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# Stability Properties of the Discrete Riccati Operator Equation

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Asymptotic stability of solutions of the discrete-time Riccati equation in Hilbert space is studied. It is shown that there exists at most one solution of the algebraic Riccati equation belonging to the so called "associated cone". If such an equilibrium operator exists then any solution of the Riccati equation starting from the associated cone tends to it geometrically in the operator norm. Explicite characterization of the associated cone is given.

## INTRODUCTION

The discrete Riccati – matrix or operator – equations arise in connection with optimal control of stochastic discrete-time systems as well as with filtering problem. The discrete-time matrix Riccati equation has been studied by many authors (the most recent references are [1]–[6]). The discrete Riccati equation in Hilbert space was investigated in papers [7]–[9].

The general discrete-time Riccati equation and the general algebraic Riccati equation can be written respectively as:

(RE) $$K_{n+1} = \mathscr{A}(K_n)\,, \quad K_0 \geqq 0\,, \quad n = 0, 1, \ldots,$$

(ARE) $$K = \mathscr{A}(K)\,, \quad K \geqq 0\,,$$

where $\mathscr{A}$ is a transformation from the cone $\mathscr{K}$, of all selfadjoint nonnegative operators on a given Hilbert space $X$ into $\mathscr{K}$ given by the formula:

$$\mathscr{A}(K) = Q + \pi_1(K) + \Phi^* K(I + D(R + \pi_2(K))^{-1} D^*K)^{-1}, \ \Phi\,, \ K \in \mathscr{K}\,.$$

In this formula $\Phi$ and $Q$ are linear operators defined on $X$ with $Q \geqq 0$. $D$ is a linear operator from a Hilbert space $U$ into $X$ and $R$ is invertible positive definite operator on $U$. Finally $\pi_1$, $\pi_2$ are linear transformations defined on the Banach space $\mathscr{E}$ of all selfadjoint operators on $X$ into respectively $\mathscr{E}$ itself and into the space $\mathscr{E}_{U}$, of all selfadjoint operators on $U$. It is assumed that they are monotonic:

$$\pi_1(K) \geqq 0, \ \pi_2(K) \geqq 0, \quad \text{for all} \quad K \geqq 0 .$$

It was proved in [9, Sec. 3], that if for some $n$, $\mathscr{A}^n(0)$ is a positive definite, invertible operator, shortly if $\mathscr{A}^n(0) > 0$, then:

(1) (ARE) has at most one solution in $\mathscr{K}$;

(2) if $\bar{K} \in \mathscr{K}$ is the solution of (ARE) then $K_n \to \bar{K}$ geometrically in the operator norm.

A necessary and sufficient condition for the existence of a non-negative solution to (ARE) was given in [9] also.

The condition: $\mathscr{A}^n(0) > 0$ for some $n$, is trivially satisfied if $Q > 0$ but in general is not very constructive. In the first section of the paper we shall formulate easily checked condition equivalent to $\mathscr{A}^n(0) > 0$.

In the second section we introduce the so called "associated cone" $\tilde{\mathscr{K}} \subset \mathscr{K}$ and investigate its basic properties. If $\mathscr{A}^n(0) > 0$ for some $n$ then, it turns out, $\tilde{\mathscr{K}} = \mathscr{K}$.

In the third section we prove that properties (1) and (2) are true if, in their formulations, the cone $\mathscr{K}$ is replaced by $\tilde{\mathscr{K}}$. This result and the explicit characterization of the cone $\tilde{\mathscr{K}}$ are the main results of the paper. As a byproduct we obtain some new results even for the matrix Riccati equation. For instance we show that:

If $\dim X < +\infty$ and $\bar{K}$ is the minimal solution of (ARE) then $\mathscr{A}^n(0) \to \bar{K}$ geometrically as $n \to +\infty$.

As an application, in the fourth section, we shall construct an (ARE) with many nonegative solutions.

## 1. PRELIMINARY RESULTS

In this section we prove a theorem which plays an important role in the characterization of the "associated cone" introduced in Sec. 2. In Corollary 1 we obtain easily checked condition equivalent to the condition $\mathscr{A}^n(0) > 0$.

