Now the optimal decoupled control problems considered in the paper can be formally defined as follows.

(6.47) *Decoupling and stable time optimal control*

Given a system $\mathscr{S}$ which is a minimal realization of

$$S = B_1 A_2^{-1} = A_1^{-1} B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}$$

and a reference input sequence

$$W = \frac{Q}{p} \in \mathfrak{F}_{l,1}\{z^{-1}\}\,.$$

Find a controller $\mathscr{R}$ which is a minimal realization of some

$$R \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

such that the closed-loop system is stably decoupled, the control sequence U is stable, and the error component e_i, $i = 1, 2, \ldots, l$ vanishes in a minimum time $k_{i\min}$ and thereafter.

(6.49) *Decoupling and finite time optimal control*

Given a system $\mathscr{S}$ which is a minimal realization of

$$S = B_1 A_2^{-1} = A_1^{-1} B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}$$

and a reference input sequence

$$W = \frac{Q}{p} \in \mathfrak{F}_{l,1}\{z^{-1}\}\,.$$

Find a controller $\mathscr{R}$ which is a minimal realization of some

$$R \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

such that the closed-loop system is stably decoupled, the control sequence U is finite, and the error component e_i, $i = 1, 2, \ldots, l$ vanishes in a minimum time $k_{i\min}$ and thereafter.

(6.50) *Decoupling and least squares control*

Given a system $\mathscr{S}$ which is a minimal realization of

$$S = B_1 A_2^{-1} = A_1^{-1} B_2 \in \mathfrak{F}_{l,m}\{z^{-1}\}$$

and a reference input sequence

$$W = \frac{Q}{p} \in \mathfrak{F}_{l,1}\{z^{-1}\} .$$

Find a controller $\mathscr{R}$ which is a minimal realization of some

$$R \in \mathfrak{F}_{m,l}\{z^{-1}\}$$

such that the closed-loop system is stably decoupled, the control sequence U is stable, and the quadratic norm $\|e_i\|^2$ of the i-th error e_i, $i = 1, 2, \ldots, l$, is minimized.

The solution of these problems is given in the following three subsections.

Stable time optimal decoupled control

Theorem 6.6. *Let $\mathfrak{F}$ be an arbitrary field with valuation $\mathscr{V}$ and let the closed-loop system can be stably decoupled. Then problem (6.48) has a solution if and only if the linear Diophantine equations*

$$(6.51) \qquad b_i^- x_i + a_{i0}^- p_i y_i = q_i^+ , \quad i = 1, 2, \ldots, l ,$$

have solutions x_i^0, y_i^0 such that

$$\partial y_i^0 = \min , \quad i = 1, 2, \ldots, l ,$$

subject to

$$(6.52) \qquad \frac{s_i}{r_i} = \frac{a_{i0}^+ x_i^0}{p_{i0} b_i^+ y_i^0} , \quad i = 1, 2, \ldots, l ,$$

and to stability of the resulting control sequence

$$U = A_2 M_1 \frac{Q}{p} .$$

The optimal controller is not unique, in general, and all optimal controllers are given as minimal realizazions of (6.46), where the matrices involved satisfy (6.42) through (6.45) and (6.47), (6.52).

Moreover,

$$e_i = a_{i0}^- q_i^- y_i^0 , \quad i = 1, 2, \ldots, l ,$$

and

$$k_{i\min} = 1 + \partial a_{i0}^- + \partial q_i^- + \partial y_i^0 .$$

Proof. If the system can be stably decoupled then all decoupling controllers are given by (6.46). It remains to further specify the M_1, N_1 and M_2, N_2 by choosing the D_1 and D_2 so as to make the i-th error component e_i, $i = 1, 2, \ldots, l$, vanish in a minimum time by application of a stable control sequence U.

Viewing

$$S_i = \frac{b_i}{a_i}$$

as a virtual single-input single -output subsystem and applying Theorem 1 in [32], the controller

$$R_i = \frac{a_{i0}^{+} x_i}{p_{i0} b_i^{+} y_i}$$

where x_i, y_i is any solution of equation (6.51), acomplishes the job.

We have denoted

$$D_1 D_2^{-1} = \operatorname{diag}\left\{\frac{s_i}{r_1}, \frac{s_2}{r_2}, \ldots, \frac{s_l}{r_l}\right\},$$

where the s_i/r_i plays the role of the virtual controller R_i. Hence we have to restrict D_1 and D_2 so that

$$\frac{s_i}{r_i} = \frac{a_{i0}^{+} x_i}{p_{i0} b_i^{+} y_i}, \quad i = 1, 2, \ldots, l. \tag{6.53}$$

It follows that

$$e_i = a_{i0}^{-} q_i^{-} y_i .$$

In order to make the e_i vanish in a minimum time, we have to choose the solution x_i^0, y_i^0 of equation (6.51) that satisfies $\partial y_i^0 = \min$ subject to (6.53) and to stability of the resulting control sequence

$$U = A_2 M_1 \frac{Q}{p},$$

as required. Then

$$k_{i\min} = 1 + \partial e_i = 1 + \partial a_{i0}^{-} + \partial q_i^{-} + \partial y_i^0, \quad i = 1, 2, \ldots, l.$$

Therefore, only the D_1 and D_2 satisfying (6.47) and (6.52) can be used in expressions (6.45) and (6.46) for the optimal controller. □

