KYBERNETIKA — VOLUME 13 (1977), NUMBER 1

On a Problem of Evasion

MILAN MEDVEĎ

A strategy of evasion for a class of nonlinear differential game is constructed.

B. N. Pchenitchny [1] has solved a differential game described by the system of differential equations

$$\dot{z} = f(z, u, v); \tag{1}$$

where $z \in R^n$, $u \in U \subset R^r$, $v \in V \subset R^s$, $f : R^n \times R^r \times R^s \to R^n$. He suppose that the function f has continuous derivatives with respect to z of sufficiently high orders, satisfying the Lipschitz condition with respect to all their arguments on arbitrary compact set. Furthermore, the function f is assumed to be of the form $f(z, u, v) = f_0(z, u) + f_1(z, u)\, v$, where $f_0 \in R^n$ and f_1 is an $n \times s$ matrix, i.e. the function f is convex in the variable v.

We shall construct a strategy of evasion for a class of nonlinear games described by the system (1), where

$$f(z, u, v) = \sum_{j=0}^{m-1} g_j(z, u, v_1, v_2, \ldots, v_{j+1}),$$

where $g_j(z, u, v_1, v_2, \ldots, v_{j+1}) = f_{1j}(z, u, v_1, \ldots, v_j) + f_{2j}(z, u, v_1, \ldots, v_j)\, v_{j+1}$, $j = 1, 2, \ldots, m-1$, $g_0(z, u, v_1) = f_{10}(z, u) + f_{20}(z, u)\, v_1$, $z \in R^n$, $u \in R^r$, $v = (v_1, v_2, \ldots, v_m)$, $v_i \in R^{q_i}$, $i = 1, 2, \ldots, m$, $f_{1j}(z, u, v_1, \ldots, v_j) \in R^n$ and $f_{2j}(z, u, v_1, \ldots, v_j)$, $j = 0, 1, \ldots, m-1$ are $n \times q_j$ matrices. The function $f(z, u, v)$ need not be convex in v, but it is convex in v_m only. We shall construct a strategy of evasion $v(t) = (v_1(t), v_2(t), \ldots, v_m(t))$ in such way that first we shall construct $v_1(t)$ and then one after the other $v_i(t)$, $i = 2, 3, \ldots, m$, where for the construction of each $v_i(t)$, $i = 1, 2, \ldots, m$ we shall use the method of Pchenitchny. For $m = 1$ we get the result of Pchenitchny [1].

We shall suppose that the terminal set M is a subspace of R^n of dimension $\leqq n - 2$.

Definition. A mapping $E : R^n \times U \times [0, \infty) \to R^s$ is said to be a strategy, if for every absolutely continuous function $x(t)$, $0 \leqq t < \infty$, and for every measurable function $u(t) \in U$, $0 \leqq t < \infty$, the function $E(x(t), u(t), t)$ is a measurable function with values in V. This strategy is called a strategy of evasion, if for arbitrary $z_0 \notin M$ and for arbitrary measurable function $u(t)$, $0 \leqq t < \infty$, the solution $z(t)$, $0 \leqq t < \infty$, of the equation

$$\dot{z}(t) = f(z(t), u(t), E(z(t), u(t), t))$$

with initial condition $z(0) = z_0$ does not intersect the subspace M for any $t \geqq 0$.

We shall assume that

(1) U is a compact set and $V = V_1 \times V_2 \times \ldots \times V_m$, where $V_i \subset R^{q_i}$ are compact convex sets, $\sum_{i=1}^{m} q_i = s$, int $V_i \neq \emptyset$ in R^{q_i}.

(2) We suppose that the function $f(z, u, v)$ has the above form where the functions $g_j(z, u, v_1, v_2, \ldots, v_{j+1})$, $j = 0, 1, \ldots, m - 1$, have continuous derivatives with respect to z of sufficiently high orders, satisfying the Lipschitz condition with respect to all their arguments on arbitrary compact set.

(3) There exists a constant $C > 0$ such that $|(z, f(z, u, v))| \leqq C(1 + \|z\|^2)$ for all $(z, u, v) \in R^n \times U \times V$, where we denote by (x, y) the scalar product of the vectors x and y and $\|z\|$ is the euclidean norm of the vector z.