Let us introduce a transformation $F$ from $E$ into $E$ by the formula:

$$F(K) = Q + \pi_1(K) + \Phi^* K \Phi .$$

**Theorem 1.** For every $n = 0, 1, \ldots$ and $K \in \mathscr{K}$

**(a)** $$F^n(K) \geqq \mathscr{A}^n(K) ,$$

(b) $$\mathscr{A}^n(K) \geqq \gamma F^n(K) \quad \text{for some} \quad \gamma = \gamma(n, K) > 0 .$$

The proof will be based on the following lemma.

**Lemma 1.** Let $K$, $S$, $T$ be operators on $X$ such that $K \geqq 0$, $S \geqq T \geqq 0$, then

$$K(I + SK)^{-1} \leqq K(I + TK)^{-1} .$$

Proof. It is easy to check that $A(I + BA)^{-1} = (I + AB)^{-1} A$, provided $I + BA$ or $I + AB$ are invertible operators. Thus

$$K(I + SK)^{-1} = \sqrt{K}\sqrt{(K)}\,(I + S\sqrt{K}\sqrt{K})^{-1} = \sqrt{(K)}\,(I + \sqrt{(K)}\,S\sqrt{K})^{-1}\sqrt{K} ,$$
$$K(I + TK)^{-1} = \sqrt{(K)}\,(I + \sqrt{(K)}\,T\sqrt{K})^{-1}\sqrt{K} .$$

Inequality $T \leqq S$ implies:

$$I + \sqrt{(K)}\,T\sqrt{K} \leqq I + \sqrt{(K)}\,S\sqrt{K} ,$$
$$(I + \sqrt{(K)}\,S\sqrt{K})^{-1} \leqq (I + \sqrt{(K)}\,T\sqrt{K})^{-1} ,$$

Therefore:

$$\sqrt{(K)}\,(I + \sqrt{(K)}\,SK)^{-1}\sqrt{K} \leqq \sqrt{(K)}\,(I + \sqrt{(K)}\,T\sqrt{K})^{-1}\sqrt{K} ,$$

or equivalently

$$K(I + SK)^{-1} \leqq K(I + TK) ,$$

which is the required inequality.

Proof of the theorem. We prove the theorem by induction on $n$. Put $S = D^*(R + \pi_2(K))^{-1} D \geqq 0$, then from Lemma 1:

$$K = K(I + OK)^{-1} \geqq K(I + SK)^{-1} .$$

Thus (a) holds for $n = 1$. To prove (b) choose $\delta > 0$ such that

$$S = D^*(R + \pi_2(K))^{-1} D \leqq \delta I ,$$

then $K(I + SK)^{-1} \geqq K(I + \delta K)^{-1}$ and consequently

$$\mathscr{A}(K) \geqq Q + \pi_1(K) + \Phi^* K(I + \delta K)^{-1} \Phi .$$

Let now $\varepsilon$ be a positive number such that $K \leqq ((1 - \varepsilon)/\varepsilon\delta) I$ (such number always exists), then $K(I + \delta K)^{-1} \geqq \varepsilon K$ and therefore

$$\mathscr{A}(K) \geqq Q + \pi_1(K) + \varepsilon\Phi^* K \Phi \geqq \varepsilon F(K)$$

because $\varepsilon < 1$. This proves (b) for $n = 1$. If (a) and (b) hold for some $n$ then monotony of the transformation $F$ and $\mathscr{A}$ and the first part of the proof imply

$$\mathscr{A}(\mathscr{A}^n(K)) \leqq \mathscr{A}(F^n(K)) \leqq F(F^n(K)),$$

this is exactly inequality (a) for $n + 1$, and

$$\mathscr{A}(\gamma\, F^n(K)) \leqq \mathscr{A}^{n+1}(K),$$
$$\tilde{\gamma}\, F(\gamma\, F^n(K)) \leqq \mathscr{A}^{n+1}(K),$$

for some positive $\tilde{\gamma}$. Since $\gamma \leqq 1$, therefore $\tilde{\gamma}\, F(\gamma\, F^n(K)) > \tilde{\gamma}\gamma\, F^{n+1}(K)$. Thus (b) holds for $n + 1$ too. The proof of the theorem is completed.

