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A Note on Characterizations of Entropies

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In this note we discuss and characterize four entropies each involving a parameter, by using the concept of generalized probability distributions. We shall call them entropies of order α, β, γ and δ.

1. GENERALIZED PROBABILITY DISTRIBUTIONS

Let $(\Omega, \mathcal{R}, \mathbf{P})$ be a probability space, that is, Ω an arbitrary non-empty set, called the set of elementary events, $\mathcal{R}$ a σ-field of subsets of Ω and $\mathbf{P}$ a probability measure defined on $\mathcal{R}$. Let us call a function $\xi = \xi(w)$ which is defined for $w \in \Omega_1$ where $\Omega_1 \in \mathcal{R}$ and $\mathbf{P}(\Omega_1) > 0$, and which is measurable w.r.t. $\mathcal{R}$, a *generalized random variable*. If $\mathbf{P}(\Omega_1) = 1$ we call ξ an *ordinary* (or *complete*) random variable, while if $0 < \mathbf{P}(\Omega_1) < 1$ we call ξ an *incomplete* random variable. Clearly, an incomplete random variable can be interpreted as a quantity describing the result of an experiment depending on chance which is not always observable. The distribution of a generalized random variable will be called a *generalized probability distribution*. In particular, in the case when ξ takes on only a finite number of different values $x_1, x_2, \ldots, x_n$ the distribution of ξ consists of the set of numbers $p_k = \mathbf{P}\{\xi = x_k\}$ for $k = 1, 2, \ldots, n$. Thus a finite discrete generalized probability distribution is simply a sequence $p_1, p_2, \ldots, p_n$ of non-negative numbers such that putting $\mathsf{P} = (p_1, p_2, \ldots, p_n)$ and $W(\mathsf{P}) = \sum_{k=1}^{n} p_k$, we have, $0 < W(\mathsf{P}) \leqq 1$.

We shall call $W(\mathsf{P})$ the *weight* of the distribution. Thus the weight of an ordinary distribution is equal to 1. A distribution which has a weight less than 1 will be called an *incomplete* distribution. If $\mathsf{P} = (p_1, p_2, \ldots, p_n)$ and $\mathsf{Q} = (q_1, q_2, \ldots, q_m)$ are two finite discrete generalized probability distributions, then $\mathsf{P} * \mathsf{Q}$ is the *direct product* of the distributions P and Q and is a finite discrete generalized probability distribution consisting of the numbers $p_j q_k$ with $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, m$.

If $W(\mathsf{P}) + W(\mathsf{Q}) \leqq 1$, we put

$$\mathsf{P} \cup \mathsf{Q} = (p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m).$$

2. CHARACTERIZATION OF ENTROPIES

Let Δ denote the set of all finite discrete generalized probability distributions, that is, Δ is the set of all sequences $\mathsf{P} = (p_1, p_2, \ldots, p_n)$ of non-negative numbers such that $0 < \sum_{k=1}^{n} p_k \leqq 1$. It is assumed that the entropy $H[\mathsf{P}]$ of a finite discrete generalized probability distribution is defined for all $\mathsf{P} \in \Delta$. We define the following postulates:

Postulate 1. $H[\mathsf{P}]$ is a symmetric function of the elements of P.

Postulate 2. If $\{p\}$ denotes the generalized probability distribution consisting of the probability p, then $H[\{p\}]$ is a continuous function of p in the interval $0 < p \leqq 1$.

Postulate 3. $H[\{\frac{1}{2}\}] = 1$.

Postulate 4. $H[\{1\}] = 0$.

Postulate 5. If $\mathsf{P} = (p_1, p_2, \ldots, p_n)$ and $\mathsf{Q} = (q_1, q_2, \ldots, q_m)$, then $H[\mathsf{P} * \mathsf{Q}] = H[\mathsf{P}] + H[\mathsf{Q}]$.