Example 6.6. Given a minimal realization of

$$S = \frac{\begin{bmatrix} z^{-1} & -z^{-2} \\ 0{\cdot}5z^{-2} & z^{-1} - z^{-2} - 0{\cdot}25z^{-3} \end{bmatrix}}{1 - 0{\cdot}5z^{-1}} =$$

$$= \begin{bmatrix} z^{-1} & 0 \\ 0{\cdot}5z^{-2} & z^{-1}(1 - 0{\cdot}5z^{-1}) \end{bmatrix} \begin{bmatrix} 1 - 0{\cdot}5z^{-1} & z^{-1} \\ 0 & 1 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1 - 0{\cdot}5z^{-1} & 0 \\ -0{\cdot}5z^{-1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} & -z^{-2} \\ 0 & z^{-1}(1 - 0{\cdot}5z^{-1}) \end{bmatrix}$$

over the field $\mathfrak{R}$ valuated by (2.25), solve problem (6.48) for the reference input sequence

$$W = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1 - 0{\cdot}5z^{-1}} .$$

We first solve the decoupling problem. Compute

$$B_{11} = \begin{bmatrix} z^{-2} & 0 \\ 0{\cdot}5z^{-2} & z^{-1}(1 - 0{\cdot}5z^{-1}) \end{bmatrix}, \quad \operatorname{adj} B_{11} = \begin{bmatrix} z^{-1}(1 - 0{\cdot}5z^{-1}) & 0 \\ -0{\cdot}5z^{-2} & z^{-1} \end{bmatrix},$$

$$\det B_{11} = z^{-2}(1 - 0{\cdot}5z^{-1}),$$

$$b_{11} = z^{-1}, \qquad b_{12} = z^{-1},$$

$$b_{01} = z^{-1}(1 - 0{\cdot}5z^{-1}), \quad b_{02} = z^{-1}(1 - 0{\cdot}5z^{-1})$$

and

$$A_1 = \begin{bmatrix} 1 - 0{\cdot}5z^{-1} & 0 \\ -0{\cdot}5z^{-1} & 1 \end{bmatrix}, \quad \operatorname{adj} A_1 = \begin{bmatrix} 1 & 0 \\ 0{\cdot}5z^{-1} & 1 - 0{\cdot}5z^{-1} \end{bmatrix},$$

$$\det A_1 = 1 - 0{\cdot}5z^{-1},$$

$$a_{11} = 1, \qquad a_{12} = 1$$

$$a_{01} = 1 - 0{\cdot}5z^{-1}, \quad a_{02} = 1 - 0{\cdot}5z^{-1}.$$

Equation (6.44) becomes

$$\begin{bmatrix} z^{-1}(1 - 0{\cdot}5z^{-1}) & 0 \\ 0 & z^{-1}(1 - 0{\cdot}5z^{-1}) \end{bmatrix} D_1 + D_2 \begin{bmatrix} 1 - 0{\cdot}5z^{-1} & 0 \\ 0 & 1 - 0{\cdot}5z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

and hence

$$D_1 = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}, \; D_2 = \begin{bmatrix} \dfrac{1}{1 - 0{\cdot}5z^{-1}} - z^{-1}t_1 & 0 \\ 0 & \dfrac{1}{1 - 0{\cdot}5z^{-1}} - z^{-1}t_2 \end{bmatrix}$$

for arbitrary $t_1, t_2 \in \mathfrak{R}^+\{z^{-1}\}$. This equation is equivalent to the first equation (6.42) together with (6.45).

The second equation (6.42) becomes

$$\begin{bmatrix} 1 - 0{\cdot}5z^{-1} & z^{-1} \\ 0 & 1 \end{bmatrix} N_2 + M_2 \begin{bmatrix} z^{-1} & -z^{-2} \\ 0 & z^{-1}(1 - 0{\cdot}5z^{-1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and it yields

$$N_2 = \begin{bmatrix} 1 + z^{-1}v_{11} & -z^{-1} - z^{-2}v_{11} + z^{-1}v_{12} \\ z^{-1}v_{21} & 1 - z^{-2}v_{21} + z^{-1}(1 - 0{\cdot}5z^{-1})\, v_{22} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0{\cdot}5 - (1 - 0{\cdot}5z^{-1})\, v_{11} - z^{-1}v_{21} & -v_{12} - z^{-1}v_{22} \\ -v_{21} & -v_{22} \end{bmatrix}$$

for arbitrary $v_{ij} \in \mathfrak{R}^+\{z^{-1}\}$.

The mutual relations (6.43) then give

$$v_{11} = \frac{0{\cdot}5}{1 - 0{\cdot}5z^{-1}} - t_1, \quad v_{12} = 0,$$

$$v_{21} = \frac{z^{-1}}{1 - 0{\cdot}5z^{-1}}(t_1 - t_2), \quad v_{22} = -t_2.$$

Therefore, all controllers that stably decouple the closed-loop system are given as minimal realizations of

$$R = \begin{bmatrix} 1 - 0{\cdot}5z^{-1} & z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 - 0{\cdot}5z^{-1} & 0 \\ -0{\cdot}5z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 - 0{\cdot}5z^{-1} & 0 \\ 0 & 1 - 0{\cdot}5z^{-1} \end{bmatrix}^{-1} D_1 D_2^{-1}$$

by (6.39).

