KYBERNETIKA — VOLUME 13 (1977), NUMBER 4

Functorial Algebras and Automata

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This paper lays down algebraic foundations to the theory of machines in a category, initiated by Arbib and Manes. Functorial algebras (which are machines without outputs) are investigated: natural algorithms for free algebras and for colimits are exhibited; the existence of colimits and their preservation by the underlying objects is investigated. A corollary: state behaviour processes often coincide with the adjoint processes.

INTRODUCTION

Arbib and Manes investigate machines over a given functor $F : \mathscr{K} \to \mathscr{K}$. Their idea is to give a general unified theory for a number of concepts of a machine, including sequential, tree, linear, bilinear machines etc. The aim of the present paper is to lay down algebraic foundations of this theory – similarly as the theory of semigroups underlies the theory of sequential machines.

We define F-algebras, which are in fact F-machines without outputs (i.e., the Medvedev machines or semiautomata). The fundamental concept of Arbib and Manes is the free F-algebra (which is Σ^* in case of sequential Σ-machines); we investigate the question of the existence of free algebras and its relation to the existence of colimits of F-algebras. We exhibit a construction of colimits, naturally generalizing the usual algebraic construction, and closely connected to the free-algebra construction, studied previously. We show, roughly speaking, that if free algebras are "constructive" then so are colimits.

We study the question of colimits, agreeing with the underlying objects. As a corollary, state-behaviour processes of Arbib and Manes are shown to coincide quite often with the adjoint processes.

Some constructions, mentioned in this paper, have been used in [2, 3].

I. ALGEBRAS AND FREE ALGEBRAS

I,1 Given a category $\mathscr{K}$ and a functor $F : \mathscr{K} \to \mathscr{K}$, Arbib and Manes [5] define an F-machine in the category $\mathscr{K}$. This is just an F-algebra, as defined below, together with an additional information (the output and the initialization). It is not necessary to assume, as Arbib and Manes do, that F has any special properties (that it is an input process); machines over an arbitrary functor are studied e.g. in [4,15]. We shall thus exhibit an algebraic theory for an arbitrary functor F.

By an F-algebra is meant a pair (A, r) where A is an object of $\mathscr{K}$ and $r : FA \to A$ is a morphism. *F-algebras* form a new category, denoted by $\mathscr{K}(F)$, where morphisms from (A, r) to (B, s) are those $\mathscr{K}$-morphisms $f : A \to B$ for which the diagram in Fig. 1 commutes. These morphisms are called F-homomorphisms.

$$\begin{tikzcd} FA \arrow[r, "r"] \arrow[d, "Ff"'] & A \arrow[d, "f"] \\ FB \arrow[r, "s"'] & B \end{tikzcd}$$

Fig. 1.

The most interesting examples for an automata theorist will be SET $(\times\ \Sigma)$, VECT$(\oplus\ \Sigma)$ and VECT$(\otimes\ \Sigma)$. Here SET denotes the category of sets and mappings and the considered functor F is defined by $X \to X \times \Sigma$ and $f \to f \times \mathrm{id}_\Sigma$. Thus, F-machines are pairs (A, r) where $r : A \times \Sigma \to A$, i.e. they are sequential Σ-machines without outputs. Homomorphisms correspond to the usual simulations. Analogously, VECT denotes the category of vector spaces (over a given field) and $\oplus$, $\otimes$ stands for the direct and tensor product, respectively. Thus, F-algebras are linear, resp. bilinear, sequential Σ-machines without outputs.

By $\mathscr{U} : \mathscr{K}(F) \to \mathscr{K}$ will be denoted the forgetful functor (sending (A, r) to A).

I,2 We consider a factorization system $(\mathscr{E}, \mathscr{M})$ in $\mathscr{K}$ (see [8]). I.e., $\mathscr{E}$ is a class of epis, $\mathscr{M}$ is a class of monics and

(i) $\mathscr{E} \cap \mathscr{M}$ is the class of all isomorphisms;

(ii) every morphism f in $\mathscr{K}$ can be written as

$$f = (\mathrm{im}\, f) \circ (\mathrm{coim}\, f)\,, \quad \mathrm{im}\, f \in \mathscr{M}\,, \quad \mathrm{coim}\, f \in \mathscr{E}\,;$$

(iii) (the diagonal fill-in) for every commuting square

$$\begin{tikzcd} \cdot \arrow[r, "e \in \mathscr{E}"] \arrow[d] & \cdot \arrow[d] \arrow[dl, dashed, "d"'] \\ \cdot \arrow[r, "m \in \mathscr{M}"'] & \cdot \end{tikzcd}$$

there exists a diagonal d, rendering both triangles commuting.

Factorization systems are well-known to have a number of natural properties. E.g., $\mathscr{E}$ is closed to composition (it is a subcategory) and is right cancellative (if $e_1 \circ e_2 \in \mathscr{E}$ then $e_1 \in \mathscr{E}$); analogously, $\mathscr{M}$ is a left cancellative subcategory.

If F preserves $\mathscr{E}$ (i.e. $F(\mathscr{E}) \subset \mathscr{E}$, in other words, $Fe \in \mathscr{E}$ for each $e \in \mathscr{E}$) then this factorization system "transfers" to F-algebras (cf. Fig. 2): given a homomorphism

$$\begin{tikzcd}
FA \arrow[r, "r"] \arrow[d, "F(\mathrm{coim}\, f)"'] & A \arrow[d, "\mathrm{coim}\, f"] \\
FC \arrow[r, dashed, "t"] \arrow[d, "F(\mathrm{im}\, f)"'] & C \arrow[d, "\mathrm{im}\, f"] \\
FB \arrow[r, "s"'] & B
\end{tikzcd}$$

Fig. 2.

$f : (A, r) \to (B, s)$ we have $F(\mathrm{coim}\, f) \in \mathscr{E}$, $\mathrm{im}\, f \in \mathscr{M}$, thus, by the diagonal fill-in, there is $t : FC \to C$ making both $\mathrm{coim}\, f$ and $\mathrm{im}\, f$ a homomorphism. All other axioms are clear.

As usual, we say that $\mathscr{K}$ is *locally small* if every object X has, up to isomorphism, only a set of subobjects. More in detail, consider the class of all subobjects, i.e. monics $m : A \to X$, ordered by *inclusion* ($m \subset m'$ iff there exists $f : A \to A'$ with $m = m' . f$); this class has no proper subclass of pairwise incompatible elements. Dually: $\mathscr{K}$ is *co-locally small* if every object has, up to isomorphism, only a set of quotient-objects.

$\mathscr{K}$ is said to be complete if it has all limits, and cocomplete if it has all colimits.