Postulate 6. If $\mathsf{P} = (p_1, p_2, \ldots, p_n)$ and $\mathsf{Q} = (q_1, q_2, \ldots, q_m)$, such that $W(\mathsf{P}) + W(\mathsf{Q}) \leqq 1$, then

$$H[\mathsf{P} \cup \mathsf{Q}] = \frac{W(\mathsf{P})\, H[\mathsf{P}] + W(\mathsf{Q})\, H[\mathsf{Q}]}{W(\mathsf{P}) + W(\mathsf{Q})}.$$

It has been shown by Renyi [8] that if $H[\mathsf{P}]$ satisfies the Postulates 1, 2, 3, 5 and 6, then

$$H[\mathsf{P}] = \frac{\sum_{k=1}^{n} p_k \log_2 (1/p_k)}{\sum_{k=1}^{n} p_k},$$

which is a well known Shannon's entropy. We define another postulate as follows:

Postulate 7. There exists a strictly monotonic and continuous function $y = g(x)$ such that if $P = (p_1, p_2, \ldots, p_n)$, $Q = (q_1, q_2, \ldots, q_m)$ and $W(P) + W(Q) \leqq 1$, then

$$H[P \cup Q] = g^{-1}\left[\frac{W(P)\, g(H[P]) + W(Q)\, g(H[Q])}{W(P) + W(Q)}\right].$$

It was also shown by Renyi [8] that if $g(x) = g_\alpha(x) = 2^{(1-\alpha)x}$ where $\alpha > 0$, $\alpha \neq 1$, then Postulates 1, 2, 3, 5 and 7 characterize Renyi's entropy of order α. In other words if $H[P]$ satisfies postulates 1, 2, 3, 5 and 7 with $g(x) = g_\alpha(x) = 2^{(1-\alpha)x}$, $\alpha > 0$ and $\alpha \neq 1$, then, for $P = (p_1, p_2, \ldots, p_n)$, we have

$$H_\alpha[P] = \frac{1}{1-\alpha} \log_2 \left[\left(\sum_{k=1}^{n} p_k^\alpha\right) \Big/ \left(\sum_{k=1}^{n} p_k\right)\right],$$

which is well known Renyi's entropy.

M. Behara and P. Nath [3] introduced the following postulate for characterizing a new class of entropies.

Postulate 5'. For every $P = (p_1, p_2, \ldots, p_n) \in \Delta$, $n = 1, 2, 3, \ldots$ and $Q = \{q\}$

$$H[(p_1, p_2, \ldots, p_n) * q] = H[p_1 q, p_2 q, \ldots, p_n q] = a\, H[p_1, p_2, \ldots, p_n]\, H[q] + b\, H[p_1, p_2, \ldots, p_n] + b\, H[q] + c\,, \quad \text{where} \quad ac = b^2 - b \quad \text{and} \quad a \neq 0\,.$$

They proved that if $H[P]$ is defined for all $P \in \Delta$ and satisfies the Postulates 2, 3, 4, 5' and 6, then

$$H[P] = H_\beta[P] = \frac{1 - \left(\sum_{k=1}^{n} p_k^\beta \big/ \sum_{k=1}^{n} p_k\right)}{1 - 2^{1-\beta}}, \quad \beta \neq 1 \quad \text{and} \quad \beta > 0\,.$$

This entropy was first of all proposed by Havrda and Charvát [7]. Later on, Daroczy [6] and M. Behara and P. Nath [2, 3] studied these entropies. Vajda [9] also characterized this entropy for finite discrete generalized probability distributions.

In the following theorems we characterize two more entropies and to avoid confusion we call them entropies of order γ and δ.

Theorem 2.1. If $H[P]$ is defined for all $P \in \Delta$ and satisfies Postulates 2, 3, 4 and 7 with $g(x) = g_\gamma(x) = [1 - x(1 - 2^{\gamma-1})]^{1/\gamma}$, $\gamma \neq 1$, $\gamma > 0$ and the following postulate

Postulate 5''. $H[pq] = a H[p]\, H[q] + b H[p] + b H[q] + c$, where $ac = b^2 - b$ and $a \neq 0$,

then $H[P] = H_\gamma[P]$, where, putting $P = (p_1, p_2, \ldots, p_n)$, we have

$$H_\gamma[P] = \left\{1 - \left[\left(\sum_{k=1}^{n} p_k^{1/\gamma}\right) \Big/ \left(\sum_{k=1}^{n} p_k\right)\right]^\gamma\right\} \Big/ (1 - 2^{\gamma-1})\,.$$