For control purposes we introduce the virtual systems

$$(6.54) \qquad S_1 = \frac{z^{-1}(1 - 0{\cdot}5z^{-1})}{1 - 0{\cdot}5z^{-1}} = z^{-1}, \quad S_2 = \frac{z^{-1}(1 - 0{\cdot}5z^{-1})}{1 - 0{\cdot}5z^{-1}} = z^{-1}$$

and the virtual controllers

$$R_1 = \frac{(1 - 0{\cdot}5z^{-1})\, t_1}{1 - z^{-1}(1 - 0{\cdot}5z^{-1})t_1}, \quad R_2 = \frac{(1 - 0{\cdot}5z^{-1})\, t_2}{1 - z^{-1}(1 - 0{\cdot}5z^{-1})\, t_2}.$$

Partitioning W conformably, we obtain

$$W_1 = \frac{1}{1 - 0{\cdot}5z^{-1}}, \quad W_2 = \frac{1}{1 - 0{\cdot}5z^{-1}}$$

The equations (6.51) read

$$z^{-1}x_i + (1 - 0{\cdot}5z^{-1})\, y_i = 1, \; i = 1, 2,$$

and the general solutions are

$$x_i = 0{\cdot}5 + (1 - 0{\cdot}5z^{-1})\, v_i,$$
$$y_i = 1 - z^{-1}v_i$$

for any $v_i \in \Re[z^{-1}]$, $i = 1, 2$.

By (6.52) we obtain the equations

$$\frac{(1 - 0{\cdot}5z^{-1})\, t_i}{1 - z^{-1}(1 - 0{\cdot}5z^{-1})\, t_i} = \frac{0{\cdot}5 + (1 - 0{\cdot}5z^{-1})\, v_i}{(1 - 0{\cdot}5z^{-1})(1 - z^{-1}v_i)}, \quad i = 1, 2,$$

which necessitate the choice

$$t_i = \frac{0{\cdot}5}{1 - 0{\cdot}5z^{-1}} + v_i, \quad i = 1, 2.$$

To minimize the degree of y_i, we take $v_i = 0$, $i = 1, 2$. Then

$$x_i^0 = 0{\cdot}5, \quad y_i^0 = 1,$$
$$t_i = \frac{0{\cdot}5}{1 - 0{\cdot}5z^{-1}}, \quad i = 1, 2,$$

and hence

$$D_1 = \begin{bmatrix} \dfrac{0{\cdot}5}{1 - 0{\cdot}5z^{-1}} & 0 \\ 0 & \dfrac{0{\cdot}5}{1 - 0{\cdot}5z^{-1}} \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It follows that

$$R = \frac{0{\cdot}5}{1 - 0{\cdot}5z^{-1}} \begin{bmatrix} 1 - z^{-1} - 0{\cdot}25z^{-2} & z^{-1} \\ -\,0{\cdot}5\, z^{-1} & 1 \end{bmatrix}$$

is the unique solution and

$$U = \frac{0{\cdot}5}{1 - 0{\cdot}5z^{-1}} \begin{bmatrix} 1 + 0{\cdot}5z^{-1} \\ 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k_{1\min} = 1, \quad k_{2\min} = 1.$$

The reader can verify that the (coupled) stable time optimal control would give the same U and E but, of course, a greater variety of controllers would be available.

Example 6.7. Given a minimal realization of

$$S = \frac{\begin{bmatrix} z^{-1} & 0 \\ z^{-1} & z^{-1}(1-2z^{-1}) \end{bmatrix}}{1-z^{-1}} =$$

$$= \begin{bmatrix} z^{-1} & 0 \\ z^{-1} & z^{-1}(1-2z^{-1}) \end{bmatrix} \begin{bmatrix} 1-z^{-1} & 0 \\ 0 & 1-z^{-1} \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1-z^{-1} & 0 \\ -1+z^{-1} & 1-z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1-2z^{-1}) \end{bmatrix}$$

over the field $\mathfrak{R}$ valuated by (2.25), solve problem (6.48) for the reference input sequence

$$W = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1-z^{-1}} .$$

Compute

$$\operatorname{adj} B_{11} = \begin{bmatrix} z^{-1}(1-2z^{-1}) & 0 \\ -z^{-1} & z^{-1} \end{bmatrix},$$

$$b_{11} = z^{-1}, \qquad b_{12} = z^{-1},$$

$$b_{01} = z^{-1}(1-2z^{-1}), \quad b_{02} = z^{-1}(1-2z^{-1})$$

and

$$\operatorname{adj} A_1 = \begin{bmatrix} 1-z^{-1} & 0 \\ 1-z^{-1} & 1-z^{-1} \end{bmatrix},$$

$$a_{11} = 1-z^{-1}, \quad a_{12} = 1-z^{-1},$$

$$a_{01} = 1-z^{-1}, \quad a_{02} = 1-z^{-1}.$$

Equations (6.42) become

$$\begin{bmatrix} z^{-1} & 0 \\ z^{-1} & z^{-1}(1-2z^{-1}) \end{bmatrix} M_1 + N_1 \begin{bmatrix} 1-z^{-1} & 0 \\ -1+z^{-1} & 1-z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1-z^{-1} & 0 \\ 0 & 1-z^{-1} \end{bmatrix} N_2 + M_2 \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1-2z^{-1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and have the solutions

$$M_1 = \begin{bmatrix} 1 + (1 - z^{-1})(t_{11} - t_{12}) & (1 - z^{-1}) t_{12} \\ 1 + (1 - z^{-1})(t_{21} - t_{22}) & -1 + (1 - z^{-1}) t_{22} \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 1 - z^{-1}t_{11} & -z^{-1}t_{12} \\ 1 - z^{-1}t_{11} - z^{-1}(1 - 2z^{-1}) t_{21} & 1 + 2z^{-1} - z^{-1}t_{12} - (z^{-1} - 2z^{-2}) t_{22} \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 1 + z^{-1}v_{11} & z^{-1}(1 - 2z^{-1}) v_{12} \\ z^{-1}v_{21} & 1 + 2z^{-1} + z^{-1}(1 - 2z^{-1}) v_{22} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 - (1 - z^{-1}) v_{11} & -(1 - z^{-1}) v_{22} \\ -(1 - z^{-1}) v_{21} & -1 - (1 - z^{-1}) v_{22} \end{bmatrix}$$

for arbitrary $t_{ij}, v_{ij} \in \mathfrak{R}^+\{z^{-1}\}$.

