t(x_m) > t(x_i)$ when $2 \leqq i \leqq m - 1$.

Now $t(x_1) = f(x_1)\, g(x_1)$ and $t(x_2) = \max\{f(x_2)\, g(x_1),\ f(x_2)\, g(x_2),\ g(x_2) f(x_1),\ g(x_2) f(x_2)\}$. If $f(x_2) \geqq f(x_1)$, then $t(x_2) \geqq f(x_2)\, g(x_1) \geqq f(x_1)\, g(x_1) = t(x_1)$, which is a contradiction to $t(x_2) < t(x_1)$. Hence $f(x_1) > f(x_2)$ and $g(x_1) > g(x_2)$. Since μ_A and μ_B are convex fuzzy grades and $t(x_2) = t(x_3) = \ldots = t(x_{m-1})$, $f(x_1) > f(x_2) \geqq f(x_3) \geqq \ldots \geqq f(x_{m-1})$ and $g(x_1) > g(x_2) \geqq g(x_3) \geqq \ldots \geqq g(x_{m-1})$. Further, $t(x_m) = \max\{f(x_m)\, g_{\max},\ f_{\max}\, g(x_m)\}$, and $f_{\max} = \max\{f(x_1), f(x_m)\}$ and

$g_{\max} = \max\{g(x_1),\ g(x_m)\}$. If $f(x_m) \geqq f(x_1)$ or $g(x_m) \geqq g(x_1)$, we obtain a contradiction to the convexity of μ_A or μ_B. So, let $f(x_m) < f(x_1)$ and $g(x_m) < g(x_1)$. But then $\max\{f(x_m)\, g(x_1),\ f(x_1)\, g(x_m)\} > t(x_{m-1}) = t(x_2)$, whence $f(x_m) > f(x_2)$ or $g(x_m) > g(x_2)$. This contradicts the convexity of μ_A or μ_B again. Hence, the relation $t(x_q) < \min\{t(x_r),\ t(x_s)\}$ does not hold for any indices $r,\ q,\ s,\ r \leqq q \leqq s$, from which the convexity of $\mu_A \cup \mu_B$ follows.

The proof is analogous for $\mu_A \cap \mu_B$.

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Dr. Juhani Nieminen, Research assistant. Finnish Academy, Department of technical sciences, Lauttasaarentie 1, 00200 Helsinki 20. Finland.