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On Bargaining in Games

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The presented paper deals with the problems of bargaining in coalition-games with side-payments. The autor's aim is to bring a survey of situations which can be described by such games, and to suggest a general game-model, including all modifications of the basic situation. In such a general model, including, e.g., even the possibility of information about the intentions of anti-players, the process of bargaining is investigated, and the expected behaviour of players is described. This attitude, which enables us to explain and forecast, on the base of adequate general model, the behaviour of players, is in [10] called the prediction of the bargaining. We keep this term here as well.

The model of game proposed in this paper includes a wide scale of conflict situations. Not only the process of coalitions forming, as well as the process of side-payments bargaining, is considered here. The investigated type of game includes also the possibility of electing strategies for all coalitions, and, moreover, according to this model each coalition is able to obtain some information (incomplete, may be) on the designs of the anti-coaliton. The proposed model may be applied, as shown in the conclusive part of this work, even in the case of non-cooperative games. The results for two-player games are comparable with the classical ones.

The model suggested here is rather general for obtaining more than the fundamental results. But, it can be specified for many special situations, for which, according to additional assumptions, some interesting results can be obtained. Some of these special types of games are briefly investigated in this work, as well.

The paper is divided into four parts. The first of them introduces the general model of game with its fundamental and obvious properties. The second part deals with the bargaining model in such general game. For simplifying the explanation and the notation, the model is defined for a special type of games with exactly one possible strategy of each admissible coalition. The third part of the work is oriented to special type of games with no information about the intentions of anti-players.

The bargaining model suggested in the second part, can be applied without changes even to the general coalition-games with more than one strategy of any coalition. This fact is shown in the fourth part of this paper. In such a case also the investigation of games with fixed coalitions, and the prediction of strategies only, has its sense, and it is also investigated in the fourth part. Finally, the application of the suggested model to the two-player games is presented.

The presented paper will be divided into a few volumes of the journal, and, consequently, it is reasonable to introduce here some basic information, important for the simplier orientation in the text, even if it ought to be traditionally located in the closing parts of the paper. First of all, the list of parts and paragraphs is included.

The notation used in this paper, is generally consistent in all parts. However, some generalizations and simplification of the basic model cause also some simplifications of used symbols, applied in some parts of the work. For the better reading of all parts we introduce here also the list of important symbols used in the text.

Symbol	Introduced in Part/Section	Comment
N	I 1	
D_L, $\mathbf{D}$, $\mathscr{D}_{\mathscr{K}}$	1, 3	side payments
$\pi(M)$	1	
$\Vert\mathscr{M}\Vert$, $\Vert\mathbf{M}\Vert$	1	
$S(M)$, s	1	
$s^{[m]}$	1	
I	I 2	set of players
$\mathbf{K}$	2	class of all coalition structures
$\mathscr{J}, \mathscr{K}, \mathscr{L}, \mathscr{M}, \ldots$	2	coalition structures
$J, K, L, M, \ldots$	2	coalitions
$A_K, S_K, S_{\mathscr{K}}$	2	sets of strategies
$\mathfrak{s}, s_K$	2	strategies
$(\mathscr{K}, \mathfrak{s}), (K, s_K)$	2	strategic structure
$\mathbf{K}_s$	2	set of strategic structures
$\mathscr{R}_K, R$	2	identification partition and identification set
v_i	2	pay-off function
Γ	2	game
φ_i	I 3	
h_i	3	
(K, s_K, D_K)	3	agreement
$C = C(\mathscr{K}, \mathfrak{s}, \mathbf{D})$	3	configuration
$\mathscr{C}$	3	class of configurations
k	3	
$c = c(C; k)$	3	sub-configuration
x_i	3	profit

Finally, even if it is not usual, we introduce here also the list of references used in the work. It ought to simplify the reader's orientation in them without expecting the publication of the last part of the work.

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PART I: GENERAL COALITION-GAME

0. Descriptive Introduction

In the following section some ideas are given, intuitive in their nature, on which the investigated model of conflict situation, as well as the proposed prediction method, is based.

The basic notion in this paper is that of *game*, defined in the section 2 as a quintuple, containing the following parts: a *set of players*, a *class of the admissible coalition structures*, a *class of the sets of coalition strategies*, a *collection of coalition identification-partitions* and a *class of pay-off functions* of the players.

With the exception of the identification-partitions these notions are well-known and used in the literature (as an admissible coalition structure a partition of the set of players into disjoint admissible coalitions is meant).

The notion of an identification-partition of a given admissible coalition is based on the intuitive imagination, that every coalition is able to obtain, during the process of bargaining, some information, greater or smaller, about the anti-players' action and, in this way, also an information about the expected pay-off, following from the negotiated cooperation. This gives a possibility for the participants of the coalition to react more sensitively on the arisen situations in the game and to obtain, in such a way, a higher profit.

In its nature, the mechanism of the identification lies in the fact, that for every admissible coalition a partition of the set of all game situations in which the coalition can occur (i.e. the set of all coalition structures, containing this coalition, with their vectors of strategies) into a finite number of subsets is defined, supposing that the members of the coalition are able to decide, in every stage of the process of bargaining, in which among these subsets the form of cooperation, negotiated in this moment, can be found.

We suppose, in the presented paper, that the particular parts of a game satisfy the following *game postulates*:

players:

– the set of players is non-empty and finite;

coalition structures:

– the class of admissible coalition structures is always non-empty and finite,
– every coalition structure, composed from admissible coalitions is admissible in the considered game;

strategies:

– every admissible coalition has a non-empty set of strategies to its disposal,

- supposing a new coalition is formed by the union of a number of smaller coalitions, then this new coalition has to its disposal at least all the strategies, the former coalitions had before their union;

identification-partitions:

- an admissible coalition does not change its information about the anti-players action supposing it changes only its strategy,
- supposing a new coalition is formed by the union of a number of smaller coalitions, also the information of particular of the coalition about the anti-players are united;

pay-off functions:

- supposing a new coalition is formed by the union of a number of smaller coalitions, the whole pay-off of its members is at least the same as the total of the whole pay-offs of the members of the former coalitions.

As we suppose that the members of the coalitions are allowed to support the goodwill of their partners to cooperate with them by the side payments, the former postulates can be enriched by the following *side payments postulates:*

- the side payments are negotiated during the process of bargaining, they are supposed to be fixed amounts, independent from the actualy obtained final payoffs of the players;
- the side payments are given to members of a coalition by another members of the same coalition from the private means of the later.

In its nature, the bargaining prediction is based on the idea that the game consists of two stages. In the first stage the players are looking for suitable coalitions, negotiating the possible common strategies and offering side payments. It is possible that a negotiated form of cooperation is changed supposing there occur some other, more suitable, ones. In some stage the negotiation process is finished by the formation of a negotiated form of cooperation and the second stage of the game occurs, in which the definitively negotiated coalitions use their strategies having been negotiated before. After the pay-offs are obtained, the members of the coalition pay to each other the side payments having promissed before.