The mutual relations (6.43) yield

$$v_{ij} = -t_{ij}, \quad i, j = 1, 2.$$

Equation (6.44) becomes

$$\begin{bmatrix} z^{-1}(1 - 2z^{-1}) & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} D_1 + D_2 \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and yields

$$(6.56) \quad D_1 = \begin{bmatrix} -1 + (1 - z^{-1}) t_1 & 0 \\ 0 & -1 + (1 - z^{-1}) t_2 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 1 + 2z^{-1} - z^{-1}(1 - 2z^{-1}) t_1 & 0 \\ 0 & 1 + 2z^{-1} - z^{-1}(1 - 2z^{-1}) t_2 \end{bmatrix}$$

for any $t_1, t_2 \in \mathfrak{R}^+\{z^{-1}\}$. Relations (6.45) then give

$$t_{11} = -2 + (1 - 2z^{-1}) t_1 + t_2, \quad t_{12} = t_2,$$

$$t_{21} = t_1 + t_2, \qquad t_{22} = t_2$$

and all controllers that stably decouple the closed-loop system are given as minimal realization of

$$R = \begin{bmatrix} 1 - 2z^{-1} & 0 \\ -1 & 1 \end{bmatrix} D_1 D_2^{-1},$$

where D_1 and D_2 are given in (6.56).

To solve the control problem, consider

$$S_i = \frac{z^{-1}(1 - 2z^{-1})}{1 - z^{-1}}, \quad W_i = \frac{1}{1 - z^{-1}}, \quad i = 1, 2,$$

and solve the equation (6.51),

$$z^{-1}(1 - 2z^{-1})\, x_i + (1 - z^{-1})\, y_i = 1, \quad i = 1, 2.$$

The general solutions are

$$x_i = -1 + (1 - z^{-1})\, u_i,$$

$$y_i = 1 + 2z^{-1} - z^{-1}(1 - 2z^{-1})\, u_i$$

for any $u_i \in \Re[z^{-1}]$, $i = 1, 2$.

Relations (6.52) yield

$$\frac{-1 + (1 - z^{-1})\, \boldsymbol{t}_i}{1 + 2z^{-1} - z^{-1}(1 - 2z^{-1})\, \boldsymbol{t}_i} = \frac{-1 + (1 - z^{-1})\, u_i}{1 + 2z^{-1} - z^{-1}(1 - 2z^{-1})\, u_i},$$

that is,

$$\boldsymbol{t}_i = u_i, \quad i = 1, 2.$$

To minimize the degree of y_i, we take $u_i = 0$. Then

$$x_i^0 = -1, \quad y_i^0 = 1 + 2z^{-1}, \quad i = 1, 2,$$

$$\boldsymbol{D}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \boldsymbol{D}_2 = \begin{bmatrix} 1 + 2z^{-1} & 0 \\ 0 & 1 + 2z^{-1} \end{bmatrix}$$

and the optimal controller is given as a minimal realization of

$$\boldsymbol{R} = \begin{bmatrix} 1 - 2z^{-1} & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 - 2z^{-1} & 0 \\ 0 & -1 - 2z^{-1} \end{bmatrix}^{-1}$$

and it is unique. The resulting control

$$\boldsymbol{U} = \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix} \begin{bmatrix} -1 + 2z^{-1} & 0 \\ 1 & -1 \end{bmatrix} \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1 - z^{-1}} = \begin{bmatrix} -1 + 2z^{-1} \\ 0 \end{bmatrix}$$

is stable, as required, and the error becomes

$$\boldsymbol{E} = \begin{bmatrix} 1 + 2z^{-1} \\ 1 + 2z^{-1} \end{bmatrix}, \quad k_{1\min} = k_{2\min} = 2.$$

We also obtain

$$K_1 = \begin{bmatrix} -z^{-1}(1 - 2z^{-1}) & 0 \\ 0 & -z^{-1}(1 - 2)z^{-1} \end{bmatrix}.$$

It is easy to verify that the (coupled) stable time optimal control is obtained when using

$$R = \begin{bmatrix} 1 & (1 - z^{-1})\, w_1 \\ 0 & -1 + (1 - z^{-1})\, w_2 \end{bmatrix} \begin{bmatrix} 1 & -z^{-1} w_1 \\ 1 & 1 + 2z^{-1} - z^{-1} w_1 - z^{-1}(1 - z^{-1}) w_2 \end{bmatrix}^{-1}$$

where $w_1, w_2 \in \Re^+\{z^{-1}\}$ arbitrary. It follows that

$$U = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k_{\min} = 1$$

but the associated matrix

$$K_1 =$$

$$= \begin{bmatrix} z^{-1} - z^{-1}(1 - z^{-1})\, w_1 & z^{-1}(1 - z^{-1})\, w_1 \\ 2z^{-1}(1 - z^{-1}) - z^{-1}(1 - z^{-1})\, w_1 - & -z^{-1}(1 - 2z^{-1}) + z^{-1}(1 - z^{-1})\, w_1 + \\ - z^{-1}(1 - z^{-1})(1 - 2z^{-1})\, w_2 & + z^{-1}(1 - z^{-1})(1 - 2z^{-1})\, w_2 \end{bmatrix}$$

cannot be made diagonal for any choice of w_1, w_2.