Definition. By a *subalgebra* of an F-algebra (A, r) is meant an F-algebra (A_1, r_1) together with an $\mathscr{M}$-monic $m : (A_1, r_1) \to (A, r)$. By a *homomorphic image* of (A, r) is meant an F-algebra (A_1, r_1) together with an $\mathscr{E}$-epi $e : (A, r) \to (A_1, r_1)$.

Thus, a homomorphism theorem holds for F-algebras as soon as $F(\mathscr{E}) \subset \mathscr{E}$: every homomorphism decomposes into its homomorphic image, followed by a subalgebra.

I,3 If $\mathscr{K}$ is a complete category then so is $\mathscr{K}(F)$ and the limits are created (in the sense of [11]) by the forgetful functor $\mathscr{U} : \mathscr{K}(F) \to \mathscr{K}$. For example, the product of algebras (A_i, r_i), $i \in I$ is simply constructed as follows: let $A = \Pi A_i$ be the product in $\mathscr{K}$ with projections $\pi_i : A \to A_i$. Then there exists a unique operation on A, for which each π_i is a homomorphism, viz., the morphism $r : FA \to A$, defined by

$$\pi_i \circ r = r_i \circ F\pi_i \quad (i \in I).$$

Then (A, r) is the product in $\mathscr{K}(F)$ with projections π_i, again.

Definition. An F-algebra (A, r) is said to be *generated by a subobject* $i : X \to A$, $i \in \mathscr{M}$, if no proper subalgebra contains i. More precisely, i generates (A, r) if every subalgebra $m : (A_1, r_1) \to (A, r)$ with $i \subset m$ is an isomorphism.

If $\mathscr{K}$ has big intersections of $\mathscr{M}$-subobjects, so does $\mathscr{K}(F)$. Then, given an F-algebra (A, r), a subobject $i : X \to A$ generates a unique subalgebra of (A, r): the intersection of all subalgebras, containing it.

I,4 A crucial notion for the development of a theory of functorial machines is the following (see [7, 5]):

Definition. A *free F-algebra*, generated by an object X is an algebra $(X^\#, \varphi^X)$ together with a morphism $s^X : X \to X^\#$ which has the following universal property

$$\begin{tikzcd} FX^\# \arrow[r, "\varphi^X"] \arrow[d, "Ff_r^\#"'] & X^\# \arrow[d, "f_r^\#"] & X \arrow[l, "s^X"'] \arrow[dl, "f"] \\ FA \arrow[r, "r"'] & A & \end{tikzcd}$$

Fig. 3.

(cf. Fig. 3): every morphism $f : X \to A$ to an algebra (A, r) can be uniquely extended to a homomorphism $f_r^\#$. I.e., there exists a unique homomorphism $f_r^\# : (X^\#, \varphi^X) \to$ $\to (A, r)$ with $f_r^\# \circ s^X = f$.

Arbib and Manes call F an *input process* provided that every object X generates a free F-algebra.

For a morphism $f : X \to Y$ we denote by $f^\# : X^\# \to Y^\#$ the homomorphism $(s^Y \circ f)^\#_{\varphi^Y}$. Notice that given $g : Z \to X$, then $(f \circ g)^\# = f^\# \circ g^\#$ (proof: since $f^\# \circ g^\#$ is a homomorphism, it suffices to verify that $(f^\# \circ g^\#) \circ s^Z = s^Y \circ f \circ g$ because then $f^\# \circ g^\#$ is an extension of $s^Y \circ f \circ g$ which is unique; now, $f^\# \circ g^\# \circ s^Z = f^\# \circ s^X \circ g =$ $= s^Y \circ f \circ g$); also $1_X^\# = 1_{X^\#}$. Furthermore, given $h_r^\# : (X^\#, \varphi^X) \to (A, r)$ and given $g : Z \to X$, then $h_r^\# \circ g^\# = (h \circ g)_r^\#$ (proof: $h_r^\# \circ g^\#$ is a homomorphic extension of $h \circ g$). Also, for an arbitrary homomorphism $k : (A, r) \to (A', r')$, $k \circ h_r^\# =$ $= (k \circ h)_{r'}^\#$ (proof: $k \circ h_r^\#$ is a homomorphic extension of $k \circ h$).

I,5 Among functors, preserving monics, input processes were characterized in [16] (under additional assumption on $\mathscr{M}$, see I, 6) as those which are non-excessive on all objects X. A functor F is non-excessive on X if there exists an object Y isomorphic to the sum $FY \vee X$. Moreover, these functors were proved to be constructive in the following sense, defined in [1], as a generalization of [10].

By a *chain* T in a category $\mathscr{K}$ we mean a functor from the (ordered) category of all ordinals (or some interval) to $\mathscr{K}$; then T_i denotes the i-th object and $T_{i,j}$, $i \leqq j$, the morphism from T_i to T_j.

Free-algebra construction. Given an object X, define a chain W in K by transfinite induction:

(a) $W_0 = X$; $W_1 = X \vee FX$; $W_{0,1} : X \to X \vee FX$ is canonical,

(b) $W_{i+1} = X \vee FW_i$ and $W_{i+1,i+2} = 1_X \vee FW_{i,i+1} : X \vee FW_i \to X \vee FW_{i+1}$,

(c) if γ is a limit ordinal then

- (c_1) W_γ and $W_{i,\gamma} : W_i \to W_\gamma$ $(i < \gamma)$ is the colimit of the chain $(W_i)_{i<\gamma}$,
- (c_2) $W_{\gamma,\gamma+1} : W_\gamma \to X \vee FW_\gamma$ is determined as follows: $W_{\gamma,\gamma+1} \circ W_{0,\gamma} : X \to X \vee FW_\gamma$ is canonical, $W_{\gamma,\gamma+1} \circ W_{i+1,\gamma} = 1_X \vee FW_{i,\gamma}$ for $i < \gamma$.

The construction is said to *stop after* α *steps* if $W_{\alpha,\alpha+1}$ is an isomorphism (then, clearly, all $W_{\alpha,\alpha+i}$ are isomorphisms as well).

F is a *constructive input process* if the free algebra construction stops for every object X (see [1]). This construction naturally generalizes the iterative definition used in universal algebra, see e.g. [14].

I,6 In this section we assume that

1) $\mathscr{M}$ as a subcategory of $\mathscr{K}$ (with the same objects) is closed in $\mathscr{K}$ to finite sums and colimits of chains and

2) $\mathscr{K}$ is $\mathscr{M}$-locally small.

Theorem [16]. For every functor $F : \mathscr{K} \to \mathscr{K}$ with $F(\mathscr{M}) \subset \mathscr{M}$ the following conditions on an object X (in $\mathscr{K}$) are equivalent:

(i) a free F-algebra over X exists;

(ii) the free algebra construction stops for X;

(iii) F is non-excessive on X.

It is an interesting question, whether subfunctors of input processes are again input processes. In general, this is not true, but it is so for monics-preserving functors.