This serial deals with the first stage of the game, with the bargaining, supposing that as the form of cooperation (obtained in every stage of the process of bargaining) a triple is considered, containing a coalition structure, a vector of strategies of its coalitions and a matrix, representing the negotiated side payments. Those triples are called *configurations.*

There is the goal of the prediction to find such a relation, or system of relations, on the class of all admissible configurations, which would enable to choose a nonempty sub-class of possible and, from the point of view of the players, acceptable results of bargaining. It is useful, moreover, to clasify the elements of the sub-class obtained in such a way, at least roughly, into several groups, according to their stability, i.e. according to the measure, in which they reflect the interests and possibilities of the players.

Such a procedure is proposed in sections 5 and 6, and generalized in section 9.

The basic quantity, which gives the motivation for the action of the players during the bargaining and which must be investigated during the bargaining prediction is the guaranted *profit of the players*. It is defined, for every player, as a function on the class of all admissible configurations and represents the sum of the expected payment of the player, under the given form of cooperation in the coalition and under the given information about the action of the anti-players, with the negotiated side payments.

In order we were able to predict effectively the process and result of the bargaining in a given game, some degree of rationality must be ascribed to the players, rationality, which gives motivation to the players' action during the whole process of bargaining. These requests can be formulated, following the paper [10], in the *postulates on rationality and motivation*:

- the given game is known correctly to every player, i.e. he knows the quintuple defining it,
- a player accepts an offer to cooperate with a coalition, supposing this cooperation guarantees to him at least such guaranteed profit he had before this offer,
- a player prefers the result of bargaining, guaranteeing him the highest profit with no respect to the other players,
- a coalition prefers one result to another supposing that at least some members of this coalition prefer this result and that there are no members prefering the other result to the first.

1. Notations

A great deal of notions and symbols used in this paper are those usually used in the elementary set theory and theory of functions. Only the following notations need for special remark.

The symbol N denotes the set of all natural numbers.

For $n \in N$, the n-dimensional Euclidian space is caaled, for the sake of shortness, n-space.

The symbol $\{m : m \textit{ has the property } P\}$ denotes the set of all the elements having the property P; $\{m\}$ denotes the one-element set containing exactly the element m;

$$\{m_1, m_2, \dots, m_u\} = \{m_r\}_{r=1,\dots,u}$$

is the finite set the only elements of which are $m_1, m_2, \dots, m_u$.

If $M \neq \emptyset$ is a set and $\mathscr{M}$ is a disjoint class of its non-empty subsets such that $\bigcup_{L \in \mathscr{M}} L = M$, then $\mathscr{M}$ is called a *partition* of the set M. If $\mathscr{K}, \mathscr{L}$ are two partitions of the same set $M \neq \emptyset$ such that for every set $L \in \mathscr{L}$ there is a set $K \in \mathscr{K}$ so that $L \subset K$, then $\mathscr{L}$ is called a *subpartition* of $\mathscr{K}$.

(1.1) Let $\mathscr{M}$ be a partition of a finite non-empty set M, let a square matrix $D_L = (d_{ij}^{(L)})_{i \in L, j \in L}$ of real numbers be defined for every subset $L \in \mathscr{M}$. Then the square matrix $\mathbf{D} = (d_{ij})_{i \in M, j \in M}$ such that

$$d_{ij} = d_{ij}^{(L)} \quad \text{for all} \quad L \in \mathscr{M},\ i \in L,\ j \in L, \quad \text{and}$$

$$d_{ij} = 0 \quad \text{for} \quad i \in K,\ j \in L,\ K \in \mathscr{M},\ L \in \mathscr{M},\ K \neq L,$$

is denoted by $\mathbf{D} = (D_L)_{L \in \mathscr{M}}$.

For every finite set M, $\pi(M) \in N$ denotes the cardinal number of the set M.

(1.2) For every non-empty class $\mathscr{M}$ of sets the symbol $\|\mathscr{M}\|$ denotes the set $\bigcup_{M \in \mathscr{M}} M$. In similar way, for every collection $\mathbf{M}$ of classes of sets the symbol $\|\mathbf{M}\|$ denotes the class $\bigcup_{\mathscr{M} \in \mathbf{M}} \mathscr{M}$.

(1.3) For every finite non-empty set M the symbol $S(M)$ denotes the set of all transformations s of the set M into the set of all real numbers such that for all $m \in M$ is $0 \leqq s(m) \leqq 1$ and $\sum_{m \in M} s(m) = 1$.

(1.4) If M_i, $i = 1, \ldots, r$ are finite non-emty sets, if $M = \times_{i=1}^{r} M_i$ is a Cartesian product, if $s \in S(M)$, $s^{(i)} \in S(M_i)$, $i = 1, \ldots, r$, then we write $s = s^{(1)} \times s^{(2)} \times \ldots \times s^{(r)}$, supposing that for every $m = (m_1, m_2, \ldots, m_r) \in M$

$$s(m) = \prod_{i=1}^{r} s^{(i)}(m_i)\,.$$

(1.5) By $s^{[m]}$, $m \in M$, the transformation from $S(M)$ is denoted, which satisfies the conditions:

$$s^{[m]}(m) = 1\,, \quad s^{[m]}(m') = 0\,, \quad m' \in M\,, \quad m' \neq m\,.$$

2. Coalition-Game with Finite Identification

Let I be a finite non-empty set. Let $\mathbf{K}$ be a non-empty collection of the partitions of the set I, satisfying the condition:

(2.1) If $\mathscr{K}_1, \ldots, \mathscr{K}_n \in \mathbf{K}$, if $\mathscr{K}$ is a partition of I, and if for every set $K \in \mathscr{K}$ is $K \in \bigcup_{r=1}^{n} \mathscr{K}_r$, then $\mathscr{K} \in \mathbf{K}$.

Let $K \in \|\mathbf{K}\|$, let a finite non-empty set A_K be given such that

(2.2) For $J^{(1)}, \ldots, J^{(n)} \in \|\mathbf{K}\|$, $J^{(r)} \cap J^{(t)} = \emptyset$ for $r \neq t$, $r = 1, \ldots, n$ $t = 1, \ldots, n$,

$$\bigcup_{r=1}^{n} J^{(r)} = K \in \| \mathbf{K} \|$$

the inclusion

$$A_K \supset \bigtimes_{r=1}^{n} A_{J^{(r)}}$$

holds.

Let be ascribed, to every set A_K, $K \in \|\mathbf{K}\|$, the class of transformations $\mathbf{S}_K = \mathbf{S}(A_K)$, satisfying (1.3).