Note that the decoupled control is inferior to the coupled one due to the requirement of diagonality.

Finite time optimal decoupled control

Theorem 6.7. *Let $\mathfrak{F}$ be an arbitrary field with valuation $\mathcal{V}$ and let the closed-loop system can be stably decoupled. Then problem (6.49) has a solution if and only if the linear Diophantine equations*

(6.57) $$b_i x_i + a_{i0}^- p_i y_i = q_i^+ , \quad i = 1, 2, \ldots, l ,$$

have solutions x_i^0, y_i^0 such that

$$\partial y_i^0 = \min , \quad i = 1, 2, \ldots, l ,$$

subject to

(6.58) $$\frac{s_i}{r_i} = \frac{a_{i0}^+ x_i^0}{p_{i0} y_i^0} , \quad i = 1.\, 2, \ldots, l ,$$

and to finiteness of the resulting control sequence

$$U = A_2 M_1 \frac{Q}{p}.$$

The optimal controller is not unique, in general, and all optimal controllers are given as minimal realizations of (6.46) *where the matrices involved satisfy* (6.42) *through* (6.45) *and* (6.47), (6.58).

Moreover,

$$e_i = a_{i0}^- q_i^- y_i^0, \quad i = 1, 2, \ldots, l,$$

and

$$k_{i\min} = 1 + \partial a_{i0}^- + \partial q_i^- + \partial y_i^0.$$

Proof. If the system can be stably decoupled then all decoupling controllers are given by (6.46). It remains to further specify the M_1, N_1 and M_2, N_2 by choosing the D_1 and D_2 so as to make the i-th error component e_i, $i = 1, 2, \ldots, l$, vanish in a minimum time by application of a finite control sequence U.

Viewing

$$S_i = \frac{b_i}{a_i}, \quad i = 1, 2, \ldots, l,$$

as a virtual single-input single-output subsystem and applying Theorem 2 in [32], the reasoning analogous to Theorem 6.6 proves our claim. □

Example 6.8. Consider the finite automaton that is a minimal realization of

$$S = \frac{\begin{bmatrix} z^{-1} & z^{-1} & z^{-1} \\ 0 & 0 & z^{-1} + z^{-1} \end{bmatrix}}{1 + z^{-1}} = \begin{bmatrix} z^{-1} & 0 & 0 \\ 0 & z^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 + z^{-1} & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1 + z^{-1} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} & z^{-1} & z^{-1} \\ 0 & 0 & z^{-1} \end{bmatrix}$$

over the field $\mathfrak{Z}_2$ valuated by (2.24) and solve problem (6.49) for the reference input sequence

$$w = \frac{\begin{bmatrix} 1 \\ z^{-1} + z^{-2} \end{bmatrix}}{1 + z^{-1}}.$$

Equations (6.42) become

$$\begin{bmatrix} z^{-1} & 0 & 0 \\ 0 & z^{-1} & 0 \end{bmatrix} \boldsymbol{M}_1 + \boldsymbol{N}_1 \begin{bmatrix} 1 + z^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 + z^{-1} & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{N}_2 + \boldsymbol{M}_2 \begin{bmatrix} z^{-1} & z^{-1} & z^{-1} \\ 0 & 0 & z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and have the solutions

$$\boldsymbol{M}_1 = \begin{bmatrix} 1 + (1 + z^{-1})\boldsymbol{t}_{11} & \boldsymbol{t}_{12} \\ (1 + z^{-1})\boldsymbol{t}_{21} & \boldsymbol{t}_{22} \\ \boldsymbol{t}_{31} & \boldsymbol{t}_{32} \end{bmatrix}, \quad \boldsymbol{N}_1 = \begin{bmatrix} 1 + z^{-1}\boldsymbol{t}_{11} & z^{-1}\boldsymbol{t}_{12} \\ z^{-1}\boldsymbol{t}_{21} & 1 + z^{-1}\boldsymbol{t}_{22} \end{bmatrix},$$

$$\boldsymbol{N}_2 = \begin{bmatrix} 1 + z^{-1}\boldsymbol{v}_{11} & 1 + z^{-1}\boldsymbol{v}_{11} & 1 + z^{-1}(\boldsymbol{v}_{11} + \boldsymbol{v}_{12}) \\ z^{-1}\boldsymbol{v}_{21} & z^{-1}\boldsymbol{v}_{21} & 1 + z^{-1}(\boldsymbol{v}_{21} + \boldsymbol{v}_{22}) \\ z^{-1}\boldsymbol{v}_{31} & 1 + z^{-1}\boldsymbol{v}_{31} & z^{-1}(\boldsymbol{v}_{31} + \boldsymbol{v}_{32}) \end{bmatrix},$$