**Corollary 1.** The condition $\mathscr{A}^n(0) > 0$ is equivalent to the condition $F^n(0) > 0$.

**Example 1.** Assume that $\pi_1 \equiv 0$, $\pi_2 \equiv 0$ then the condition $F^n(0) > 0$ is exactly the observability condition:

$$\sum_{i=0}^{n-1} \Phi^{*i} Q\, \Phi^i > 0\,.$$

**Example 2.** If $\pi_1(K) = A^*KA$, where $A$ is a linear operator from $X$ into $X$ then the conditions:

$$F^1(0) > 0\,, \quad F^2(0) > 0\,, \quad F^3(0) > 0$$

are respectively:

$$Q > 0\,, \quad Q + A^*QA + \Phi^*Q\,\Phi > 0\,,$$
$$Q + A^*QA + A^{*2}QA^2 + \Phi^*Q\,\Phi + \Phi^{*2}Q\,\Phi^2 + A^*\,\Phi^*Q\,\Phi A + \Phi^*A^*QA\,\Phi > 0\,.$$

## 2. DEFINITION AND CHARACTERIZATIONS OF THE ASSOCIATED CONE

Le us define a set $\bar{\mathscr{K}}$ by the formula $\bar{\mathscr{K}} = \{K \in \mathscr{K};\, K \leqq \gamma\mathscr{A}^n(0)$, for some $\gamma > 0$ and $n\}$. It is obvious that $\bar{\mathscr{K}}$ is a *convex cone*. Moreover it is *invariant* with respect to the transformation $\mathscr{A}$. To see this, assume that $K \leqq \gamma\mathscr{A}^n(0)$ for some $\gamma > 1$ and $n$. Monotony and concavity of $\mathscr{A}$ imply:

$$\mathscr{A}^{n+1}(0) = \mathscr{A}\left(\frac{1}{\gamma}\gamma\mathscr{A}^n(0) + \left(1 - \frac{1}{\gamma}\right)0\right) \geqq \frac{1}{\gamma}\mathscr{A}(\gamma\mathscr{A}^n(0)) + \left(1 - \frac{1}{\gamma}\right)\mathscr{A}(0)\,.$$

Thus $\mathscr{A}(K) \leqq \mathscr{A}(\gamma\mathscr{A}^n(0)) \leqq \gamma\mathscr{A}^{n+1}(0)$ and therefore $\mathscr{A}(K) \in \bar{\mathscr{K}}$.

We will call the cone $\bar{\mathscr{K}}$: the *associated cone* to the Riccati equation (RE). 

We now proceed to prove the following theorems:

**Theorem 2.** Let $\bar{\mathscr{K}}$ be the cone associated to the Riccati equation (RE), then an operator $K \in \mathscr{K}$ belongs to $\bar{\mathscr{K}}$ if and only if:

$$K \leqq \gamma F^n(0) \quad \text{for some} \quad \gamma > 0 \quad \text{and} \quad n , \tag{3}$$

and if and only if

$$\text{Range } K^{1/2} \leqq \text{Range } (F^n(0))^{1/2} \quad \text{for some} \quad n . \tag{4}$$

**Theorem 3.** Assume that $\dim X = \bar{n} < +\infty$, then nonnegative operator $K$ belongs to $\bar{\mathscr{K}}$ if and only if

$$K \leqq \gamma F^{\bar{n}}(0) \quad \text{for some} \quad \gamma > 0 , \tag{5}$$

and if and only if

$$\text{Ker } K \supset \text{Ker } F^{\bar{n}}(0) . \tag{6}$$

Theorem 1 implies that the inequality $K \leqq \gamma F^n(0)$ is true for some $\gamma > 0$ if and only if the inequality $K \leqq \gamma' \mathscr{A}^n(0)$ is true for some $\gamma' > 0$ thus the first part of the theorem follows. The second part is a consequence of the following general result due to R. G. Douglas [10].

Let $A$ and $B$ be two operators on a Hilbert space $X$ then the following two conditions:

$$\text{Range } A \subset \text{Range } B \tag{7}$$

$$\text{there exists} \quad \gamma > 0 \quad \text{such that} \quad AA^* \leqq \gamma BB^* \tag{8}$$

are equivalent.

To obtain (4) it is sufficient to put $A = K^{1/2}$ and $B = (F^n(0))^{1/2}$.

To prove Theorem 3 we shall need the following lemma:

**Lemma 2.** If $A$ and $B$ are nonnegative operators on a finite dimensional space $X$ then conditions:

$$\text{Ker } A \supset \text{Ker } B \tag{9}$$

$$\text{there exists } \gamma > 0 \quad \text{such that} \quad A \leqq \gamma B \tag{10}$$

are equivalent.