If $\mathscr{K} \in \mathbf{K}$ then we denote by $\mathbf{S}_{\mathscr{K}}$ the Cartesian product

$$\bigtimes_{K \in \mathscr{K}} \mathbf{S}_K \; ; \quad \text{if} \quad \mathbf{s} = (s_K)_{K \in \mathscr{K}} \in \mathbf{S}_{\mathscr{K}} \, ,$$

then the set of pairs

$$\{K, s_K) : K \in \mathscr{K} \, , \; \mathbf{s} = (s_K)_{K \in \mathscr{K}}\}$$

is denoted by the symbol $(\mathscr{K}, \mathbf{s})$. By $\mathbf{K}_S$ we denote the class of sets

$$\{(\mathscr{K}, \mathbf{s}) : \mathscr{K} \in \mathbf{K}, \; \mathbf{s} \in \mathbf{S}_{\mathscr{K}}\} \, .$$

To every set $K \in \|\mathbf{K}\|$, a partition $\mathscr{R}_K$ of the class $\mathbf{K}_S$ into finite number of subclasses is ascribed, satisfying the following conditions:

(2.3) If $(\mathscr{K}, \mathbf{s}) \in \mathbf{K}_S$, $(\mathscr{K}, \mathbf{s}') \in \mathbf{K}_S$, $\mathbf{s} = (s_L)_{L \in \mathscr{K}}$, $\mathbf{s}' = (s'_L)_{L \in \mathscr{K}}$, if $K \in \mathscr{K}$, and $s_L = s'_L$ for $L \in \mathscr{K}$, $L \neq K$, then $(\mathscr{K}, \mathbf{s})$ and $(\mathscr{K}, \mathbf{s}')$ belong to the same set $\mathbf{R} \in \mathscr{R}_K$.

(2.4) If $J^{(1)}, \ldots, J^{(n)} \in \|\mathbf{K}\|$, $J^{(r)} \cap J^{(t)} = \emptyset$ for $r \neq t$, $r = 1, \ldots, n$, $t = 1, \ldots, n$,

$$\bigcup_{r=1}^{n} J^{(r)} = K \in \|\mathbf{K}\| \, ,$$

if $(\mathscr{K}, \mathbf{s}) \in \mathbf{K}_S$, $(\mathscr{K}', \mathbf{s}') \in \mathbf{K}_S$, $K \in \mathscr{K}$, $J^{(r)} \in \mathscr{K}'$, $r = 1, \ldots, n$, $\mathbf{s} = (s_L)_{L \in \mathscr{K}}$, $\mathbf{s}' = (s'_J)_{J \in \mathscr{K}'}$,

$$(\mathscr{K}, \mathbf{s}) \cap (\mathscr{K}', \mathbf{s}') = (\mathscr{K}, \mathbf{s}) - \{(K, s_K)\} \, ,$$

if sets $\mathbf{R} \in \mathscr{R}_K$, $\mathbf{R}_{(r)} \in \mathscr{R}_{J^{(r)}}$, $r = 1, \ldots, n$ are given, so that $(\mathscr{K}, \mathbf{s}) \in \mathbf{R}$, $(\mathscr{K}', \mathbf{s}') \in \mathbf{R}_{(r)}$ $r = 1, \ldots, n$, then

$$\mathbf{R} = \bigcap_{r=1}^{n} \{(\mathscr{K}'', \mathbf{s}'') : (\mathscr{K}'', \mathbf{s}'') \in \mathbf{K}_S, \mathbf{s}'' = (s''_L)_{L \in \mathscr{K}''}, K \in \mathscr{K}'', (((\mathscr{K}'', \mathbf{s}'') -$$

$$- \{(K, s''_K)\}) \cup (\bigcup_{t=1}^{n} \{(J^{(r)}, s'_{J^{(r)}})\})) \in \mathbf{R}_{(r)}\} \, ;$$

if it is the case, we write, abbreviately:

$$R = \mathrm{Comp}\{R_{(r)};\ r = 1, \dots, n\}\,.$$

Finally, let be ascribed, to every $i \in I$, a bounded realvalued function v_i defined on the set of all triples

$$(K, s_K. R_K)\,,\quad i \in K \in \|\mathbf{K}\|\,,\quad s_K \in S_K\,,\quad R_K \in \mathscr{R}_K\,,$$

satisfying the conditions:

(2.5) If

$$J^{(1)}, \dots, J^{(n)} \in \|\mathbf{K}\|\,,\quad \bigcup_{r=1}^{n} J^{(r)} = K \in \|\mathbf{K}\|\,,$$

$$J^{(r)} \cap J^{(t)} = \emptyset \quad \text{for} \quad r \neq t\,,\ r = 1, \dots, n,\ t = 1, \dots, n\,,$$

if

$$s_K \in S_K\,,\quad s_{(r)} \in S_{J^{(r)}}\,,\quad r = 1, \dots, n\,,$$

$$s_K = s_{(1)} \times \dots \times s_{(n)}$$

(c.f. (1,4)), and if

$$R_K \in \mathscr{R}_K,\ R_{(r)} \in \mathscr{R}_{J^{(r)}}\,,\quad r = 1, \dots, n,\quad R_K = \mathrm{Comp}\,\{R_{(r)};\ r = 1, \dots, n\}\,,$$

then the inequality

$$\sum_{i \in K} v_i(K, s_K, R_K) \geqq \sum_{t=1}^{n} \sum_{i \in J^{(t)}} v_i(J^{(t)}, s_{(t)}, R_{(t)})$$

holds.

Then the ordered quintuple

$$\Gamma = (I, \mathbf{K}, \{S_K\}_{K \in \|\mathbf{K}\|}, \{\mathscr{R}_K\}_{K \in \|\mathbf{K}\|}, \{v_i\}_{i \in I})$$

is called a *coalition-game with finite identification.* According to the usual interpretation, the following terms are used for its particular components.

The set I is the *set of players*, its partitions are called *coalition structures* and the collection $\mathbf{K}$ is called a *collection of admissible coalition structures.* The sets $K \subset I$ belonging to $\|\mathbf{K}\|$ are called *coalitions* (or admissible coalitions) and the transformations $s_K \in S_K$ are called *strategies* (or mixed strategies) of the coalition K. The partition $\mathscr{R}_K$ is called *identification partition* of the coalition K and the ascribed sets $R_K \in \mathscr{R}_K$ are called *identification sets* of the coalition K. The sets $(\mathscr{K}, \mathbf{s}) \in \mathbf{K}_S$ are called *strategic structures* and the functions v_i, $i \in I$, are called *pay-off functions* of the players.

3. Further Notions and Properties

It can be useful to concern our attention to several elementary properties of the particular components of a strategic coalition-game with finite identification and to introduce several other notions serving to the description of the development and result of the process of bargaining in a game. In whole the rest of this section, we shall consider a given game

$$\Gamma = \left(I, \mathbf{K}, \{S_K\}_{K\in\|\mathbf{K}\|}, \ \{\mathscr{R}_K\}_{K\in\|\mathbf{K}\|}, \ \{v_i\}_{i\in I}\right).$$

Remark 3.1. If $K \in \|\mathbf{K}\|$, $\mathscr{K} \in \mathbf{K}$, then the sets $\boldsymbol{S}_K$ and $\boldsymbol{S}_{\mathscr{K}}$ are closed convex polyhedra in the $\pi(A_K)$-space and in the $(\sum_{L\in\mathscr{K}} \pi(A_L))$-space.