(4) Let $\varphi : R^n \to R^n$ be a C^1 function. Denote

(2) $$\nabla_z \varphi(z) = D\, \varphi(z) f(z, u, v),$$

where $D\, \varphi(z)$ is the matrix of the first derivatives of $\varphi(z)$ at z. We shall suppose that

(A) there is a subspace $W \subset L$ (L is the orthogonal complement of M in R^n) of dimension $q \geqq 2$ and an ineger k such that all functions $\varphi^0(z) = \pi z$, $\varphi^i(z) = \nabla_z \varphi^{i-1}(z)$, $i = 1, 2, \ldots, k - 1$ do not depend on u and v, where $\pi : R^n \to W$ is the orthogonal projection.

(B) The function $f^k(z, u, v) = \nabla_z \varphi^{k-1}(z)$ depends on u and v. The assumption (2) implies that $f^k(z, u, v) = \sum_{j=0}^{m-1} g_j^k(z, u, v_1, v_2, \ldots, v_{j+1})$, where $g_j^k(z, u, v_1, v_2, \ldots, v_{j+1}) = f_{1j}^k(z, u, v_1, v_2, \ldots, v_j) + f_{2j}^k(z, u, v_1, v_2, \ldots, v_j)\, v_{j+1}$, $j = 0, 1, \ldots, m - 1$. It is clear that $f^k(z, u, v) \in W$.

(C) Denote

$$F_0(z) = \bigcap_{u \in U} g_0^k(z, u, V_1),$$

$$F_j(z) = \bigcap_{\substack{(u, v_1, \ldots, v_j) \in \\ \in U \times V_1 \times \ldots \times V_j}} g_j^k(z, u, v_1, \ldots, v_j, V_{j+1})$$

$j = 1, 2, \ldots, m - 1$. Let there exist continuous functions $\varphi_j^k : R^n \to R^n$ and $\varepsilon : R^n \to R^1$ such that for all $z \in R^n$ $\varepsilon(z) > 0$ and

$$\varphi_j^k(z) + \varepsilon(z)\,\pi S \subset F_j(z)\,, \quad j = 0, 1, \ldots, m - 1\,, \tag{3}$$

where S is the unit sphere in R^n.

Theorem. Under the assumptions (1)–(4) there exists a strategy of evasion.

Before proving this theorem consider the following equations

$$f_{1j}^k(z, u, v_1, v_2, \ldots, v_j) + f_{2j}^k(z, u, v_1, v_2, \ldots, v_j)\, v_{j+1} = \tag{4}$$

$$= \varphi_j^k(z_0) + \frac{1}{m}\,\varepsilon(z_0)\,\xi_0\,, \quad j = 0, 1, \ldots, m - 1\,,$$

in a neighbourhood of a point $(z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0) \in R^n \times U \times V_1 \times \ldots \times V_j \times \times \pi S$ in v_{j+1} for $j = 0, 1, \ldots, m - 1$. The assumption (C) implies that for arbitrary such point, there exists a point $v_{j+1}^0 \in V_{j+1}$ that $f_{1j}^k(z_0, u_0, v_1^0, \ldots, v_j^0) + f_{2j}^k(z_0, u_0, v_1^0, \ldots, v_j^0)\, v_{j+1}^0 = \varphi_j^k(z_0) + (1/m)\,\varepsilon(z_0)\,\xi_0$, $j = 0, 1, \ldots, m - 1$.

Lemma 1. Let X be a compact set in R^n. Then for $j = 0, 1, \ldots, m - 1$ there exists a number $\varepsilon_X^j > 0$ such that for arbitrary $(z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0) \in X \times U \times V_1 \times \ldots \times V_j \times \pi S$ there exists a continuous function $v_j(z, u, v_1, \ldots, v_j, \xi | z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0)$ with values in V_{j+1}, which is the solution of the equation (4) for all $(z, u, v_1, \ldots, v_j, \xi) \in \{(\bar{z}, \bar{u}, \bar{v}_1, \ldots, \bar{v}_j, \bar{\xi}) \mid \max(\|\bar{z} - z_0\|, \|\bar{u} - u_0\|, \|\bar{v}_1 - v_1^0\|, \ldots, \|\bar{v}_j - v_j^0\|, \|\bar{\xi} - \xi_0\|) \leqq \varepsilon_X^j\}$. Moreover $v_j(z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0 \mid z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0 \in \operatorname{int} V_{j+1}$.

Proof. The proof is almost the same as the proof of [1, Lemma 3] and therefore we shall sketch it only. Let $V(z, u, v_1, \ldots, v_j, \xi) = \{v_{j+1} \in V_{j+1} \mid g_j^k(z, u, v_1, \ldots, v_{j+1}) = \varphi_j^k(z) + (1/m)\,\varepsilon(z)\}$. By the same procedure as in the proof of [1, Lemma 1] it is possible to prove that $V(z, u, v_1, \ldots, v_j, \xi) \cap (\operatorname{int} V_{j+1}) \neq \emptyset$ for arbitrary $\xi \in \pi S$.