Proof. From Postulate 5″, we have

$$a\,H[pq] = a^2\,H[p]\,H[q] + ab\,H[p] + ab\,H[q] + ac \Rightarrow$$
$$\Rightarrow a\,H[pq] = a^2\,H[p]\,H[q] + ab\,H[p] + ab\,H[q] + b^2 - b \Rightarrow$$
$$\Rightarrow (a\,H[pq] + b) = (a\,H\,[p] + b)\,(a\,H[q] + b)\,.$$

Replacing $a\,H[p] + b$ by $h[p]$, we obtain

(1) $$h[pq] = h[p]\,h[q]\,.$$

The continuity of $H[p] \Rightarrow$ continuity of $h[p]$. Hence non-identically vanishing continuous solutions of (1) are of the form

$$h[p] = p^{1-\gamma}\,, \quad \gamma \neq 1$$

so that

$$H[p] = \frac{p^{1-\gamma} - b}{a}\,, \quad \gamma \neq 1\,.$$

Using Postulates 3 and 4, it can be easily seen that $b = 1$ and $a = 2^{\gamma-1} - 1$, $\gamma \neq 1$, so that

$$H[p] = H_\gamma[p] = \frac{1 - p^{1-\gamma}}{1 - 2^{\gamma-1}}\,.$$

From Postulate 7, we have

$$H[p_1, p_2, \ldots, p_n] = g^{-1}\left|\frac{\sum_{k=1}^{n} p_k\,g(H[p_k])}{\sum_{k=1}^{n} p_k}\right| =$$

$$= g^{-1}\left|\frac{\sum_{k=1}^{n} p_k\,g\left(\frac{1 - p_k^{1-\gamma}}{1 - 2^{\gamma-1}}\right)}{\sum_{k=1}^{n} p_k}\right| = g^{-1}\left|\frac{\sum_{k=1}^{n} p_k\left\{1 - (1 - 2^{\gamma-1})\,\frac{1 - p_k^{1-\gamma}}{1 - 2^{\gamma-1}}\right\}^{1/\gamma}}{\sum_{k=1}^{n} p_k}\right| =$$

$$= g^{-1}\left|\frac{\sum_{k=1}^{n} p_k p_k^{(1-\gamma)/\gamma}}{\sum_{k=1}^{n} p_k}\right| = g^{-1}\left|\frac{\sum_{k=1}^{n} p_k^{1/\gamma}}{\sum_{k=1}^{n} p_k}\right| =$$

$$= \left\{1 - \left(\frac{\sum_{k=1}^{n} p_k^{1/\gamma}}{\sum_{k=1}^{n} p_k}\right)^{\gamma}\right\} \Big/ (1 - 2^{\gamma-1});\ \gamma \neq 1,\ \gamma > 0\,.$$

 Because, if $y = g(x) = [1 - x(1 - 2^{\gamma-1})]^{1/\gamma}$, then

$$y^\gamma = 1 - x(1 - 2^{\gamma-1}) \Rightarrow x(1 - 2^{\gamma-1}) = 1 - y^\gamma \Rightarrow x = \frac{1 - y^\gamma}{1 - 2^{\gamma-1}} \Rightarrow$$

$$\Rightarrow g^{-1}(y) = \frac{1 - y^\gamma}{1 - 2^{\gamma-1}}.$$

The above entropy of order γ has been studied by M. Behara and J. S. Chawla [4].

Suguru Arimoto [1] while investigating the finite parameter estimation problem deduced the entropy of order γ using the concept of generalized information measure and discussed its relationship with Renyi's entropy of order α.

Theorem 2.2. If $H[\mathsf{P}]$ is defined for all $\mathsf{P} \in \Delta$ and satisfies the Postulates 2, 3, 6 and the following:
Postulate 5‴. $H^{1/\delta}[pq] = H^{1/\delta}[p] + H^{1/\delta}[q]$; $\delta \neq 1$, $\delta > 0$,
then, for $\mathsf{P} = (p_1, p_2, \ldots, p_n)$

$$H[\mathsf{P}] = H_\delta[\mathsf{P}] = \sum_{k=1}^{n} p_k |\log_2 p_k|^\delta / \sum_{k=1}^{n} p_k .$$

Proof. Replacing $H^{1/\delta}[p]$ by $h[p]$ in Postulate 5‴, we obtain

$$h[pq] = h[p] + h[q]. \tag{2}$$

Now, by Postulate 2, $H[p]$ is a continuous function of p, $p \in (0, 1]$ and hence $h[p]$ is a continuous function of p, $p \in (0, 1]$. Thus the only continuous solutions of (2) are of the form $h[p] = c \log_2 p$ and hence $H[p] = c^\delta (\log_2 p)^\delta$.