Remark 3.2. It can be easily seen, that a transformation φ_i can be ascribed to every player $i \in I$, such that to every strategic structure $(\mathscr{K}, \mathbf{s}) \in \mathbf{K}_S$ the triple $(K, s_K, \boldsymbol{R}_K)$, where $K \in \|\mathbf{K}\|$, $s_K \in \boldsymbol{S}_K$, $\boldsymbol{R}_K \in \mathscr{R}_K$, satisfying $i \in K$, $(K, s_K) \in (\mathscr{K}, \mathbf{s})$, $(\mathscr{K}, \mathbf{s}) \in \boldsymbol{R}_K$, is ascribed. Then the composed transformation $\boldsymbol{v}_i(\varphi_i)$ of the class $\mathbf{K}_S$ to the set of all real numbers can be defined for every $i \in I$ in such a way that

$$\boldsymbol{v}_i(\varphi_i(\mathscr{K}, \mathbf{s})) = \boldsymbol{v}_i(K, s_K, \boldsymbol{R}_K),$$

if

$$\varphi_i(\mathscr{K}, \mathbf{s}) = (K, s_K, \boldsymbol{R}_K).$$

As there is no danger, in the following of misunderstanding between the strategic structure $(\mathscr{K}, \mathbf{s})$ and the triple $(K, \mathbf{s}_K, \boldsymbol{R}_K) = \varphi_i(\mathscr{K}, \mathbf{s})$, we shall write

$$\boldsymbol{v}_i(\mathscr{K}, \mathbf{s}) = \boldsymbol{v}_i(K, s_K, \boldsymbol{R}_K)$$

for $\boldsymbol{v}_i(\varphi_i(\mathscr{K}, \mathbf{s}))$, iff $(K, s_K, \boldsymbol{R}_K) = \varphi_i(\mathscr{K}, \mathbf{s})$. This symbolization makes some procedures, used below, more simple.

Remark 3.3. In literature, such games are often investigated, in which the pay-off functions of players are considered to be real-valued functions $\boldsymbol{h}_i$, $i \in I$, defined on the set

$$\bigcup_{\mathscr{K}\in\mathbf{K}} \boldsymbol{S}_{\mathscr{K}}$$

and satisfying:

(3.1) If $\mathbf{s} = (s_K)_{K\in\mathscr{K}} \in \boldsymbol{S}_{\mathscr{K}}$, then for all $i \in I$

$$\boldsymbol{h}_i(\mathbf{s}) = \sum_{a=(a_K)_{K\in\mathscr{K}}, a_K\in A_K, K\in\mathscr{K}} \prod_{K\in\mathscr{K}} (\mathbf{s}_K(a_K) \cdot \boldsymbol{h}_i(\mathbf{s}^{[a]})),$$

where

$$\mathbf{s}^{[a]} = (\mathbf{s}_K^{[a]})_{K \in \mathscr{K}}, \quad \mathbf{s}_K^{[a]}(a_K) = 1,$$

$$\mathbf{s}_K^{[a]}(a_K) = 0, \quad a'_K \in A_K, \quad a'_K \neq a_K, \quad K \in \mathscr{K}$$

(c.f. (1.5)).

In such a case the formerly introduced functions $\boldsymbol{v}_i$, $i \in I$ may be interpreted as the *expected value* of the pay-off under the given vector of strategies $\mathbf{s}$ and under the given information of the players about their anti-players action; e.g. $\boldsymbol{v}_i$ may be derived from $\boldsymbol{h}_i$ by the relation:

(3.2) If $(\mathscr{K}, \mathbf{s}) \in \mathbf{K}_S$, $(K, s_K) \in (\mathscr{K}, \mathbf{s})$, $(\mathscr{K}, \mathbf{s}) \in \boldsymbol{R}_K$, $i \in K$, then

$$\boldsymbol{v}_i(K, s_K, \boldsymbol{R}_K) = \boldsymbol{v}_i(\mathscr{K}, \mathbf{s}) =$$
$$= \inf \{\boldsymbol{h}_i(\mathbf{s}') : \mathbf{s}' \in \boldsymbol{S}_{\mathscr{K}}, (\mathscr{K}, \mathbf{s}') \in \boldsymbol{R}_K, S'_K = S_K\}.$$

In such a case the condition (2.5) is satisfied (being a consequence of (2.6)), even the more strong form holds:

$$\boldsymbol{v}_i(K, s_K, \boldsymbol{R}_K) \geqq \boldsymbol{v}_i(J^{(t)}, s_{(t)}, \boldsymbol{R}_{(t)}) \quad \text{for all} \quad i \in J^{(t)} \subset K, 1 \leqq t \leqq n.$$

In the case of necessity the function $\boldsymbol{v}_i$ can be chosen even in a less pesimistic way than that described in (3.2), according to an actual interpretation. The condition (2.5), however, should be satisfied in every case.

In order to obtain a complete description of the bargaining, the introduction of another notions may be useful, in the first order the notion of side payments, described, in heuristic way, in Section 0.

If $K \in \|\mathbf{K}\|$, then a matrix $\boldsymbol{D}_K = (d_{ij}^{(K)})_{i \in K, j \in K}$ of real numbers, satisfying

$$d_{ij}^{(K)} = - d_{ji}^{(K)}, \; i \in K, j \in K,$$

is called *side payments matrix in* K. Analogously, if $\mathscr{K} \in \mathbf{K}$, and if for every $K \in \mathscr{K}$ the side payments matrix $\boldsymbol{D}_K$ is given, then the matrix

$$\mathbf{D} = (\boldsymbol{D}_K)_{K \in \mathscr{K}}$$

(c.f. (1.1)) is called *side payments matrix in* $\mathscr{K}$. The symbols $\mathscr{D}_K$ and $\mathscr{D}_{\mathscr{K}}$ denote the sets of all side payments matrices in K and in $\mathscr{K}$.

If $K \in \|\mathbf{K}\|$, $s_K \in \boldsymbol{S}_K$ and $\boldsymbol{D}_K \in \mathscr{D}_K$, then the triple

$$(K, s_K, \boldsymbol{D}_K)$$

is called an *agreement*.

If $\mathscr{K} \in \mathbf{K}$, $\mathbf{s} = (s_K)_{K \in \mathscr{K}} \in \boldsymbol{S}_{\mathscr{K}}$, $\mathbf{D} = (\boldsymbol{D}_K)_{K \in \mathscr{K}} \in \mathscr{D}_{\mathscr{K}}$, then the set

$$\mathbf{C} = \{(K, s_K, \boldsymbol{D}_K) : K \in \mathscr{K}\}$$

is called a *configuration*; also the symbol

$$C = C(\mathscr{K}, \mathbf{s}, \mathbf{D})$$

will be used. By $\mathscr{C}$ the class of all configurations, admissible in the considered game, will be denoted.

If $\mathscr{K} \in \mathbf{K}$, if $\boldsymbol{k} \subset \mathscr{K}$, $\boldsymbol{k} \neq \mathscr{K}$, if $C = C(\mathscr{K}, \mathbf{s}, \mathbf{D}) \in \mathscr{C}$,

then the set

$$c = \{(K, s_K, D_K) : (K, s_K, D_K) \in C, K \in \boldsymbol{k}\} = c(C; \boldsymbol{k})$$

will be called a *sub-configuration* of the configuration C.