If $\alpha(v_{j+1})$ is a continuous function, then by [1, Lemma 2] the function $\beta_j(z, u, v_1, \ldots, v_j, \xi) = \max_{v_{j+1}} \{\alpha(v_{j+1}) \mid v_{j+1} \in V(z, u, v_1, \ldots, v_j, \xi)\}$ is a continuous function of the variables z, $u \in U$, $v_k \in V_k$, $k = 0, 1, \ldots, j$, $\xi \in \pi S$.

Let $\alpha(v_{j+1}) = \min_{v_{j+1}} \{\|\bar{v}_{j+1} - v_{j+1}\| \mid \bar{v}_{j+1} \in \partial V_{j+1}\}$, where ∂V_{j+1} is the boundary of the convex set V_{j+1}. Since $V(z, u, v_1, \ldots, v_j, \xi) \cap (\operatorname{int} V_{j+1}) \neq \emptyset$, then $\beta_j(z, u, v_1, \ldots, v_j, \xi) > 0$. This means that if X is a compact set in R^n, then there exists a number $r_X^j > 0$ such that for arbitrary $z \in X$, $u \in U$, $v_k \in V_k$, $k = 1, 2, \ldots, j$ there is a point $v_{j+1}^0 \in V(z, u, v_1, \ldots, v_j, \xi)$ which is contained in the interior of the set $V(z, u, v_1, \ldots, v_j, \xi)$ together with the ball with center v_{j+1}^0 and radius r_X^j.

Consider the equation defining the set $V(z, u, v_1, \ldots, v_j, \xi) : f_{1j}^k(z, u, v_1, \ldots, v_j) + f_{2j}^k(z, u, v_1, \ldots, v_j)\, v_{j+1} = \varphi_j^k(z) + (1/m)\, \varepsilon(z)\, \xi$ in a neighbourhood of $(z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0)$. This equation is solvable in v_{j+1} for $z = z_0$, $u = u_0$, $v_k = v_k^0$, $k = 1, 2, \ldots, j$ for arbitrary $\xi \in \pi S$ and therefore there exist ν linearly independent columns of the matrix $f_{2j}^k(z, u, v_1, \ldots, v_j)$, where $\nu = \dim W$. Let J_j denote the set of indices of arbitrary chosen columns of the matrix $f_{2j}^k(z, u, v_1, \ldots, v_j)$ and let $f_{2J_j}^k(z, u, v_1, \ldots, v_j)$ be the corresponding matrix. Denote

$$m(z, u, v_1, \ldots, v_j) = \max_{J_j} \det\left(f_{2J_j}^{k*}(z, u, v_1, \ldots, v_j)\, f_{2J_j}^k(z, u, v_1, \ldots, v_j)\right),$$

where A^* means the transpose of a matrix A. Let J_{0j} be the set of such indices for which

$$\max_{J_j} \det\left(f_{2J_j}^{k*}(z_0, u_0, v_1^0, \ldots, v_j^0)\, f_{2J_j}^k(z_0, u_0, v_1^0, \ldots, v_j^0)\right) = \det\left(f_{2J_{0j}}^{k*}(z_0, u_0, v_1^0, \ldots, v_j^0)\, f_{2J_{0j}}^k(z_0, u_0, v_1^0, \ldots, v_j^0)\right).$$

Then

$$\det\left(f_{2\ J_{0j}}^{k*}(z, u, v_1, \ldots, v_j)\, f_{2J_{0j}}^k(z, u, v_1, \ldots, v_j)\right) > 0$$

in some neighbourhood of the point $(z_0, u_0, v_1^0, \ldots, v_j^0)$. Let $v_{J_{0j}}$ be a vector with components of the vector v_{j+1}^0 with indices from J_{0j}.