Theorem. Let F, F' be functors with $F(\mathscr{M}) \subset \mathscr{M}$ and $F'(\mathscr{M}) \subset \mathscr{M}$ and let a transformation $\mu : F' \to F$ exist with $\mu^X \in \mathscr{M}$ for all X. Then the existence of a free F-algebra over an object implies the existence of a free F'-algebra.

Proof. Denote by W_i (W_i') the i-th step in the free-algebra construction over X for F (F'). If a free F-algebra over X exists then there exists α with all W_γ, $\gamma < \alpha$, isomorphic. To prove the theorem it suffices to find a transformation h of the chains with $h_i : W_i' \to W_i$ in $\mathscr{M}$ for all i. Then the free-algebra construction for F' stops, too, because the morphisms $W_{\alpha,\gamma}^{-1} \circ h_\gamma$ present for $\gamma \geqq \alpha$ a proper class of $\mathscr{M}$-subobjects of W_α.

Put $h_0 = 1_X : X \to X$, $h_{i+1} = 1_X \vee \mu^{W_i} \circ F'h_i$; for γ limit, h_γ is defined by $h_\gamma \circ W'_{i,\gamma} = W_{i,\gamma} \circ h_i$, $i < \gamma$. Since sums and chain colimits of $\mathcal{M}$-monics are $\mathcal{M}$-monics, clearly $h_i \in \mathcal{M}$ for all i. That concludes the proof.

Analogously as above, it can be proved that subobjects of objects with a free F-algebra also have a free F-algebra.

Proposition. Let $F : \mathcal{K} \to \mathcal{K}$ fulfil $F(\mathcal{M}) \subset \mathcal{M}$. If a free F-algebra over X exists and if $m : Y \to X$ is in $\mathcal{M}$, then a free F-algebra over Y exists, too.

I,7 Analogous questions, concerning epi-transformations and quotient objects (in place of mono-transformations and subobjects) are investigated by Reiterman in [13]. He proves, e.g., if $F(\mathcal{E}) \subset \mathcal{E}$ and a free algebra over X exists then for every $e : X \to Y$ in $\mathcal{E}$ also a free algebra over Y exists.

I,8 We conclude this section by a criterion on the existence of free algebras, involving Freyd's adjoint functor theorem [11]. The phrase "X generates, up to isomorphism, only a set of algebras" means that for the object X there is a set $m_i : X \to (A_i, r_i)$, $i \in I$, of algebras (A_i, r_i), generated by X, such that for every algebra (A, r), generated by X; $m : X \to (A, r)$, there is an isomorphism $t_i : (A_i, r_i) \to (A, r)$ (in $\mathcal{K}(F)$) with $m = t_i \circ m_i$.

Theorem. Let $\mathcal{K}$ be a complete, locally and co-locally small category. A functor F with $F(\mathcal{E}) \subset \mathcal{E}$ is an input process iff every object generates, up to isomorphism, only a set of algebras.

Proof. 1) Sufficiency is an application of the adjoint functor theorem: for a given object X consider all algebras generated by all quotients of X. Clearly, a representative solution set with respect to the forgetful functor $\mathcal{U}$ can be chosen from these algebras.

2) Necessity. Let a free algebra over X exist, say $(X^\#, \varphi^X)$ and $s^X : X \to X^\#$. Given an algebra (A, r) generated by $m : X \to A$, it is easy to prove that $m_r^\# \in \mathcal{E}$. Since $\mathcal{K}$ is co-locally small it now suffices to verify that, given another algebra (A', r'), generated by $m' : X \to A'$, and given an isomorphism $i : A \to A'$ (in $\mathcal{K}$) with $i \circ m_r^\# = m_{r'}'^\#$, then necessarily i is a homomorphism. But this is easy, since $Fm_r^\#$ is an epi ($F(\mathcal{E}) \subset \mathcal{E}$!) and $i \circ r \circ Fm_r^\# = r' \circ F(i \circ m_r^\#)$. Therefore $i \circ r = r' \circ Fi$ – i is a homomorphism.

II. COLIMITS

II,1 Throughout this section we assume that a category $\mathcal{K}$ with a factorization system $(\mathcal{E}, \mathcal{M})$ and a functor $F : \mathcal{K} \to \mathcal{K}$ is given, and that $\mathcal{K}$ has *big co-unions*. The last means that (cf. Fig. 4), given $\mathcal{E}$-epis $e_i : X \to X_i$, $i \in I$ (I may be a proper class) there exists $e : X \to Y$ in $\mathscr{E}$ such that

$$\begin{tikzcd}
T \arrow[rr, "f"] \arrow[dd, "k"'] & & X \arrow[dl, "e_i"'] \arrow[dd, "e"] \\
& X_i & \\
S \arrow[ur, "k_i"] \arrow[rr, "\bar{k}"'] & & Y \arrow[ul, "e'_i"']
\end{tikzcd}$$

Fig. 4.

1) there exist $e'_i : Y \to X_i$ with $e'_i \circ e = e_i$

2) given $f : T \to X$ and an $\mathscr{E}$-epi $k : T \to S$ such that $k_i \circ k = e_i \circ f$ for $k_i : S \to X_i$ then there is $\bar{k} : S \to Y$ with $\bar{k} \circ k = e \circ f$.

This is the dual notion to unions, as defined by Mitchell [12]. If $\mathscr{K}$ is cocomplete and $\mathscr{E}$-co-locally small, then it has co-unions. We denote $e = \bigcup^* e_i$; if $I = \emptyset$ then $\bigcup^* e_i$ denotes the canonical morphism from X to a terminal object (whose existence we thus assume).

II,2 Theorem. Let $F(\mathscr{E}) \subset \mathscr{E}$. A diagram of *F-algebras*, $D : \mathscr{D} \to \mathscr{K}(F)$, has a colimit as soon as colim $\mathscr{U} \circ D$ exists in $\mathscr{K}$ and generates a free F-algebra.

Proof. Let $D : \mathscr{D} \to \mathscr{K}(F)$ be a diagram with $Dd = (A_d, r_d)$ for each object d of $\mathscr{D}$. By hypothesis there exists a colimit A; $v_d : A_d \to A$ of $\mathscr{U} \circ D$ in $\mathscr{K}$ as well as a free F-algebra $(A^\#, \varphi^A)$, $s^A : A \to A^\#$.

Denote by $\mathscr{W}$ the collection of all bounds $w = ((Q, q), h)$ of the diagram $D : (Q, q)$ is an F-algebra and $h = \{h_d : (A_d, r_d) \to (Q, q)\}$ is a compatible family of homomorphisms. For each $w \in \mathscr{W}$ there exists a unique $h^* : A \to Q$ with $h^* \circ v_d = h_d$ (for each object d of $\mathscr{D}$). Denote by $k : A^\# \to Q_0$ the co-union of coim $(h^*)^\#_q$ where w runs through $\mathscr{W} : k = \bigcup^*_{((Q,q),h) \in \mathscr{W}} (h^*)^\#_q$.