For every player $i \in I$, a real-valued function $\mathscr{K}_i$ can be defined on the class $\mathscr{C}$ by the relation

(3.3)
$$\boldsymbol{x}_i(C) = \boldsymbol{v}_i(\mathscr{K}, \mathbf{s}) + \sum_{j \in I} d_{ij} = \\ = \boldsymbol{v}_i(K, s_K, \boldsymbol{R}_K) + \sum_{j \in K} d_{ij}$$

where

$$C = C(\mathscr{K}, \mathbf{s}, \mathbf{D}), \quad i \in K \in \mathscr{K}, \quad (\mathscr{K}, \mathbf{s}) \in \boldsymbol{R}_K \in \mathscr{R}_K,$$
$$\mathbf{D} = (D_L)_{L \in \mathscr{K}} = (d_{ij})_{i \in I, j \in I} \in \mathscr{D}_{\mathscr{K}},$$
$$\mathbf{s} = (s_L)_{L \in \mathscr{K}} \in S_{\mathscr{K}}.$$

This function $\boldsymbol{x}_i$ will be called the *profit* of the player i.

(3.4) For every $M \subset I$, $C \in \mathscr{C}$, $C' \in \mathscr{C}$ the symbol

$$\boldsymbol{x}_M(C) > \boldsymbol{x}_M(C')$$

denotes that

$\boldsymbol{x}_j(C) \geqq \boldsymbol{x}_j(C')$ for all $j \in M$, and,

$\boldsymbol{x}_i(C) > \boldsymbol{x}_i(C')$ for some $i \in M$.

Remark 3.4. If $(\mathscr{K}, \mathbf{s}) \in \mathbf{K}_S$, $(\mathscr{K}', \mathbf{s}) \in \mathbf{K}_S$, $K \in \mathscr{K} \cap \mathscr{K}'$, and if the equality

$$\sum_{i \in K} \boldsymbol{v}_i(\mathscr{K}, \mathbf{s}) = \sum_{i \in K} \boldsymbol{v}_i(\mathscr{K}', \mathbf{s}')$$

holds, then there exists, for every configuration $C = C(\mathscr{K}, \mathbf{s}, \mathbf{D}) \in \mathscr{C}$, such a $C' = C'(\mathscr{K}', \mathbf{s}', \mathbf{D}') \in \mathscr{C}$, that

$$\boldsymbol{x}_i(C) = \boldsymbol{x}_i(C')$$

for all $i \in K$. It is sufficient, for this purpose, to choose $\mathbf{D}' = (d'_{ij})_{i\in I, j\in I} \in \mathscr{D}_{\mathscr{K}'}$ in such a way that

$$d'_{ij} = (\pi(K))^{-1} \cdot (x_i(C) - v_i(\mathscr{K}', \mathbf{s}') + v_j(\mathscr{K}', \mathbf{s}') - x_j(C)),$$
$$i \in K, j \in K.$$

Lemma 3.1. If $C' = C'(\mathscr{K}', \mathbf{s}', \mathbf{D}') \in \mathscr{C}$, $(\mathscr{K}, \mathbf{s}) \in \mathbf{K}_S$, then there exist $\mathbf{D} \in \mathscr{D}_{\mathscr{K}}$ and $C = C(\mathscr{K}, \mathbf{s}, \mathbf{D}) \in \mathscr{C}$ such that

$$x_K(C) > x_K(C')$$

if and only if

$$\sum_{i\in K} v_i(\mathscr{K}, \mathbf{s}) > \sum_{i\in K} x_i(C'). \tag{3.5}$$

Proof. Supposing the inequality (3.5) holds for $K \in \mathscr{K}$, then it will be sufficient, for our purposes, to choose

$$\mathbf{D} = (D_L)_{L\in\mathscr{K}} \in \mathscr{D}_{\mathscr{K}}, \quad D_K = (d_{ij}^{(K)})_{i\in K, j\in K},$$
$$d_{ij}^{(K)} = (\pi(K))^{-1} \cdot (x_i(C') - v_i(\mathscr{K}, \mathbf{s}) + v_j(\mathscr{K}, \mathbf{s}) - x_j(C')),$$
$$i \in K, j \in K.$$

Then $d_{ij}^{(K)} = -d_{ji}^{(K)}$, $i \in K$, $j \in K$, and

$$x_i(C) = v_i(\mathscr{K}, \mathbf{s}) + \sum_{j\in K} d_{ij}^{(K)} = v_i(\mathscr{K}, \mathbf{s}) + \pi(K) \cdot ((\pi(K))^{-1} \cdot x(_i(C') -$$
$$- v_i(\mathscr{K}, \mathbf{s})) + (\pi(K))^{-1} \cdot (\sum_{j\in K} v_j(\mathscr{K}, \mathbf{s}) - \sum_{j\in K} x_j(C')) =$$
$$= x_i(C') + (\pi(K))^{-1} \cdot (\sum_{j\in K} v_j(\mathscr{K}, \mathbf{s}) - \sum_{j\in K} x_i(C')) > x_i(C')$$

for $i \in K$. Therefore

$$x_K(C) > x_K(C').$$

The reverse implication follows immediately from (3.4) □

4. Special Types of Games

A coalition-game may take various special forms, the investigation of which may be interesting in the following.

Supposing the collection $\mathbf{K}$ contains exactly one admissible coalition structure, the game is called a *game with fixed cooperation.*

Supposing for every pair $\mathscr{K} \in \mathbf{K}$, $\mathscr{L} \in \mathbf{K}$, $\mathscr{K}$ is a subpartition of $\mathscr{L}$ or $\mathscr{L}$ is a subpartition of $\mathscr{K}$, the game is called a *game with strictly bounded cooperation.*

Supposing $\mathbf{K}$ contains all the a priori possible partitions of the set I, the game is called a *game with free cooperation.*

Supposing for every coalition $K \in \|\mathbf{K}\|$ the set S_K contains exactly one strategy (i.e. supposing all the sets A_K, $K \in \|\mathbf{K}\|$ contain exactly one element), the game is called *coalition game with no choice of strategies*, or, abbreviately, *coalition-game*.

Supposing for every coalition $K \in \|\mathbf{K}\|$ the identification partition $\mathscr{R}_K$ contains exactly one identification set

$$R_K = \{(\mathscr{K}, \mathbf{s}) : (\mathscr{K}, \mathbf{s}) \in \mathbf{K}_S, K \in \mathscr{K}\}, \quad \mathscr{R}_K = \{R_K\},$$

the game is called a *game without identification*.

In Section 2, the investigated model of game was defined to be a game with finite identification, i.e. to be a game in which all the identification partitions $\mathscr{R}_K$, $K \in \|\mathbf{K}\|$ are finite. This definition includes all the cases of games with no choice of strategies. In case such games are investigated, that their sets of strategies contain more elements, even games with infinite identification partitions can be considered, and the presumption of a finite identification seems to be a limitation. We can see, however, considering the limited possibilities of recognizing of real players, that the games with finite identification contain a substantial part of all real coalition-games. Moreover, the apparatus developed in the case of the coalition-games can be applied in the case of the games with finite identification, what would not be possible in the case of the coalition games with generally infinite identification. Further discussion of this subject c.f. in Conclusive remarks.