Consider the following equation

$$\text{(6)} \qquad f_{1j}^k(z, u, v_1, \ldots, v_j) + f_{2j}^k(z, u, v_1, \ldots, v_j)\, v_{j+1}^0 + f_{2J_{0j}}^k(z, u, v_1, \ldots, v_j)\left(v_{J_{0j}} - v_{0J_{0j}}\right) = \varphi_j^k(z) + \frac{1}{m}\, \varepsilon(z)\, \xi\,.$$

The condition (5) implies that the equation (6) is equivalent to the following one:

$$\text{(7)} \qquad f_{2J_{0j}}^{k*}(z, u, v_1, \ldots, v_j)\, f_{2J_{0j}}^k(z, u, v_1, \ldots, v_j)\left(v_{J_{0j}} - v_{0J_{0j}}\right) = f_{2J_{0j}}^{k*}(z, u, v_1, \ldots, v_j)\left[\varphi_j^k(z) + \frac{1}{m}\, \varepsilon(z)\, \xi - f_{1j}^k(z, u, v_1, \ldots, v_j) - f_{2j}^k(z, u, v_1, \ldots, v_j)\, v_{j+1}^0\right].$$

The equation (7) has the unique solution $v_{J_{0j}}(z, u, v_1, \ldots, v_j, \xi)$ which is continuous in all its arguments and $v_{J_{0j}}(z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0) = v_{0J_{0j}}$. It is easy to see that the vector $v_{j+1}(z, u, v_1, \ldots, v_j, \xi)$ constructed from the components of the vector $v_{J_{0j}}(z, u, v_1, \ldots, v_j, \xi)$ completed with the remaining components of the vector v_{j+1}^0 is a solution of the equation (4). We shall denote it by $v_{j+1}(z, u, v_1, \ldots, v_j, \xi \mid z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0)$. In the same way as in the proof [1, Lemma 3] it is possible to prove that there exists a number $\varepsilon_X^j > 0$ which is the same for all $(z_0, u_0, v_1^0, \ldots$

$\ldots, v_j^0, \xi_0) \in X \times V_1 \times \ldots \times V_j \times \pi S$ such that the function $v_{j+1}(z, u, v_1, \ldots$ $\ldots, v_j, \xi \mid z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0)$ is defined and continuous for all $(z, u, v_1, \ldots, v_j, \xi)$ from the ε_X^j-neighbourhood of the point $(z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0)$. From the construction of the function $v_{j+1}(z, u, v_1, \ldots, v_j, \xi \mid z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0)$ it is clear that

$$v_{j+1}(z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0 \mid z_0, u_0, v_1^0, \ldots, v_j^0, \xi_0) \in \operatorname{int} V_{j+1} .$$

The proof is complete.

Denote $v_{j+1}(z, u, v_1, \ldots, v_j, \xi \mid z_0) = v_{j+1}(z, u, v_1, \ldots, v_j, \xi \mid z_0, u, v_1, \ldots, v_j, \xi)$. In the same way as [1, Lemma 3] it is possible to prove the following lemma.

Lemma 2. The functions $v_{j+1}(z, u, v_1, \ldots, v_j, \xi)$, $j = 0, 1, \ldots, m - 1$ are defined and continuous for all z, $\|z - z_0\| \leqq \frac{1}{2}\varepsilon_X^j$, $u \in U$, $v_i \in V_i$, $i = 1, 2, \ldots, j$, $\xi \in \pi S$.

Let $z_0 \notin M$. Consider the following function

$$\varphi(t, \xi) = \sum_{i=1}^{k-1} \frac{t^i}{i!} \varphi^i(z_0) + (\varphi_0^k(z_0) + \varphi_1^k(z_0) + \ldots + \varphi_{m-1}^k(z_0)) \frac{t^k}{k!} + \\ + \int_0^t (t - \tau)^{k-1} \xi(\tau) \, d\tau ,$$

where $\xi(\tau)$, $0 \leqq \tau \leqq t$ is a measurable function with values in $(1/m)\, \varepsilon(z_0)\, \pi S$.

Lemma 3. (cf. [1, § 3]). Let $\lambda > 0$. There exists a measurable function $\xi(\tau)$, $0 \leqq \tau \leqq$ $\leqq \lambda$ with values in $(1/m)\, \varepsilon(z_0)\, \pi S$ such that $\varphi(t, \xi) \neq 0$ for $0 \leqq t \leqq \lambda$.

Proof of the Theorem. Let $z_0 \notin M$ and let $u(t) \in U$, $v(t) \in V$ be measurable controls. Then by the assumptions (2) and (4) the corresponding solution $z(t)$ of the equation (1) is such that $\pi\, z(t)$ is of the class C^k and

$$\frac{d^i}{dt^i} \pi\, z(t)\big|_{t=0} = \varphi^i(z_0) , \quad i = 0, 1, \ldots, k - 1 ,$$

and by Taylor's formula

$$\pi\, z(t) = \sum_{i=0}^{k-1} \frac{t^i}{i!} \varphi^i(z_0) + \frac{1}{(k-1)!} \int_0^t (t - \tau)^{k-1} f^k(z(\tau), u(\tau), v(\tau)) \, d\tau = \\ = \sum_{i=0}^{k-1} \frac{t^i}{i!} \varphi^i(z_0) + \Big(\sum_{i=0}^{m-1} \varphi_i^k(z_0)\Big) \frac{t^k}{k!} + \\ + \frac{1}{(k-1)!} \int_0^t (t - \tau)^{k-1} \Big[f^k(z(\tau), u(\tau), v(\tau)) - \sum_{i=0}^{m-1} \varphi_i^k(z_0)\Big] \, d\tau . \tag{8}$$