Proof. We prove only that (9) implies (10) because the opposite implication is obvious. Define $X_0 = \operatorname{Ker} B$ and let $X_0^{\perp}$ be the orthogonal complement to $X_0$. The subspace $X_0^{\perp}$ is invariant with respect to both operators $A$ and $B$. Since the operator $B$ is invertible on $X_0^{\perp}$, we have, for some $\gamma > 0$ and all $x_1 \in X_0^{\perp}$,

$$(Ax_1, x_1) \leqq \gamma(Bx_1, x_1).$$

Any element $x \in X$ can be represented in the form $x = x_1 + x_2$, $x_1 \in X_0^{\perp}$, $x_2 \in X_0$ and it is easy to see that

$$(Ax, x) = (Ax_1, x_1) \leqq \gamma(Bx_1, x_1) = \gamma(Bx, x).$$

This proves the lemma.

Proof of Theorem 3. Let us remark that the sequence $(F^n(0))_{n=1,2,\ldots}$ is increasing and if, for some $\gamma > 1$ and $n$

$$F^{n+1}(0) \leqq \gamma\, F^n(0) \tag{11}$$

then

$$F^{n+k}(0) \leqq \gamma^k\, F^n(0) \quad \text{for all} \quad k = 1, 2, \ldots. \tag{12}$$

We show (12) only for $k = 2$; the general case follows easily by induction. Since $F$ is a monotonic transformation therefore (11) implies:

$$F^{n+2}(0) \leqq F(\gamma\, F^n(0)) \leqq Q + \gamma\pi_1(F^n(0)) + \gamma\Phi^*\, F^n(0)\, \Phi \leqq$$
$$\leqq \gamma\, (Q + \pi_1(F^n(0)) + \gamma\Phi^*\, F^n(0)\, \Phi) \leqq \gamma\, F^{n+1}(0).$$

Applying (11) again we obtain

$$F^{n+2}(0) \leqq \gamma^2\, F^n(0).$$

From Lemma 2 we have that the sequence of subspaces $\operatorname{Ker} F^n(0)$ is decreasing. Let $\tilde{n}$ be the first natural number such that:

$$\operatorname{Ker} F^{\tilde{n}}(0) = \operatorname{Ker} F^{\tilde{n}+1}(0). \tag{13}$$

Since $\dim X < +\infty$ therefore such number $\tilde{n}$ always exists and $\tilde{n} \leqq \dim X = \bar{n}$. From (13) it follows that for some $\gamma > 0$ $F^{\tilde{n}+1}(0) \leqq \gamma\, F^{\tilde{n}}(0)$ and consequently for all $k : F^{\tilde{n}+k}(0) \leqq \gamma^k\, F^n(0) \leqq \gamma^k\, F^{\bar{n}}(0)$. This proves the first part of the theorem. Lemma 2 and the condition (5) imply (6). The proof of Theorem 3 is complete.

**Example 3.** If $\pi_1 \equiv 0$ and $\Phi = I$ then $\bar{\mathscr{K}} = \{K \geqq 0;\ K \leqq \gamma Q, \text{ for some } \gamma > 0\}$. In this case $\bar{\mathscr{K}} = \mathscr{K}$ if and only if $Q$ is an invertible operator.

**Corollary 2.** If $\dim X < +\infty$ then the associated cone $\bar{\mathscr{K}}$ is closed in the operator norm and in the strong topology. To see this assume that $K_n \to K$ strongly and $K_n \in \bar{\mathscr{K}}$. If $x \in \operatorname{Ker} F^{\bar{n}}(0)$ then $K_n x = 0$ for all $n$, therefore $Kx = 0$ and $\operatorname{Ker} K \supset \operatorname{Ker} F^{\bar{n}}(0)$. Thus $K \in \bar{\mathscr{K}}$.

**Remark 1.** If $\dim X = +\infty$ then, in general, the cone $\bar{\mathscr{K}}$ is not closed in strong topology. The corresponding example can be found in [9, Proposition 3.1].

## 3. THE RICCATI OPERATOR EQUATION IN THE ASSOCIATED CONE

The following theorem is a generalization of Theorem 3.3 in [9] in which the cone $\mathscr{K}$ is replaced by the associated cone $\bar{\mathscr{K}}$.