PART II. BARGAINING IN A GAME WITHOUT CHOOSING STRATEGIES

This part, the most important of the presented paper, introduces the idea of bargaining solution in the investigated type of game. For the simplification of notations and for easier reading of proofs, the bargaining model is introduced here for a special case of games with exactly one strategy in each coalition. It will be shown in the fourth part of this paper, that the restriction is not essential, and that it does not influence the generality of obtained results.

The simplification of the investigated game model enables us to simplify the notations introduced above. In this part

$$\Gamma = (I, \mathbf{K}, \{S_K\}_{K \in \|\mathbf{K}\|}, \{\mathscr{R}_K\}_{K \in \|\mathbf{K}\|}, \{v_i\}_{i \in I})$$

is considered to be a coalition-game and we shall use the notation $(K, \mathbf{D}_K)$ (resp. $C(\mathscr{K}, \mathbf{D})$) for agreements (resp. configurations) instead of $(K, s_K, \mathbf{D}_K)$ resp. $C(\mathscr{K}, \mathbf{s}, \mathbf{D})$. Every strategic structure $(\mathscr{K}, \mathbf{s}) \in \mathbf{K}_S$ is unambiguously defined by the relevant coalition structure $\mathscr{K} \in \mathbf{K}$, and the identification partitions of the coalitions then can be understood as partitions of the collection $\mathbf{K}$ of all the admissible coalition structures. The condition (2.3) is satisfied automatically, in this case, and the condition (2.4) can be formulated in the form:

If $J^{(1)}, \ldots, J^{(n)} \in \|\mathbf{K}\|$,

$$\bigcup_{r=1}^{n} J^{(r)} = K \in \|\mathbf{K}\| ,$$

$J^{(r)} \cap J^{(t)} = \emptyset$ for $r \neq t, r = 1, \ldots, n, \; t = 1, \ldots, n, \; \mathscr{K} \in \mathbf{K}, \; \overline{\mathscr{K}} \in \mathbf{K}, \; K \in \mathscr{K}$,

$J^{(r)} \in \overline{\mathscr{K}}$ for $r = 1, \ldots, n$ and if $\boldsymbol{R} \in \mathscr{R}_K$ and $\boldsymbol{R}_{(r)} \in \mathscr{R}_{J^{(r)}}$, $r = 1, \ldots, n$ then

$$\boldsymbol{R} = \bigcap_{r=1}^{n} \{\mathscr{K}' : \mathscr{K}' \in \mathbf{K} , \quad K \in \mathscr{K}' , \quad ((\mathscr{K}' - \{K\}) \cup (\bigcup_{t=1}^{n} \{J^{(r)}\})) \in \boldsymbol{R}_{(r)}\} .$$

Every game without choosing strategies is automatically a game with finite indentification.

In a similar way, the pay-of functions $\boldsymbol{v}_i$, $i \in I$, are functions of only two variables. K and $\boldsymbol{R}_K$, $i \in K \in \|\mathbf{K}\|$, $\boldsymbol{R}_K \in \mathscr{R}_K$. According to Remark 3·3, the pay-off functions $\boldsymbol{v}_i$, $i \in I$, can be defined also on the collection $\mathbf{K}$ when the relation $\boldsymbol{v}_i(\mathscr{K}) = \boldsymbol{v}_i(\mathscr{K}, \mathbf{s})$, $\mathscr{K} \in \mathbf{K}$, $(\mathscr{K}, \mathbf{s}) \in \mathbf{K}_S$ is used.

5. Auxiliary relations

The purpose of the following section is to introduce several notions which can be useful when a bargaining model is introduced. The most important among them is that of rationality and the strong domination relation (definitions 1 and 4). The other relations, introduced in this section, have purely preliminary character, and serve to a preliminary analysis of agreements and configurations.

Let us introduce, at the first time, the following symbol:

(5.1) If $C \in \mathscr{C}$, $C' \in \mathscr{C}$, if $c \subset C \cap C'$ is their sub-configuration, if there exists an agreement $(K, \boldsymbol{D}_K) \in C - c$ satisfying the relation $\boldsymbol{x}_K(C) > \boldsymbol{x}_K(C')$, then we shall write

$$C \; \delta \; C' \bmod c .$$

Remark 5.1. It follows from Lemma 3.1 that for no configurations $C = C(\mathscr{K}, \mathbf{D})$, $C' = C'(\mathscr{K}, \mathbf{D}')$ with the same coalition structure $\mathscr{K}$, and for no their sub-configuration $c \subset C \cap C'$ the relation $C \; \delta \; C' \bmod c$ holds.

Definition 1. Let us consider a $C = C(\mathscr{K}, \mathbf{D}) \in \mathscr{C}$. An *agreement* $(K, \boldsymbol{D}_K) \in C$ is called *rational in* C supposing there does not exist $C' \in \mathscr{C}$, such that

$$C' \; \delta \; C \bmod c(C; \mathscr{K} - \{K\}) .$$

By the symbol

$$\mathsf{C}_{\mathrm{rat}}$$

the set of all agreements rational in C will be denoted.

Remark 5.2. It follows from (5.1) and from Definition 1 that if $\mathsf{C} = \mathsf{C}(\mathscr{K}, \mathbf{D}) \in \mathscr{C}$, $(K, \boldsymbol{D}_K) \in \mathsf{C} - \mathsf{C}_{\mathrm{rat}}$ then thre exist $\mathsf{C}' \in \mathscr{C}$ and $(L, \boldsymbol{D}'_L) \in \mathsf{C}'_{\mathrm{rat}}$ satisfying the relations

$$\mathsf{C}' \;\delta\; \mathsf{C} \bmod \mathsf{c}(\mathsf{C}; \mathscr{K} - \{K\}),\; x_L(\mathsf{C}') > x_L(\mathsf{C}), \text{ and } L \subset K .$$

Remark 5.3. If $\mathsf{C} = \mathsf{C}(\mathscr{K}, \mathbf{D}) \in \mathscr{C}$, if there exists no $\mathscr{L} \in \mathbf{K}$, $\mathscr{L}$ being a subpartition of $\mathscr{K}$, then if follows, from Definition 1, that $\mathsf{C} = \mathsf{C}_{\mathrm{rat}}$.

Lemma 5.1. Let $\mathscr{K} \in \mathbf{K}$, $K \in \mathscr{K}$, Let us denote by $\mathbf{K}(K)$ the collection

$$\mathbf{K}(K) = \{\mathscr{L} : \mathscr{L} \in \mathbf{K},\, \mathscr{L} \cap \mathscr{K} \supset \mathscr{K} - \{K\}\} .$$

Let there exists a matrix $\mathbf{D} = (\boldsymbol{D}_L)_{L \in \mathscr{K}}$ and a configuration $\mathsf{C} = \mathsf{C}(\mathscr{K}, \mathbf{D}) \in \mathscr{C}$ such that $(K, \boldsymbol{D}_K) \in \mathsf{C}_{\mathrm{rat}}$. Then the inequality

$$(5.2) \qquad \pi(\mathbf{K}(K)) \sum_{i \in K} v_i(\mathscr{K}) \geqq \sum_{\mathscr{L} \in \mathbf{K}(K)} \; \sum_{L \in \mathscr{L}, L \subset K} \max \{ \sum_{i \in L} v_i(\mathscr{J}) : L \in \mathscr{J} ,\; \mathscr{J} \in \mathbf{K}(K)\} ,$$

holds.