$$\boldsymbol{M}_2 = \begin{bmatrix} 1 + (1 + z^{-1})\,\boldsymbol{v}_{11} + \boldsymbol{v}_{21} + \boldsymbol{v}_{31} & (1 + z^{-1})\,\boldsymbol{v}_{12} + \boldsymbol{v}_{22} + \boldsymbol{v}_{32} \\ \boldsymbol{v}_{31} & \boldsymbol{v}_{32} \\ \boldsymbol{v}_{21} & \boldsymbol{v}_{22} \end{bmatrix}$$

for arbitrary $\boldsymbol{t}_{ij}, \boldsymbol{v}_{ij} \in \mathfrak{Z}_2^+\{z^{-1}\}$. The mutual relations (6.43) yield

$$\boldsymbol{t}_{11} = \boldsymbol{v}_{11}, \qquad \boldsymbol{t}_{12} = \boldsymbol{v}_{12},$$

$$\boldsymbol{t}_{21} = \boldsymbol{v}_{21}, \qquad \boldsymbol{t}_{22} = \boldsymbol{v}_{22},$$

$$\boldsymbol{t}_{31} = \boldsymbol{v}_{31}(1 + z^{-1}), \quad \boldsymbol{t}_{32} = \boldsymbol{v}_{32}.$$

Now compute

$$B_{11} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix}, \quad \operatorname{adj} B_{11} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix},$$

$$\det B_{11} = z^{-1},$$

$$b_{11} = z^{-1}, \quad b_{12} = z^{-1},$$

$$b_{01} = z^{-1}, \quad b_{02} = z^{-1}$$

and

$$A_1 = \begin{bmatrix} 1 + z^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \quad \operatorname{adj} A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 + z^{-1} \end{bmatrix},$$

$$\det A_1 = 1 + z^{-1},$$

$$a_{11} = 1, \qquad a_{12} = 1 + z^{-1},$$

$$a_{01} = 1 + z^{-1}, \quad a_{02} = 1.$$

Then equation (6.44) can be written as

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} D_1 + D_2 \begin{bmatrix} 1 + z^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and its diagonal solution is

$$D_1 = \begin{bmatrix} 1 + (1 + z^{-1})\, t_1 & 0 \\ 0 & t_2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 + z^{-1} t_1 & 0 \\ 0 & 1 + z^{-1} t_2 \end{bmatrix}$$

for any $t_1, t_2 \in \mathfrak{Z}_2^+\{z^{-1}\}$.

Since

$$M_{11} = \begin{bmatrix} 1 + (1 + z^{-1})\, t_{11} & t_{12} \\ (1 + z^{-1})\, t_{21} & t_{22} \end{bmatrix},$$

equations (6.45) yield

$$t_{11} = t_1, \quad t_{12} = 0,$$

$$t_{21} = 0, \quad t_{22} = t_2$$

and all decoupling controllers are given by (6.46), where

$$M_1 = \begin{bmatrix} 1 + (1 + z^{-1})\, t_1 & 0 \\ 0 & t_2 \\ (1 + z^{-1})\, v_{31} & v_{32} \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 + z^{-1} t_1 & 0 \\ 0 & 1 + z^{-1} t_2 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 1 + z^{-1} t_1 & 1 + z^{-1} t_1 & 1 + z^{-1} t_1 \\ 0 & 0 & 1 + z^{-1} t_2 \\ z^{-1} v_{31} & 1 + z^{-1} v_{31} & z^{-1} v_{31} + z^{-1} v_{32} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 + (1 + z^{-1}) t_1 + v_{31} & t_2 + v_{32} \\ v_{31} & v_{32} \\ 0 & t_2 \end{bmatrix}.$$

To solve the control problem, we find the virtual subsystems

$$S_1 = \frac{z^{-1}}{1 + z^{-1}}, \quad S_2 = z^{-1},$$

and

$$W_1 = \frac{1}{1 + z^{-1}}, \quad W_2 = z^{-1}.$$

Then equations (6.57) read

$$z^{-1}x_1 + (1 + z^{-1})\, y_1 = 1\,,$$

$$z^{-1}x_2 + y_2 \qquad\qquad = 1$$

and the general solutions are

$$x_1 = 1 + (1 + z^{-1})\, u_1\,, \quad y_1 = 1 + z^{-1}u_1\,,$$

$$x_2 = u_2\,, \qquad\qquad\qquad y_2 = 1 + z^{-1}u_2$$

for any $u_i \in \mathfrak{Z}_2[z^{-1}]$. Equations (6.58) yield

$$t_i = u_i\,, \quad i = 1, 2\,.$$

The solution x_i^0, y_i^0 with $\partial y_i^0 = \min$ is obtained when setting $u_i = 0$, $i = 1, 2$. Then

$$x_1^0 = 1\,, \quad y_1^0 = 1\,,$$

$$x_2^0 = 0\,, \quad y_2^0 = 1\,,$$

(6.59) $$t_1 = 0\,, \quad t_2 = 0\,,$$

and the optimal controllers are given as minimal realizations of

$$R = M_2 N_1^{-1} = N_2^{-1} M_1 = \begin{bmatrix} 1 + v_{31} & v_{32} \\ v_{31} & v_{32} \\ 0 & 0 \end{bmatrix}$$

on using (6.59). The resulting controls are

$$U = \begin{bmatrix} 1 + v_{31} + z^{-1}v_{33} \\ v_{31} + z^{-1}v_{32} \\ 0 \end{bmatrix}$$

and the optimal error becomes

$$E = \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}, \quad k_{1\min} = 1\,, \quad k_{2\min} = 2\,.$$

Note also that

$$K_1 = \begin{bmatrix} z^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad I_l - K_1 = \begin{bmatrix} 1 + z^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