Using the fact that $k \in \mathscr{E}$ implies $Fk \in \mathscr{E}$, it is easy to compute that there is $q_0 : FQ_0 \to Q_0$ turning all $k \circ v_d$ into homomorphisms $k \circ v_d : (A_d, r_d) \to (Q_0, q_0)$. Then (Q_0, q_0) together with $k \circ v_d$, $d \in \mathscr{D}$ presents the colimit of D.

Corollary. Let $\mathscr{K}$ be cocomplete and let F be an input process with $F(\mathscr{E}) \subset \mathscr{E}$. Then also $\mathscr{K}(F)$ is cocomplete.

Remark. Since a functor is an input process iff it generates a free triple in the sense of Barr, the above corollary follows also from the results in [7].

II,3 Example. Let $\mathscr{K}$ have sums and $F(\mathscr{E}) \subset \mathscr{E}$. Then there exists a sum of F-algebras (A_i, r_i), $i \in I$, as soon as a free F-algebra over $\bigvee_{i \in I} A_i$ exists.

This example can be reversed, in a way. Recall that a set of objects I is a generator if every object is an $\mathscr{E}$-quotient of a sum of objects from I. The proposition below follows from the fact that a sum of free F-algebras is free and from the mentioned result of J. Reiterman (see I, 7).

Proposition. Let both $\mathscr{K}$ and $\mathscr{K}(F)$ have sums and let $F(\mathscr{E}) \subset \mathscr{E}$. If free algebras over each object of a generator of $\mathscr{K}$ exist then all free F-algebras exist.

II,4 Just as the free-algebra construction I,5 presents a natural generalization of the construction of free universal algebras, we exhibit here a transfinite construction of colimits of algebras, generalizing the usual constructions of free sums, amalgamated sums etc.

Throughout the rest of this section we assume that $\mathscr{K}$ is a cocomplete category. Nothing is assumed about F.

Let $D : \mathscr{D} \to \mathscr{K}(F)$ be an arbitrary diagram of F-algebras, say $Dd = (A_d, r_d)$ for $d \in \mathscr{D}$. Denote $A = \operatorname{colim} \mathscr{U} \circ D$, $X = \operatorname{colim} F \circ \mathscr{U} \circ D$ (in $\mathscr{K}$), with injections $v_d : A_d \to A$ and $q_d : FA_d \to X$. Then there are two natural morphisms:

$$q : X \to FA \quad \text{with} \quad q \circ q_d = Fv_d$$

$$p : X \to A \quad \text{with} \quad p \circ q_d = v_d \circ r_d .$$

II,5 Construction. Define chains U, V and a transformation $p : U \to V$, by transfinite induction, such that $FV_i = U_{i+1}$ and $FV_{i,i+1} = U_{i+1,i+2}$.

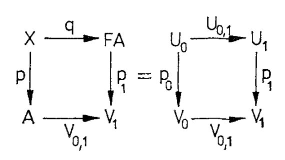

Fig. 5.

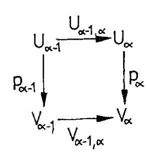

Fig. 6.

A) The first step is described by the push-out in Fig. 5.

B) Let α be an ordinal and let $V_{i,j}$, p_j be defined for $i \leqq j < \alpha$ and $U_{i,j}$ be defined for $i \leqq j - 1 < \alpha$.

(i) if α is isolated, define p_α, V_α and $V_{\alpha-1,\alpha}$ by the push-out in Fig. 6. Furthermore $U_{\alpha+1} = FV_\alpha$ and $U_{\alpha,\alpha+1} = FV_{\alpha-1,\alpha}$.

(ii) if α is limit, let $U_\alpha = \operatorname{colim} \{U_{i,j}\}_{j<\alpha}$ and $V_\alpha = \operatorname{colim} \{V_{i,j}\}_{j<\alpha}$ and let $p_\alpha : U_\alpha \to V_\alpha$ be the canonical map. Furthermore, $U_{\alpha+1} = FV_\alpha$ and $U_{\alpha,\alpha+1}$ is the canonical map: for all $i < \alpha$, $U_{\alpha,\alpha+1} \circ U_{i,\alpha} = FV_{i,\alpha} \circ U_{i+1,i} : U_i \to FV_i \to FV_\alpha = U_{\alpha+1}$.

We say that the colimit construction over the diagram D *stops* (*after* α *steps*) if $V_{\alpha,\alpha+1}$ is an isomorphism. (Then all $V_{\alpha,\alpha+i}$ are isomorphisms as well.)

II,6 Theorem. Every diagram, over which the colimit construction stops, has a colimit in $\mathscr{K}(F)$. If it stops after α steps, put $r = V^{-1}_{\alpha,\alpha+1} \circ p_{\alpha+1} : FV_\alpha \to V_\alpha$. Then (V_α, r) is the colimit of D with injections $V_{0,\alpha} \circ v_d : A_d \to V_\alpha$.

Proof. a) We must show that $f_d = V_{0,\alpha} \circ v_d$ are homomorphisms, i.e. that $f_d \circ r_d = r \circ Ff_d$, in other words, that $V_{0,\alpha+1} \circ v_d \circ r_d = p_{\alpha+1} \circ F(V_{0,\alpha} \circ v_d)$. Since $v_d \circ r_d = p_0 \circ q_d$, the left side equals to $p_{\alpha+1} \circ U_{0,\alpha+1} \circ q_d$. It can be easily proved (by induction in α) that $U_{1,\alpha+1} = FV_{0,\alpha}$. Then the right side equals to $p_{\alpha+1} \circ U_{1,\alpha+1} \circ (q \circ q_d) = p_{\alpha+1} \circ U_{0,\alpha+1} \circ q_d$.

b) Given a direct bound $g_d : (A_d, r_d) \to (B, s)$ of D, we shall find $h : (V_\alpha, r) \to (B, s)$ with $h \circ f_d = g_d$ and observe its unicity. Define $h_i : V_i \to B$ by induction, so that

$$h_i \circ p_i = s \,.\, Fh_i \circ V_{i,i+1} \,. \tag{*}$$

Then $h = h_\alpha$ fulfils all that is required and, moreover, every such h is obtained in the same way from the (unique!) $h_0 : A \to B$ for which $h_0 \circ v_d = g_d$ holds for all $d \in \mathscr{D}$. This h_0 satisfies (*), as can be easily checked.