Now, $H[\frac{1}{2}] = 1 \Rightarrow 1 = c^\delta(\log_2 \frac{1}{2})^\delta \Rightarrow 1 = c^\delta(-\log_2 2)^\delta \Rightarrow 1 = c^\delta(-1)^\delta \Rightarrow (-c)^\delta = 1 \Rightarrow c = -1$. Thus $H^{1/\delta}[p] = -\log_2 p \Rightarrow H[p] = |\log_2 p|^\delta$. For $(p_1, p_2) \in \Delta$, we have

$$H[p_1, p_2] = \frac{p_1 H[p_1] + p_2 H[p_2]}{p_1 + p_2} = \frac{p_1|\log_2 p_1|^\delta + p_2|\log_2 p_2|^\delta}{p_1 + p_2},$$

$$H[p_1, p_2, p_3] = \frac{(p_1 + p_2) H[p_1, p_2] + p_3 H[p_3]}{p_1 + p_2 + p_3} =$$

$$= \frac{(p_1 + p_2)\dfrac{p_1 H[p_1] + p_2 H[p_2]}{(p_1 + p_2)} + p_3 H[p_3]}{(p_1 + p_2 + p_3)} = \frac{\sum_{i=1}^{3} p_i |\log_2 p_i|^\delta}{\sum_{i=1}^{3} p_i}.$$

Using the method of mathematical induction, we obtain

$$H[p_1, p_2, \ldots, p_n] = \frac{\sum_{i=1}^{n} p_i |\log_2 p_i|^\delta}{\sum_{i=1}^{n} p_i}.$$

This entropy has been studied by the author [5] for complete probability distributions. It has also been shown in [5] that the entropy of order δ like Shannon entropy is also an invariant for isomorphic Bernoulli shifts.

It is important to note that if $\mathsf{P} = (p_1, p_2, \ldots, p_n)$, then

$$\lim_{\alpha \to 1} H_\alpha[\mathsf{P}] = \lim_{\beta \to 1} H_\beta[\mathsf{P}] = \lim_{\gamma \to 1} H_\gamma[\mathsf{P}] = \lim_{\delta \to 1} H_\delta[\mathsf{P}] = \frac{-\sum_{k=1}^{n} p_k \log_2 p_k}{\sum_{k=1}^{n} p_k}$$

which is Shannon's entropy for a finite discrete generalized probability distribution.

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REFERENCES

[1] Suguru Arimoto: Information-Theoretical Considerations on Estimation Problems. Informamation and Control *19* (1971), 181—194.

[2] M. Behara, P. Nath: Additive and Non-Additive Entropies of Finite Measurable Partitions. Probability and Information Theory II, 1970, Springer-Verlag, 102—138.

[3] M. Behara, P. Nath: An Axiomatic Characterization of Entropy of a Finite Discrete Generalized Probability Distribution. McMaster Univ. Math. report no. 40.

[4] M. Behara, J. S. Chawla: Generalized Gamma-Entropy: "Entropy and Ergodic Theory". Selecta Statistica Canadiana, University Press of Canada/Hindustan Publishing Corporation (India), 1974, 15—38.

[5] J. S. Chawla: Ph. D. Dissertation. McMaster University, Hamilton, Ontario, Canada 1974.

[6] Z. Daroczy: Generalized Information Functions. Information and Control *16* (1970), 1, 36—51.

[7] J. Havrda, F. Charvát: Quantification Method of Classification Processes. The Concept of Structural a-Entropy. Kybernetika *3* (1967), 1, 30—35.

[8] A. Rényi: On Measures of Entropy and Information. Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, Vol. I, 1961, 547—561.

[9] I. Vajda: Axioms of α-Entropy of a Generalized Probability Scheme. Kybernetika *4* (1968), 2, 105—110.

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