Least squares decoupled control

Theorem 6.8. *Let $\mathfrak{F}$ be a subfield of $\mathfrak{C}$ valuated by* (2.25) *and let the closed-loop system can be stably decoupled. Then problem* (6.50) *has a solution if and only if the linear Diophantine equations*

$$b_i^- x_i + a_{i0}^- p_i y_i = a_{i0}^{-\sim} q_i^* b_i^{-\sim}\,, \quad i = 1, 2, \ldots, l\,. \tag{6.60}$$

have solutions x_i^0, y_i^0 such that

$$\partial y_i^0 < \partial b_i^-\,, \quad i = 1, 2, \ldots, l\,,$$

subject to

$$\frac{s_i}{r_i} = \frac{a_{i0}^+ x_i^0}{p_{i0} b_i^+ y_i^0}\,, \quad i = 1, 2, \ldots, l\,, \tag{6.61}$$

and to stability of the resulting control

$$U = A_2 M_1 \frac{Q}{p}$$

and the error components

$$e_i = \frac{a_{i0}^- q_i^-}{a_{i0}^{-\sim} q_i^{-\sim}} \frac{y_i^0}{b_i^{-\sim}}\,, \quad i = 1, 2, \ldots, l\,.$$

The optimal controller is not unique, in general, and all optimal controllers are given as minimal realizations of (6.46), *where the matrices involved satisfy* (6.42) *through* (6.45) *and* (6.47), (6.61). *Moreover*

$$\|e_i\|_{\min}^2 = \left\langle \left(\frac{y_i^0}{b_i^-}\right)^{=} \left(\frac{y_i^0}{b_i^-}\right) \right\rangle.$$

Proof. If the system can be stably decoupled then all decoupling controllers are given by (6.46). It remains to further specify the M_1, N_1 and M_2, N_2 by choosing the D_1 and D_2 so as to minimize $\|e_i\|^2$ by application of a stable control sequence U. Viewing

$$S_i = \frac{b_i}{a_i}\,, \quad i = 1, 2, \ldots, l\,,$$

as a single-input single-output virtual subsystem and applying Theorem 3 in [32], the reasoning analogous to Theorem 6.6 proves our claim. □

It is obvious that the decoupled least squares control cannot give results better than the (coupled) least squares control. The diagonality of K_1 is rather a severe restriction.

Example 6.9. Consider a minimal realization of

$$S = \frac{z^{-1}}{1 - z^{-1}} \begin{bmatrix} 1 & 0 \\ 1 & 1 - 2z^{-1} \end{bmatrix} =$$

$$= \begin{bmatrix} z^{-1} & 0 \\ z^{-1} & z^{-1}(1 - 2^{-1}) \end{bmatrix} \begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1 - z^{-1} & 0 \\ -1 + z^{-1} & 1 - z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix}$$

over $\mathfrak{R}$ and solve problem (6.50) for the reference input sequence

$$w = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{1 - z^{-1}} .$$

We first solve equations (6.42) and (6.43). They are

$$\begin{bmatrix} z^{-1} & 0 \\ z^{-1} & z^{-1}(1 - 2z^{-1}) \end{bmatrix} M_1 + N_1 \begin{bmatrix} 1 - z^{-1} & 0 \\ -1 + z^{-1} & 1 - z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix} N_2 + M_2 \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 - z^{-1} & 0 \\ 0 & 1 - z^{-1} \end{bmatrix} M_1 = M_2 \begin{bmatrix} 1 - z^{-1} & 0 \\ -1 + z^{-1} & 1 - z^{-1} \end{bmatrix},$$

$$\begin{bmatrix} z^{-1} & 0 \\ z^{-1} & z^{-1}(1 - 2z^{-1}) \end{bmatrix} N_2 = N_1 \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1}(1 - 2z^{-1}) \end{bmatrix}$$

and they have the solutions

(6.62)

$$M_1 = \begin{bmatrix} 1 + (1 - z^{-1})(t_{11} - t_{12}) & (1 - z^{-1}) t_{12} \\ 1 + (1 - z^{-1})(t_{21} - t_{22}) & -1 + (1 - z^{-1}) t_{11} \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 1 - z^{-1} t_{11} & -z^{-1} t_{12} \\ 1 - z^{-1} t_{11} - z^{-1}(1 - 2z^{-1}) t_{21} & 1 + 2z^{-1} - z^{-1} t_{12} - z^{-1}(1 - 2z^{-1}) t_{22} \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 1 - z^{-1} t_{11} & -z^{-1}(1 - 2z^{-1}) t_{12} \\ - z^{-1} t_{21} & 1 + 2z^{-1} - z^{-1}(1 - 2z^{-1}) t_{22} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 + (1 - z^{-1}) t_{11} & (1 - z^{-1}) t_{12} \\ (1 - z^{-1}) t_{21} & -1 + (1 - z^{-1}) t_{22} \end{bmatrix}$$

for arbitrary $t_{ij} \in \mathfrak{R}^+\{z^{-1}\}$.