Given h_i for all $i < \beta$, we define h_β. It is clear how, provided that β is a limit ordinal, and also (*) follows easily. Let β be isolated. Then we have a push-out $p_\beta \circ U_{\beta-1,\beta} = V_{\beta-1,\beta} \circ p_{\beta-1}$ (see Fig. 7) and (*) for $i = \beta - 1$ yields the existence of $h_\beta : V_\beta \to B$ with $h_\beta \circ V_{\beta-1,\beta} = h_{\beta-1}$ and $h_\beta \circ p_\beta = s \circ Fh_{\beta-1}$. The latter implies (*) for $i = \beta$.

$$\begin{tikzcd}
U_{\beta-1} \arrow[r, "U_{\beta-1,\beta}"] \arrow[d, "p_{\beta-1}"'] & U_\beta = FV_{\beta-1} \arrow[d, "p_\beta"'] \arrow[dd, dashed, bend left, "s \cdot Fh_{\beta-1}"] \\
V_{\beta-1} \arrow[r, "V_{\beta-1,\beta}"] \arrow[dr, dashed, "h_{\beta-1}"'] & V_\beta \arrow[d, dashed, "h_\beta"] \\
 & B
\end{tikzcd}$$

Fig. 7.

Proposition. If F preserves colimits of chains of length α (α is a limit ordinal) then the colimit construction stops for every diagram D (after α steps).

The proof is easy. For functors, preserving epis, a more general statement will be proved below.

II,7 Theorem. Let $\mathscr{K}$ be co-locally small and let F preserve epis. Given a diagram D, for which the free-algebra construction stops over colim $\mathscr{U} \circ D$, then the colimit construction stops over D.

Corollary. If F is a constructive input process then $\mathcal{K}(F)$ is cocomplete, with "constructive" colimits.

Proof. Denote, as above, by W the free-algebra chain over $A = \operatorname{colim} \mathcal{U} \circ D$ and by U, V the chains in the colimit construction over D. It clearly suffices to find an epitransformation $e : W \to V$; then, if the free-algebra construction stops after α steps, all V_γ, $\gamma \geqq \alpha$, are quotients of W_α, thus also the colimit construction stops (via the co-local smallness).

Define $e_i : W_i \to V_i$ by induction.

a) $e_0 = 1_A$ (it is an epi)

b) if α is limit and for $i < \alpha$ there are epis $e_i : W_i \to V_i$, then the map e_α, defined by $e_\alpha \circ W_{i,\alpha} = V_{i,\alpha} \circ e_i$ is an epi, too.

c) if α is isolated (then $W_\alpha = A \vee FW_{\alpha-1}$ and $U_\alpha = FV_{\alpha-1}$), put $e_\alpha = V_{0,\alpha}$ "on A", $e_\alpha = p_\alpha \circ Fe_{\alpha-1}$ "on FW_α". Assuming $e_{\alpha-1}$ is an epi, so is $Fe_{\alpha-1}$. We prove that e_α is epi. Choose a, b with $a \circ e_\alpha = b \circ e_\alpha$, then $a \circ p_\alpha \circ Fe_{\alpha-1} = b \circ p_\alpha \circ Fe_{\alpha-1}$ and therefore $a \circ p_\alpha = b \circ p_\alpha$. Further $a \circ e_\alpha \circ W_{\alpha-1,\alpha} = b \circ e_\alpha \circ W_{\alpha-1,\alpha}$, and because $e_\alpha \circ$ $\circ W_{\alpha-1,\alpha} = V_{\alpha-1,\alpha} \circ e_{\alpha-1}$ we have $a \circ V_{\alpha-1,\alpha} \circ e_{\alpha-1} = b \circ V_{\alpha-1,\alpha} \circ e_{\alpha-1}$ and thus $a \circ V_{\alpha-1,\alpha} = b \circ V_{\alpha-1,\alpha}$. Therefore $a = b$; thus, e_α is epi.

II,8 Given a factorization system $(\mathcal{E}, \mathcal{M})$, then $\mathcal{E}$ is closed under the formation of colimits; e. g., every coequalizer belongs to $\mathcal{E}$ and opposit an $\mathcal{E}$-morphism in a push-out there is always an $\mathcal{E}$-morphism. (See [8].) We shall need another result of this sort.

Lemma. Let T be a γ-chain of $\mathcal{E}$-epis, $T_{i,j} : T_i \to T_j \in \mathcal{E}(i \leqq j < \gamma)$. Let T_γ and $t_i : T_i \to T_\gamma$, $i < \gamma$, be the colimit of T. Then

1) also $t_i \in \mathcal{E}$ for all $i < \gamma$,

2) for every bound of T lying wholly in $\mathcal{E}$ also the canonical map from T_γ lies in $\mathcal{E}$.

Proof. 1) Choose a fixed $a < \gamma$ and let $t_a = T_a \xrightarrow{e} T'_a \xrightarrow{m} T$ be the factorization of t_a — we shall prove that m is an isomorphism. Due to the diagonal fill-in, for every i, $a < \gamma$, there exists h_i with $h_i \circ T_{a,i} = e$ and $m \circ h_i = t_i$; furthermore, for $i \leqq a$ put $h_i = e \circ T_{i,a}$. Then, clearly, h_i is a bound of T and there exists a unique $k : T \to T'_a$ with $k \circ t_i = h_i$, $i < \gamma$. Then $t_i = m \circ k \circ t_i$ holds for all $i < \gamma$, thus $m \circ k = 1$ and since $m (\in \mathcal{M})$ is a monic, $k = m^{-1}$.

2) is easy.

Theorem. Let $\mathcal{K}$ be co-locally small and let $F(\mathcal{E}) \subset \mathcal{E}$. Then the colimit construction stops as soon as there exists an ordinal α with $V_{\alpha,\alpha+1} \in \mathcal{E}$.

Proof. We shall prove that all $V_{i,j}$ and $U_{i,j}$ with $i \geqq \alpha + 1$ are in $\mathcal{E}$ — then, by the co-local smallness, some $V_{i,j}$ must be an isomorphism. By hypothesis, $U_{\alpha+1,\alpha+2} =$ $= FV_{\alpha,\alpha+1}$ is in $\mathcal{E}$; so is $V_{\alpha+1,\alpha+2}$, which is opposite to $U_{\alpha+1,\alpha+2}$ in a push-out.

Assume that all $U_{i,j}$ and $V_{i,j}$ are in $\mathscr{E}$ for $i \geqq \alpha + 1$ and $j < \gamma$. If γ is a limit ordinal then $U_{i,\gamma}$ and $V_{i,\gamma}$ lie in $\mathscr{E}$, too, by the above Lemma (applied to the chains $V_{\alpha+1+i}$ and $U_{\alpha+1+i}$). Moreover, by the same Lemma, the canonical map $U_{\gamma,\gamma+1}$ belongs to $\mathscr{E}$, too. If γ is isolated and if $U_{\gamma-1,\gamma}$ belongs to $\mathscr{E}$, so does $V_{\gamma-1,\gamma}$ (opposite in a push-out) as well as $U_{\gamma,\gamma+1} = FV_{\gamma-1,\gamma}$. That concludes the proof.