**Theorem 4.** The (ARE) has at most one solution in the associated cone $\bar{\mathscr{K}}$. If $\bar{K} \in \bar{\mathscr{K}}$ is the solution of (ARE) then, for every initial condition $K_0 \in \bar{\mathscr{K}}$

$$\mathscr{A}^n(K_0) \to \bar{K}\,, \quad \text{as} \quad n \to +\infty\,,$$

geometrically in the operator norm and in the norm generated by $\bar{K}$.

Proof. The proof is a modification of the proof of Theorem 3.3 from [9].

1. *Uniqueness.* Let $K_1, K_2 \in \bar{\mathscr{K}}$ be two solutions of (ARE) and let $K_1 \nleqq K_2$. Define $\bar{t} = \sup\{s; sK_1 \leqq K_2\}$, then $\bar{t} < 1$. Let $\gamma > 0$ and $n$ be numbers such that $\mathscr{A}^n(0) \geqq \gamma K_1$. Then

$$K_2 = \mathscr{A}^n(K_2) \geqq \mathscr{A}^n(0) \geqq \gamma K_1\,,$$

therefore $\bar{t} > 0$. From the concavity of the transformation $\mathscr{A}^n(\cdot)$, (see [9, Theorem 3.1]), we obtain that

$$K_2 = \mathscr{A}^n(K_2) \geqq \mathscr{A}^n(\bar{t}K_1) \geqq \bar{t}\,\mathscr{A}^n(K_1) + (1 - \bar{t})\,\mathscr{A}^n(0) \geqq$$
$$\geqq \bar{t}K_1 + (1 - \bar{t})\,\gamma K_1 \geqq (\bar{t} + (1 - \bar{t})\,\gamma)\,K_1\,.$$

This contradicts the definiton of the number $\bar{t}$. Thus $K_1 = K_2$.

2. *Geometrical convergence.* Let us assume that there exists a solution of (ARE) belonging to $\bar{\mathscr{K}}$. We have just proved that it is unique. Let us denote it by $\bar{K}$ and define, for any $L \in \tilde{\varepsilon} = \{L_1 - L_2; L_1 \in \bar{\mathscr{K}}, L_2 \in \bar{\mathscr{K}}\}$, the norm $\|L\|$ generated by $\bar{K}$ as follows (see [11]):

$$\|L\| = \inf\{\varepsilon > 0;\ -\varepsilon\bar{K} \leqq L \leqq \varepsilon\bar{K}\}\,.$$

Since $\mathscr{A}^n(0) \leqq \bar{K}$ for all $n$ therefore $\|L\| < +\infty$ if $L \in \tilde{\varepsilon}$. If $|L|$ denotes the usual norm of operator $L \in \tilde{\varepsilon}$ then, it is not difficult to see, that

$$|L| \leqq |\bar{K}| \,.\, \|L\| \,.$$

Therefore to prove the second part of the theorem it is sufficient to show that, for any $K_0 \in \mathscr{K}$ . $\|\mathscr{A}^n(K_0) - \bar{K}\| \to 0$ geometrically as $n \to +\infty$. Since the proof is a modification of that given in [9, Theorem 3.3] we restrict ourselves to the most interesting case $K_0 = 0$. Let us define a transformation $\mathscr{W} : \mathscr{K} \to \mathscr{K}$ by the formula $\mathscr{W}(K) = \mathscr{A}^m(K)$ where $m$ is a natural number such that $\bar{K} \leqq \gamma \mathscr{A}^m(0)$ for some $\gamma > 1$. Then $\mathscr{W}$ is a monotonic and concave transformation. Define sequence $(\varrho_n)$ by the formula:

$$\varrho_n = \sup \{\delta > 0; \delta \bar{K} \leqq \mathscr{W}^n(0)\}\,, \quad n = 1, 2, \ldots.$$

By the very definition $(\varrho_n)$ is an increasing sequence $\varrho_1 \geqq 1/\gamma$ and $\mathscr{W}(\varrho_n \bar{K}) \leqq \mathscr{W}^{n+1}(0)$. Concavity of the transformation $\mathscr{W}$ implies that $\mathscr{W}(\varrho_n \bar{K}) = \mathscr{W}(\varrho_n \bar{K} + (1 - \varrho_n)0) \geqq \varrho_n \mathscr{W}(\bar{K}) + (1 - \varrho_n)\,\mathscr{W}(0) \geqq \varrho_n \bar{K} + (1 - \varrho_n)\,(1/\gamma)\bar{K}$ and consequently

$$\varrho_{n+1} \geqq \varrho_n + (1/\gamma)\,(1 - \varrho_n)\,, \quad n = 1, 2, \ldots.$$