Proof. Let $(K, \boldsymbol{D}_K) \in \mathsf{C} = \mathsf{C}(\mathscr{K}, \mathbf{D})$. Then $(K, \boldsymbol{D}_K) \in \mathsf{C}_{\mathrm{rat}}$ if and only if for no $\mathscr{L} \in \mathbf{K}(K)$ and for no matrix $\mathbf{D}' \in \mathscr{D}_{\mathscr{L}}$ the relation $\mathsf{C}' \;\delta\; \mathsf{C} \bmod \mathsf{c}(\mathsf{C}; \mathscr{K} - \{K\})$ holds for $\mathsf{C}' = \mathsf{C}'(\mathscr{L}, \mathbf{D}')$. This condition is satisfied if and only if for all $\mathscr{L} \in \mathbf{K}(K)$ and for all $L \in \mathscr{L}$, $L \subset K$ the inequality

$$\sum_{i \in L} x_i(\mathsf{C}) \geqq \sum_{i \in L} v_i(\mathscr{L})$$

holds, which is equivalent to the condition

$$(5.3) \qquad \sum_{i \in L} x_i(\mathsf{C}) \geqq \max \{ \sum_{i \in L} x_i(\mathsf{C}'') : \mathsf{C}'' = \mathsf{C}''(\mathscr{J}, \mathbf{D}'') \in \mathscr{C}, \mathscr{J} \in \mathbf{K}(K),\, L \in \mathscr{J}\} =$$
$$= \max \{ \sum_{i \in L} v_i(\mathscr{J}) : \mathscr{J} + \mathbf{K}(K),\, L \in \mathscr{J}\}$$

for all coalitions $L \in \mathscr{L}$, $L \subset K$, and for all coalition structures $\mathscr{L} \in \mathbf{K}(K)$.

Summarizing the inequalities (5.3) for all $L \subset K$, $L \in \mathscr{L}$ and $\mathscr{L} \in \mathbf{K}(K)$ gives the inequality (5.2). □

Corollary. If $\mathscr{K} \in \mathbf{K}$, $K \in \mathscr{K}$, then the last Lemma gives the following:

(a) If $K = \{i\}$, $i \in I$, then $\mathscr{D}_K$ is an one-element set, containing the zero matrix and the agreement $(K, 0)$ is rational in $C(\mathscr{K}, \mathbf{D})$ for every $\mathbf{D} \in \mathscr{D}_{\mathscr{K}}$.

(b) If $K = \{i, j\}$, $i \in I, j \in I$, $i \neq j$, then there always exists a matrix $\mathbf{D} = (D_L)_{L \in \mathscr{K}} \in \mathscr{D}_{\mathscr{K}}$ such that (K, D_K) is rational in $C(\mathscr{K}, \mathbf{D})$.

(c) If $\pi(K) = k$, and if every coalition $J \subset K$ is admissible, i.e. $J \in \|\mathbf{K}\|$, and if there exists $\mathbf{D} = (D_L)_{L \in \mathscr{K}} \in \mathscr{D}_{\mathscr{K}}$ and $C = C(\mathscr{K}, \mathbf{D}) \in \mathscr{C}$ such that $(K, D_K) \in C_{rat}$ then

$$\binom{k-1}{n-1} \sum_{i \in K} v_i(\mathscr{K}) \geqq \sum_{J \subset K, \pi(J) = n} \max \{ \sum_{i \in J} v_i(\mathscr{J}) :$$

$$: \mathscr{J} \in \mathbf{K}, \mathscr{J} \cap \mathscr{K} = \mathscr{K} - \{K\} , \quad J \in \mathscr{J} \}$$

for all $\in N$, $1 \leqq n < k$.

Definition 2. If C, C' are admissible configurations, if $c \subset C \cap C'$ is one of their sub-configurations, then we shall say that C *weakly dominates* C' *modulo* c and write

$$C \text{ dom } C' \text{ mod } c$$

if:

(a) there exists an agreement $(K, D_K) \in C_{\text{rat}} - c$ such that $x_K(C) > x_K(C')$, and

(b) there exists no $C'' \in \mathscr{C}$ such that the conditions

(b1) $$c \subset C'' ,$$

(b2) $$C''_{\text{rat}} - c \supset C_{\text{rat}} - c .$$

$$C''_{\text{rat}} - c \neq C_{\text{rat}} - c .$$

(b3) if

$$(K, D_K) \in C_{\text{rat}} - c , \quad x_K(C) > x_K(C')$$

then

$$x_K(C'') > x_K(C')$$

would hold simultaneously.

An agreement $(K, D_K) \in C_{\text{rat}} - c$, satisfying the condition (a), is called to be *active in the relation* C dom C' mod c.

The weak domination relation, just defined, enables us to choose such a configurations among all the configurations belonging to $\mathscr{C}$ and satisfying the condition $C \, \delta \, C'$ mod c for given $C' \in \mathscr{C}$, $c \subset C$, which can be seen, from the point of view of the participating players, as the most rational.

Lemma 5.2. Supposing C is an admissible configuration, $c \subset C$ one of its sub-configurations, then there exists $C' \in \mathscr{C}$ satisfying $C' \delta C \bmod c$, iff there exist $C'' \in \mathscr{C}$ satisfying C'' dom $C \bmod c$.

Proof. Supposing $C \in \mathscr{C}$, $c \subset C$, $C' \in \mathscr{C}$, $C' \delta C \bmod c$, let us denote by $(K, \boldsymbol{D}'_K) \in \in C' - c$ the agreement satisfying $x_K(C') > x_K(C)$. If $(K, \boldsymbol{D}'_K) \notin C'_{\text{rat}}$, $C'' = C''(\mathscr{K}'', \mathbf{D}'')$ as well as an agreement $(L, \boldsymbol{D}''_L) \in C''_{\text{rat}}$ can be constructed (as a consequence of Remark 5.2) such that $x\mathscr{L}(C'') > x_L(C')$ and $C'' \cap C' = C' - \{(K, \boldsymbol{D}_K)\}$, which implies $x_L(C'') > x_L(C)$ and $C'' \delta C \bmod c$. It means that the set

$$\{(J, \boldsymbol{D}''_J) : (J, \boldsymbol{D}''_J) \in C''_{\text{rat}} - c\,,\ x_J(C'') > x_J(C)\}$$

is non-empty and $C'' \in \mathscr{C}$ can be chosen in such a way that the conditions of the latter part of Definition 2 were satisfied, so C'' dom $C \bmod c$. The contrary implication in the assertion of this Lemma is an immediate consequence of Definition 2. □

Remark 5.4. If C, $C' \in \mathscr{C}$, if $c' \subset c \subset C \cap C'$ are two of their sub-configurations, and if the inclusion $c - c' \subset C_{\text{rat}}$; holds. then, according to the Definition 2,

$$C \text{ dom } C' \bmod c \quad \text{implies} \quad C \text{ dom } C' \bmod c'\,.$$

In the following steps, let us go to the strong domination relation, which is, as said in the beginning of this section, the basic relation of the proposed prediction method. The significance of the strong domination relation consists in the fact, that it enables us to pass from the analysis of agreements or sub-configurations to considerations involving the configurations as a whole. Therefore, it forms also the basis for the proper prediction relations defined on the class $\mathscr{C}$.