II,9 For this section we assume that a category $\mathscr{K}$ is given with a factorization system $(\mathscr{E}, \mathscr{M})$ and that $\mathscr{K}$ is complete and $\mathscr{E}$-co-locally small and $F(\mathscr{E}) \subset \mathscr{E}$.

Definition. A functor $F : \mathscr{K} \to \mathscr{K}$ is said to *tighten colimits* over a scheme $\mathscr{D}$ if for every diagram $D : \mathscr{D} \to \mathscr{K}$ the natural morphism colim $F \circ D \to F(\text{colim } D)$ is an $\mathscr{E}$-epi.

The above theorem has an immediate corollary.

Corollary. If F tightens colimits over a scheme $\mathscr{D}$ then the colimit construction stops for every diagram over $\mathscr{D}$. If F tightens colimits of α-chains (for some α) then $\mathscr{K}(F)$ is "constructively cocomplete", i.e. the colimit construction stops for every diagram.

In the above corollary if F tightens colimits of α-chains then the construction needs not stop after α steps of course, but it stops sometimes. An analogous theorem holds for the free-algebra construction. We omit the proof, which is an easy modification of the one above.

Theorem. If F tightens colimits of α-chains (for some α) then F is a constructive input process.

Which functors tighten colimits? Generally, this question might turn out difficult to answer. But for functor, preserving $\mathscr{E}$-epis, the situation is clear. We say that a functor F (not necessarily preserving monics) preserves unions provided that for arbitrary $\mathscr{M}$-monics $m_i : X_i \to A$, $i \in I$, we have

$$\text{im } F\big(\bigcup_{i\in I} m_i\big) = \bigcup_{i\in I} \text{im } Fm_i \, .$$

Theorem. Let F be a functor with $F(\mathscr{E}) \subset \mathscr{E}$. Then

A) F tightens coequalizers;

B) F tightens all colimits iff F preserves unions;

C) F tightens colimits of chains iff F preserves well-ordered unions.

Proof. A) Let $k : B \to C$ be the coequalizer of $f, g : A \to B$ and let $q : FB \to Q$ be the coequalizer of Ff, Fg. We are to prove that for $r : Q \to FC$ with $r \circ q = Fk$ we have $r \in \mathscr{E}$. This follows from the well-known properties of factorization systems: $\mathscr{E}$ contains all coequalizers, thus $k \in \mathscr{E}$, and since $F(\mathscr{E}) \subset \mathscr{E}$ we have $r \circ q \in \mathscr{E}$; and $\mathscr{E}$ is right cancellative, thus $r \in \mathscr{E}$.

B) Let F preserve unions. Let $D : \mathscr{D} \to \mathscr{K}$ is a diagram with colimit $k_d : Dd \to C$, let $F \circ D$ have a colimit $q_d : F \circ Dd \to Q$ and let $r : Q \to FC$ be defined by

$$r \circ q_d = Fk_d .$$

We have $1_C = \bigcup \operatorname{im} k_d$ and so $1_{FC} = \bigcup \operatorname{im} Fk_d \supset \operatorname{im} r$. Therefore, $\operatorname{im} r = 1_{FC}$, i.e. $r \in \mathscr{E}$.

Conversely, let F tighten colimits. Given $\mathscr{M}$-monics $m_i : X_i \to A$, we can assume that $\bigcup m_i = 1_A$. That is the same as $f \in \mathscr{E}$ where

$$f = \vee m_i : \vee X_i \to A .$$

We are to prove that also $\bigcup \operatorname{im} Fm_i = 1_{FA}$, i.e. that $g \in \mathscr{E}$, where

$$g = \vee Fm_i : \vee FX_i \to FA .$$

This is easy: we have $r : \vee FX_i \to F(\vee X_i)$ in $\mathscr{E}$ and clearly $g = Ff \circ r$. Since $f, r \in \mathscr{E}$ and $F(\mathscr{E}) \subset \mathscr{E}$, we get $g \in \mathscr{E}$.

C) analogous to B).

Corollary. Let $F(\mathscr{E}) \subset \mathscr{E}$. Then $\mathscr{K}(F)$ has coequalizers (constructive). If F preserves well-ordered unions then F is a constructive input process and $\mathscr{K}(F)$ is cocomplete (constructively).

III. COLIMITS, PRESERVED BY THE FORGETFUL FUNCTOR

III,1 If $\mathscr{K}$ is cocomplete and F preserves colimits then also $\mathscr{K}(F)$ is cocomplete and, moreover, the colimits of algebras agree with the colimits of their underlying objects (in $\mathscr{K}$). More precisely: the forgetful functor $\mathscr{U} : \mathscr{K}(F) \to \mathscr{K}$ then preserves colimits. This is easy. The aim of the present section is to show that, conversely, if $\mathscr{U}$ preserves colimits, so does F. We assume that $\mathscr{K}$ is a cocomplete category.

III,2 Proposition. Given an arbitrary diagram D in $\mathscr{K}(F)$ then D has a colimit, preserved by $\mathscr{U}$ iff the colimit construction stops after 0 steps for D.

Proof. The sufficiency follows from Theorem II,6. The colimit is $V_{0,0} \circ v_d : (A_d, r_d) \to (V_0, r)$ and $V_0 = A$, $V_{0,0} = 1_A$, thus the underlying bound is $v_d : A_d \to A$, i.e. the colimit of $\mathscr{U} \circ D$. Let us prove the necessity. By hypothesis, there exists a morphism $r : FA \to A$ such that $v_d : (A_d, r_d) \to (A, r)$, $d \in \mathscr{D}$ presents the colimit of $\mathscr{K}(F)$. It is our task to prove that in the push-out $V_{0,1} \circ p = p_1 \circ q$ the morphism $V_{0,1}$ is an isomorphism. For every d we have $p \circ q_d = v_d \circ r_d = r \circ Fv_d$ and, since $Fv_d = q \circ q_d$, $p \circ q_d = (r \circ q) \circ q_d$. This implies that $p = r \circ q$ and so there exists a unique $f : V_1 \to A$ satisfying $f \circ V_{0,1} = 1_A$ and $f \circ p_1 = r$. See Fig. 8. Let us prove that $V_{0,1} \circ f = 1_{V_1}$ – this proves that $V_{0,1} = f^{-1}$ is an isomorphism. Since $(V_{0,1} \circ f) \circ V_{0,1} = V_{0,1}$, it suffices to prove that $(V_{0,1} \circ f) \circ p_1 = p_1$. See Fig. 9.