From the last inequality we obtain

$$1 - \varrho_n \leqq \left(1 - \frac{1}{\gamma}\right)^n, \quad n = 1, 2, \ldots,$$

and

$$\|\bar{K} - \mathscr{W}^n(0)\| \leqq \left(1 - \frac{1}{\gamma}\right)^n.$$

But $\mathscr{W}^n(0) = \mathscr{A}^{mn}(0)$ thus for all $n = 1, \ldots,$ it holds

$$\|\bar{K} - \mathscr{A}^{mn}(0)\| \leqq \left(1 - \frac{1}{\gamma}\right)^n.$$

Therefore for any $k = 1, 2, \ldots$

$$\|\bar{K} - \mathscr{A}^k(0)\| \leqq \left(1 - \frac{1}{\gamma}\right)^{[k/m]m} \leqq \left(\frac{\gamma}{\gamma - 1}\right)^m \left(1 - \frac{1}{\gamma}\right)^k, \quad k = 1, \ldots,$$

where $\gamma$ and $m$ are numbers such that

$$\bar{K} \leqq \gamma \mathscr{A}^m(0)\,.$$

**Remark 2.** If $\bar{K} = \mathscr{A}(\bar{K})$ and $\bar{K} \in \mathscr{K}$ then $\mathscr{K} = \{K \geqq 0;\ K \leqq \gamma \bar{K},$ for some $\gamma > 0\}$ and the cone $\mathscr{K}$ is closed in the norm $\|\cdot\|$ generated by $\bar{K}$.

**Corollary 4.** Assume $\dim X < +\infty$ then the minimal solution of (ARE), if it exists, belongs to $\bar{\mathscr{K}}$ and the iterates $\mathscr{A}^n(0)$, $n = 1, 2, \ldots$, tend to it geometrically.

Proof. Since the minimal solution $\bar{K}$ is the strong limit of the sequence $(\mathscr{A}^n(0))_{n=1,\ldots}$ and since the cone $\bar{\mathscr{K}}$ is closed (see Corollary 2) therefore $\bar{K} \in \bar{\mathscr{K}}$. The remaining part of the corollary follows now from Theorem 4.

## 4. AN APPLICATION

As an application of the developed theory we shall construct (ARE) with many nonnegative solutions.

**Proposition 1.** Assume $\dim X < +\infty$, $\pi_1, \pi_2 \equiv 0$ and $\Phi = I$, $R = I$. Then

(a) $\bar{\mathscr{K}} = \{K \geqq 0;\ K \leqq \gamma Q \quad \text{for some} \quad \gamma > 0\}$ ;

(b) $\bar{\mathscr{K}} = \mathscr{K}$ if and only if $Q$ is invertible ;

(c) the unit operator $I$ is a solution of (ARE):

$$K = Q + K(I + DD^*K)^{-1}\,, \quad K \geqq 0 \tag{14}$$

if and only if

$$DD^* = Q + Q^2 + Q^3 + \ldots = (I - Q)^{-1} - I\,;$$

(d) if $DD^* + I = (I - Q)^{-1}$ and $Q$ is not an ivertible operator then the equation (14) has at least two solutions.

Proof. Since $F^n(0) = nQ$ therefore (a) follows; the point (b) is the direct consequence of (a). If we substitute in (14) the operator $I$ for $K$ we obtain $I = Q + (I + DD^*)^{-1}$. Thus $Q \leqq I$ and $I - Q$ is an invertible operator. This is possible if and only if $|Q| < 1$. We obtain also that $I + DD^* = Q + Q^2 + \ldots$. This is exactly (c). If $DD^* + I = (I - Q)^{-1}$ then the unit operator $I$ is a solution of the equation (14). If in addition $Q$ is not invertible then the operator $I$ does not belong to the cone $\bar{\mathscr{K}}$, thus it is not the minimal solution of (14). This proves (d).

## CONCLUSION

We have shown that the associated cone plays a rather important role in the study of the discrete Riccati operator equation. It seems true that all results obtained in the paper can be proved, after obvious reformulation, also for continuous time Riccati equation.

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