Let $\delta_j(z_0)$, $j = 0, 1, \ldots, m - 1$ be the diameter of the maximal sphere where the function $v_{j+1}(z, u, v_1, \ldots, v_j, \xi \mid z_0)$ is continuous (cf. Lemma 1). Denote by $\tau_j(z_0)$

the maximal time during which the solution $z(t)$, $z(0) = z_0$ of the system (1) does not leave this sphere. By Lemma 2 $\delta_j(z_0) \geqq \frac{1}{2}\varepsilon_X^j$ and by the Gronwall's lemma $\tau_j(z_0) \geqq \geqq \tau_X^j > 0$. Denote $\varepsilon_X = \min_j \varepsilon_X^j$, $z_0 = \min_j \tau_j(z_0)$, $\delta(z_0) = \min_j \delta_j(z_0)$.

By Lemma 3, it is possible to choose a measurable function $\bar{\xi}(t)$, $0 \leqq t \leqq \tau(z_0)$ with values in $(1/m)\,\varepsilon(z_0)\,\pi S$ such that $\varphi(t, \bar{\xi}) \neq 0$ on $(0, \tau(z_0)]$.

Denote $v(z, u, \xi \mid z_0) = (v_1(z, u, \xi \mid z_0), \ldots, v_2(z, u, v_1(z, u, \xi \mid z_0), \ \xi \mid z_0), \ldots \ldots, v_m(z, u, v_1(z, u, \xi \mid z_0), \ldots, \xi \mid z_0)$. By Lemma 1 this function is defined and continuous for all $\xi \in \pi S$, $u \in U$ and $z \in R^n$ such that $\|z - z_0\| \leqq \delta(z_0)$. Therefore for a given measurable function $u(t) \in U$, $0 \leqq t \leqq \tau(z_0)$ there exists a solution $z(t)$, $0 \leqq t \leqq \tau(z_0)$ of the equation

$$\dot{z} = f(z, u(t), v(z, u(t), \xi(t) \mid z_0)), \qquad z(0) = z_0 \tag{9}$$

and we can choose $v(t) = v(z(t), \ u(t), \xi(t) \mid z_0)$. The definition of $v(z, u, \xi \mid z_0)$ implies the following equalities: $g_j^k(z(\tau), u(\tau), v_1(\tau), \ldots, v_{j+1}(\tau)) - \varphi_j^k(z_0) = 1/m\,.\,\varepsilon(z_0)\,\xi(\tau)$, $j = 0, 1, \ldots, m - 1$. Now using these equalities and the formula (8), we get

$$\pi\, z(t) = \sum_{i=0}^{k-1} \frac{t^i}{i!}\, \varphi^i(z_0) + \Big(\sum_{i=0}^{m-1} \varphi_i^k(z_0)\Big) \frac{t^k}{k!} + \frac{1}{(k-1)!} \int_0^t (t - \tau)^{k-1}\, \bar{\xi}(\tau)\, d\tau\,,$$

where $\bar{\xi}(\tau) = (1/m)\,\varepsilon(z_0)\,\xi(\tau)$ and such that $\varphi(t, \bar{\xi}) = \pi\, z(t) \neq 0$ for all $0 \leqq t \leqq \tau(z_0)$ (cf. Lemma 3) and therefore $z(t) \notin M$ for all $t \in [0, \tau(z_0)]$.

For $t_1 = \tau(z_0)$ we can take $z(t_1)$ instead of the initial point and we can find the strategy of evasion on the interval $[t_1, t_1 + \tau(z_0)]$ by the same construction as before. Therefore we can extend the game for arbitrary long time. This proves the Theorem.

(Received May 6, 1976.)

REFERENCES

[1] В. Н. Пшеничный: О задаче убегания. Кибернетика 1975, 4, 120—127.

RNDr. Milan Medveď, CSc., Matematický ústav SAV (Mathematical Institute — Slovak Academy of Sciences), Obrancov mieru 49, 886 25 Bratislava. Czechoslovakia.