Put $s = p_1 \circ Ff : FV_1 \to V_1$ and let us prove that $V_{0,1} \circ v_d : (A_d, r_d) \to (V_1, s)$ is a bound of D, i.e. that $V_{0,1} \circ v_d \circ r_d = s \circ F(V_{0,1} \circ v_d)$ holds for all d. The right side equals $p_1 \circ F(f \circ V_{0,1} \circ v_d) = p_1 \circ Fv_d$. Since $v_d \circ r_d = p \circ q_d$, the left side equals

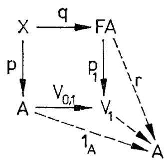

Fig. 8.

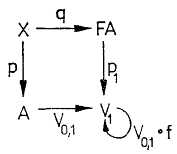

Fig. 9.

$V_{0,1} \circ p \circ q_d$ and, since $V_{0,1} \circ p = p_1 \circ q$, this is again $p_1 \circ Fv_d$. The colimit of D is $v_d : (A_d, r_d) \to (A, r)$, therefore we have $k : (A, r) \to (V_1, s)$ with $k \circ v_d = V_{0,1} \circ v_d$ for all d – in other words, $k = V_{0,1}$, which proves that $V_{0,1}$ is a homomorphism, i.e. $V_{0,1} \circ r = s \circ FV_{0,1}$.

Since $s = p_1 \circ Ff$, this means that $V_{0,1} \circ r = p_1$ and substituting $r = f \circ p_1$, we obtain $(V_{0,1} \circ f) \circ p_1 = p_1$. Together with $f \circ V_{0,1} = 1_A$ this yields $V_{0,1} = f^{-1}$, which completes the proof.

III,3 We say that a morphism $f : X \to Y$ is a *weak summand* if there exists an object A and a morphism $g : Y \to X \vee A$ such that $g \circ f : X \to X \vee A$ is the sum injection.

Lemma. Let $\mathscr{D}$ be a diagram scheme such that every diagram $D : \mathscr{D} \to \mathscr{K}(F)$ has a colimit, preserved by $\mathscr{U}$. Then for every D the mapping $q(: \operatorname{colim} F \circ \mathscr{U} \circ D \to F \operatorname{colim} \mathscr{U} \circ D)$ is a weak summand.

Proof. Denote, as in II,2, $Dd = (A_d, r_d)$ and define a new diagram $\bar{D} : \mathscr{D} \to \mathscr{K}(F)$, $\bar{D}d = (\bar{A}_d, \bar{r}_d)$ where $\bar{A}_d = A_d \vee FA_d$ (with sum injections $i_d : A_d \to \bar{A}_d$ and $j_d : FA_d \to \bar{A}_d$) and $\bar{r}_d = j_d \circ F(1 \cup r_d)$, where $1 \cup r_d : \bar{A}_d \to A_d$ is defined by $(1 \cup r_d) \circ i_d = 1_{A_d}$, $(1 \cup r_d) \circ j_d = r_d$. For a morphism $\delta : d \to d'$ in $\mathscr{D}$ put $\bar{D}\delta = D\delta \vee F \circ D\delta$. This is clearly a correctly defined diagram. Since $\{v_d : A_d \to A\}$ is a colimit of $\mathscr{U} \circ D$, $\{q_d : FA_d \to X\}$ is a colimit of $F \circ \mathscr{U} \circ D$, and since colimits commute with finite sums, it is clear that the colimit of $\mathscr{U} \circ \bar{D}$ in $\mathscr{K}$ is $\bar{v}_d : \bar{A}_d \to \bar{A}$, where $\bar{A} = A \vee X$ (with sum injections $i : A \to \bar{A}$ and $j : X \to \bar{A}$) and $\bar{v}_d = v_d \vee q_d$. In particular, $\bar{v}_d \circ i_d = i \circ v_d$ and $\bar{v}_d \circ j_d = j \circ q_d$.

By hypothesis, a colimit of $\bar{D}$ exists and it is preserved by $\mathscr{U}$. In other words, there exists a morphism $s : F\bar{A} \to \bar{A}$ such that $\{\bar{v}_d : (\bar{A}_d, \bar{r}_d) \to (\bar{A}, s)\}$ is a colimit of $\bar{D}$ in $\mathscr{K}(F)$. See Fig. 10. Then, for every d, $\bar{v}_d \circ \bar{r}_d = s \circ F\bar{v}_d$.

$$\begin{tikzcd}
& FA_d \arrow[dr, "j_d"] & & & & & \\
F\bar A_d \arrow[ur, "F(1\cup r_d)"] \arrow[rr, "\bar r_d"'] \arrow[d, "F\bar v_d"'] & & \bar A_d \arrow[d, "\bar v_d"'] & = & A_d \arrow[d, "= v_d"'] & \vee & FA_d \arrow[d, "q_d"] \\
F\bar A \arrow[rr, "s"'] & & \bar A & = & A & \vee & X
\end{tikzcd}$$

Fig. 10.

We have $(1 \cup r_d) \circ i_d = 1_{A_d}$, therefore $(\bar{v}_d \circ \bar{r}_d) \circ Fi_d = \bar{v}_d \circ j_d = j \circ q_d$. Thus $j \circ q_d = s \circ F(\bar{v}_d \circ i_d) = (s \circ Fi) \circ Fv_d$ holds for all d. Since q is defined by $q \circ q_d = Fv_d$, we have $j \circ q_d = (s \circ Fi) \circ q \circ q_d$ for all d, i.e. $j = (s \circ Fi) \circ q$. Since $j : X \to A \vee X$ is a sum injection, this proves that q is a weak summand.

III,4 In a number of categories push-outs have the property that opposite an isomorphism there is always an epimorphism. To all of these categories (e.g., the category of sets, graphs, topological spaces, unary algebras etc.) the following corollary applies.

Definition. Let us say that $\mathscr{K}$ fulfils the *push-out condition* for a class $\mathscr{M}$ (of morphisms) if in any push-out (Fig. 11) such that m' is an isomorphism and $m \in \mathscr{M}$, also m is an isomorphism.

$$\begin{tikzcd}
\cdot \arrow[r, "m"] \arrow[d, "n"'] & \cdot \arrow[d, "n'"] \\
\cdot \arrow[r, "m'"'] & \cdot
\end{tikzcd}$$

Fig. 11.

Corollary. Let $\mathscr{K}$ fulfil the push-out condition for weak summands, let $\mathscr{D}$ be a diagram scheme. Then the following conditions are equivalent:

(i) F preserves colimits of all diagrams $\mathscr{U} \circ D$ with $D : \mathscr{D} \to \mathscr{K}(F)$;

(ii) $\mathscr{K}(F)$ has colimits of all diagrams over $\mathscr{D}$, preserved by the forgetful functor $\mathscr{U}$.