Since

$$\operatorname{adj} B_{11} = \begin{bmatrix} z^{-1}(1-2z^{-1}) & 0 \\ -z^{-1} & z^{-1} \end{bmatrix}, \quad \operatorname{adj} A_1 = \begin{bmatrix} 1-z^{-1} & 0 \\ 1-z^{-1} & 1-z^{-1} \end{bmatrix},$$

$$b_{11} = b_{12} = z^{-1}, \qquad a_{11} = a_{12} = 1 - z^{-1},$$

$$b_{01} = b_{02} = z^{-1}(1-2z^{-1}), \quad a_{01} = a_{02} = 1 - z^{-1},$$

equation (6.44) becomes

$$\begin{bmatrix} z^{-1}(1-2z^{-1}) & 0 \\ 0 & z^{-1}(1-2z^{-1}) \end{bmatrix} D_1 + D_2 \begin{bmatrix} 1-z^{-1} & 0 \\ 0 & 1-z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the solution is

$$D_1 = \begin{bmatrix} -1+(1-z^{-1})t_1 & 0 \\ 0 & -1+(1-z^{-1})t_2 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 1+2z^{-1}-z^{-1}(1-2z^{-1})t_1 & 0 \\ 0 & 1+2z^{-1}-z^{-1}(1-2z^{-1})t_2 \end{bmatrix}$$

for any $t_1, t_2 \in \Re^+\{z^{-1}\}$. Equations (6.45) then yield

$$t_{11} = -2 + (1-2z^{-1})t_1, \quad t_{12} = 0, \qquad t_{21} = t_2 - t_1, \quad t_{22} = t_2. \tag{6.63}$$

Thus all decoupling controllers are given as minimal realizations of (6.46), where the matrices involved satisfy (6.62) and (6.63).

To solve the control problem, we shall solve the equations (6.60)

$$z^{-1}(1-2z^{-1})x_i + (1-z^{-1})y_i = z^{-1} - 2, \quad i = 1, 2.$$

We obtain

$$x_i = 1 + (1-z^{-1})u_i,$$

$$y_i = -2 - 2z^{-1} - z^{-1}(1-2z^{-1})u_i, \quad i = 1, 2,$$

and the solution x_i^0, y_i^0 satisfying $\partial y_i^0 < 2$ becomes

$$x_i^0 = 1, \quad y_i^0 = -2 - 2z^{-1}$$

on setting $u_i = 0$, $i = 1, 2$. Then relations (6.61) gives

$$\frac{-1+(1-z^{-1})t_i}{1+2z^{-1}-z^{-1}(1-2z^{-1})t_i} = \frac{1}{-2-2z^{-1}}, \quad i = 1, 2,$$

that is,

$$t_i = \frac{1}{2 - z^{-1}}, \quad i = 1, 2 .$$

Therefore, the optimal controller is unique and it is given as a minimal realization of (6.46), where

$$M_1 = \begin{bmatrix} 1 - 3\dfrac{1 - z^{-1}}{2 - z^{-1}} & 0 \\ 1 - \dfrac{1 - z^{-1}}{2 - z^{-1}} & -1 + \dfrac{1 - z^{-1}}{2 - z^{-1}} \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 1 + 3\dfrac{z^{-1}}{2 - z^{-1}} & 0 \\ 1 + 3\dfrac{z^{-1}}{2 - z^{-1}} & 1 + 2z^{-1} - \dfrac{z^{-1}(1 - 2z^{-1})}{2 - z^{-1}} \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 1 + 3\dfrac{z^{-1}}{2 - z^{-1}} & 0 \\ 0 & 1 + 2z^{-1} - \dfrac{z^{-1}(1 - 2z^{-1})}{2 - z^{-1}} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 - 3\dfrac{1 - z^{-1}}{2 - z^{-1}} & 0 \\ 0 & -1 + \dfrac{1 - z^{-1}}{2 - z^{-1}} \end{bmatrix}.$$

The resulting control is

$$U = M_2 A_1 \frac{Q}{p} = \frac{\begin{bmatrix} 1 - 2z^{-1} \\ 0 \end{bmatrix}}{z^{-1} - 2}$$

and the associated error

$$E = \frac{\begin{bmatrix} -2 - 2z^{-1} \\ -2 - 2z^{-1} \end{bmatrix}}{z^{-1} - 2}, \quad \|e_1\|^2_{\min} = \|e_2\|^2_{\min} = 4 .$$

The reader can verify that the (coupled) least squares control is generated by a minimal realization of (6.46), where the matrices involved are given by (6.62) with $t_{11} = 0$, $t_{21} = 0$. The resulting control and error are

$$U = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \|e_1\|^2_{\min} = \|e_2\|^2_{\min} = 1 .$$

To conclude, it can be said that the decoupled optimal control is nonsuperior to the (coupled) optimal control. Otherwise speaking, the optimal control system cannot always be made diagonal, i.e. the coupling in the closed-loop system may be essential for attaining the optimal performance.

7. CONCLUSIONS

This work has provided a new algebraic theory of discrete linear control for multivariable systems. Unlike the common approaches, the algebraic method is based exclusively on polynomial algebra. This makes it possible to reduce the synthesis procedure for all optimal control problems to solving Diophantine equations in polynomials, thus unifying the procedure and making it as simple as possible.

The present publication together with a series of papers on single-variable systems [30; 31; 32; 33; 34; 35] forms a compact theory of discrete linear control which is fairly general and computationally attractive.

We have discussed the open-loop control strategy, where all signals are known in advance and, hence, the only problem is to ensure the optimality, as well as the closed-loop control strategy, where optimality subject to stability of the closed-loop system is required. Therefore, a whole chapter has been devoted to problems of stability in closed-loop systems and the results contained there are of central importance.

Although the algebraic method has been developed for deterministic control problems, it can equally well be applied to stochastic control problems. This will be done in future publications.