Proof. (ii) $\Rightarrow$ (i): Given $D : \mathscr{D} \to \mathscr{K}(F)$, we want to prove that $F \operatorname{colim} \mathscr{U} \circ D = \operatorname{colim} F \circ \mathscr{U} \circ D$, i.e. that q is an isomorphism. This follows from the push-out condition, since $V_{0,1}$ is an isomorphism (by the above Proposition) and q is a weak summand (by the above Lemma).

(i) $\Rightarrow$ (ii): This is easy.

III,5 For the proof of the main theorem we need additional assumptions on $\mathscr{K}$. Recall that an object I is initial (a void sum) if from I there leads just one morphism to any other object; dually: terminal object. For the next theorem we assume that $\mathscr{K}$ fulfils the following conditions:

a) $\mathscr{K}$ is cocomplete and has a terminal object;

b) $\mathscr{K}$ is connected, i.e. hom $(A, B) \neq \emptyset$ for arbitrary non-initial objects A, B;

c) $\mathscr{K}$ fulfils the push-out condition for split monics.

The most restrictive assumption is b), for a), c) hold in any current category. Thus the theorem applies to various categories of topological spaces, vector spaces, lattices, abelian categories etc.

III,6 Theorem. Let $\mathscr{K}$ be a category, which fulfils *a*), *b*), *c*). Then $\mathscr{K}(F)$ has colimits, preserved by the forgetful functor $\mathscr{U}$, if and only if F preserves colimits.

Proof. A) F preserves the initial object I. By hypothesis, $\mathscr{K}(F)$ has an initial object, preserved by $\mathscr{U}$, say (I, d). For the F-algebra (FI, Fd) there exists a unique homomorphism $f : (I, d) \to (FI, Fd)$. Then $f \circ d = F(d \circ f)$. Since I has no endomorphism other than 1, we have $d \circ f = 1$ and so $f \circ d = F1 = 1$. Therefore $d : FI \to I$ is an isomorphism, $f = d^{-1}$.

B) F preserves colimits. It suffices to verify that F preserves multiple push-outs (of $f_t : A_0 \to A_t$, $t \in T$). Then F preserves sums (it suffices to put $A_0 = I$ – the initial object) as well as push-outs, therefore it preserves all colimits. Denote (for

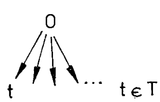

Fig. 12.

arbitrary T) by $\mathscr{D}$ the category in Fig. 12. We are to prove that F preserves the colimit of an arbitrary diagram $D_0 : \mathscr{D} \to \mathscr{K}$. By the above Corollary it suffices to find a diagram $D : \mathscr{D} \to \mathscr{K}(F)$ such that $D_0 = \mathscr{U} \circ D$.

For every object M denote by $t_M : M \to T$ the canonical map to the terminal object T. Consider two possibilities.

a) A_0 is non-initial. Then there exists a morphism $\varrho : T \to A_0$. Put $r_0 = \varrho \circ t_{FA_0} : FA_0 \to A_0$ and, for $t \in T$, $r_t = f_t \circ \varrho \circ t_{FA_t} : FA_t \to A_t$. Then clearly f_t is a homomorphism $f_t : (A_0, r_0) \to (A_t, r_t)$. This defines the diagram D we wanted: $D(O) = (A_0, r_0)$ and $D(t) = (A_t, r_t)$.

b) A_0 is initial. Via A), there exists an isomorphism $r_0 : FA_0 \to A_0$. Given $t \in T$, either A_t is initial, then there exists an isomorphism $r_t : FA_t \to A_t$, or it is non-initial, then choose $r_t : FA_t \to A_t$ arbitrarily (since $\mathscr{K}$ is connected, such r_t exists). Then, again, we have a diagram D with $\mathscr{U} \circ D = D_0$.

The same theorem holds for sums (finite sums, finite colimits) in place of all colimits. This is easy to see by going through the proof.

III,7 Arbib and Manes call F and *adjoint process* if it has a right adjoint. Using the dual to the Freyd special adjoint functor theorem we see that this is equivalent to the preservation of colimits as soon as $\mathscr{K}$ is cocomplete, co-locally small and has a generator (these are very natural conditions). By a *state-behaviour process* Arbib and Manes call such F that the forgetful functor $\mathscr{U}$ has a left as well as a right adjoint. Every adjoint process is easily seen to be a state-behaviour one. Conversely

Corollary. Let $\mathscr{K}$ be a co-locally small category with a generator, which fulfils conditions a), b) c) in III,5. Then every state-behaviour process is an adjoint process.

Proof. Since $\mathscr{U}$ has a right adjoint, it preserves colimits and so does F.

(Received July 20, 1976.)

REFERENCES

[1] J. Adámek: Free algebras and automata realizations in the language of categories. Comment. Math. Univ. Carolinae *15* (1974), 589—602.

[2] J. Adámek: Categorial automata theory and universal algebra. (In Czech.) Dissertation.

[3] J. Adámek: Automata in categories, finiteness contra minimality. Lecture Notes in Comp. Sci. 32. Springer (1975), 160—166.

[4] J. Adámek: Realization theory for automata in categories. To appear in J. Pure Appl. Algebra.

[5] M. A. Arbib, E. G. Manes: A categorist's view of automata and systems. Lecture Notes in Comp. Sci. 25. Springer (1975), 51—64.

[6] M. A. Arbib, E. G. Manes: Adjoint machines, state-behavior machines and duality. J. Pure Appl. Algebra *6*, (1975), 313—345.

[7] M. Barr: Coequalizers and free triples, Math. Z. *116*, (1970), 307—322.

[8] H. Herrlich, G. E. Strecker: Category theory. Allyn and Bacon, Boston 1973.

[9] V. Koubek, J. Reiterman: Automata in categories — input processes. Lecture Notes in Comp. Sci. 32. Springer (1975), 280—286.

[10] V. Kůrková-Pohlová, V. Koubek: When a generalized algebraic category is monadic. Comment. Math. Univ. Carolinae *15* (1974), 577—602.

[11] S. MacLane: Categories for the working mathematician. Springer 1968.

[12] B. Mitchell: Theory of categories. Academic Press 1965.

[13] J. Reiterman: A left adjoint construction related to free triples. To appear in J. Pure Appl. Algebra.

[14] J. Slominsky: The theory of abstract algebras with infinitary operations. Rozprawy Matematyczne 18 (1959).

[15] V. Trnková: On minimal realizations of behavior maps in categorial automata theory. Comment. Math. Univ. Carolinae *15* (1974), 555—566.

[16] V. Trnková, J. Adámek, V. Koubek, J. Reiterman: Free algebras, free monads and input processes. Comment. Math. Univ. Carolinae *16* (1975), 339—352.

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