A question can arise, namely, whether it is or is not possible to make the used procedure easier in the sense that the weak domination relation should be used only modulo the empty sub-configuration $\emptyset$. It will be shown in the following example that such a simplification, applied to games not being the games without identification, would lead to results, which could be hardly accepted.

Example 1. A three-players game is considered, where

$$I = \{i, j, k\}\,,\quad \mathbf{K} = \{\mathscr{K}_0, \mathscr{K}_i\}\,,\quad \mathscr{K}_0 = \{\{i\}, \{j\}, \{k\}\}\,,\quad \mathscr{K}_i = \{\{i\}, \{j, k\}\}\,,$$

$$v_i(\mathscr{K}_i) = v_j(\mathscr{K}_0) = v_k(\mathscr{K}_0) = 0\,,$$

$$v_i(\mathscr{K}_0) = v_j(\mathscr{K}_i) = v_k(\mathscr{K}_i) = 1\,.$$

If $\boldsymbol{0}$ denotes the zero side-payments matrix then the relations

$$C(\mathscr{K}_0, \boldsymbol{0}) \text{ dom } C(\mathscr{K}_i, \boldsymbol{0}) \bmod \emptyset\,,$$

$$C(\mathscr{K}_i, \boldsymbol{0}) \text{ dom } C(\mathscr{K}_0, \boldsymbol{0}) \bmod \emptyset$$

hold simultaneously, nevertheless in the first relation the agreement $(\{i\}, 0)$ is active only because of the fact that the players j and k have chosen a proceeding, which is extremally disadvantageous from their point of view, and contradicts the postulates on rationalism.

Therefore, the introduction of the domination relations modulo a non-empty sub-configuration aims, at least, to limit such cases.

Definition 3. If.

$$\{C^{(r)}\}_{r=1,\ldots,n}$$

is a finite sequence of admissible configurations, if

$$c \subset \bigcap_{r=1}^{n} C^{(r)}$$

is one of their sub-configurations and if the relation

$$C^{(r)} \text{ dom } C^{(r-1)} \bmod c\,, \quad r = 2, 3, \ldots, n$$

holds, then we say that $\{C^{(r)}\}_{r=1,\ldots,n}$ is a *weak domination sequence modulo c from $C^{(1)}$ to $C^{(n)}$.*

Remark 5.5. It follows from Definition 3 that in case $C, C', C'' \in \mathscr{C}$, if $c \subset C'' \cap C' \cap C$, and if there exist weak domination sequences modulo c from C to C' and from C' to C'', then also a weak domination sequence modulo c from C to C'' exists.

Definition 4. If C, C' are admissible configurations, if $c \subset C \cap C'$ is one of their sub-configurations, then we shall say that C *strongly dominates C' modulo c* and write

$$C \text{ Dom } C' \bmod c\,,$$

if

(a) there exists a weak domination sequence modulo c from C' to C, and

(b) there exists no weak domination sequence modulo c from C to C'.

Remark 5.6. It follows from Remark 5.5 and from Definition 3 that the strong domination relation modulo c is a partial ordering relation on the class

$$\{C : C \in \mathscr{C},\ c \subset C\}$$

and that this relation is anti-reflexive, anti-symetric and transitive.

Lemma 5.3. If $C = C(\mathscr{K}, \mathbf{D}) \in \mathscr{C}$, $(K, \mathbf{D}_K) \in C$, $c = c(C; \mathscr{K} - \{K\})$, then the following assertions hold:

(1) If $(K, \mathbf{D}_K) \in C_{\text{rat}}$, then there exists no $C' \in \mathscr{C}$ such that

$$C' \text{ Dom } C \bmod c\,,$$

and if for some $C'' \in \mathscr{C}$ the relation

$$C \,\delta\, C'' \bmod c$$

holds, then also

$$C \text{ Dom } C'' \bmod c$$

holds.

(2) If $(K, D_K) \in C - C_{\text{rat}}$ then there exists no $C'' \in \mathscr{C}$ such that

$$C \text{ Dom } C'' \bmod c\,,$$

and always such a $C' \in \mathscr{C}$ exists that

$$C' \text{ Dom } C \bmod c\,.$$

Proof. If $(K, D_K) \in C_{\text{rat}}$, then, according to Definition 1, there exists no C' such that $C' \,\delta\, C \bmod c$. According to Lemma 5.2 and Definitions 2, 3, 4, there exists no $C' \in \mathscr{C}$ such that

$$C' \text{ Dom } C \bmod c\,.$$

On the other hand, if for some $C'' \in \mathscr{C}$ the relation

$$C \,\delta\, C'' \bmod c$$

holds, then

$$C \text{ dom } C'' \bmod c$$

according to Definition 2. According to the foregoing step of the proof, there exists no $C' \in \mathscr{C}$ such that C' dom $C \bmod c$, which implies, according to Definitions 3 and 4, that

$$C \text{ Dom } C'' \bmod c\,.$$

If $(K, D_K) \in C - C_{\text{rat}}$, then, according to Definition 1 and Lemma 2, there exists $C' \in \mathscr{C}$ such that

$$C' \text{ dom } C \bmod c\,.$$

As, at the same time, according to Definition 2, the relation

$$C \text{ dom } C'' \bmod c$$

holds for no $C'' \in \mathscr{C}$, this implies that

$$C' \text{ Dom } C \bmod c$$

and that for no $C'' \in \mathscr{C}$

$$C \text{ Dom } C'' \bmod c$$

holds. □

Even in the case of strong domination relation an analogy of Definition 3 can be formulated.

Definition 5. If

$$\{C^{(r)}\}_{r=1,\dots,n}$$

is a finite sequence of admissible configurations, if

$$\{c^{(r)}\}_{r=1,\dots,n-1}$$

is a finite sequence of their sub-configurations such that

$$c^{(r)} \subset C^{(r)} \cap C^{(r+1)}$$

and

$$C^{(r+1)} \operatorname{Dom} C^{(r)} \bmod c^{(r)} \quad \text{for} \quad r = 1, 1, \dots, n-1 ,$$

then we say that the pair

$$\{C^{(r)}\}_{r=1,\dots,n}, \quad \{c^{(r)}\}_{r=1,\dots,n-1}$$

forms a *strong domination sequence from* $C^{(1)}$ *to* $C^{(n)}$.

Remark 5.7. If C, C', $C'' \in \mathscr{C}$, if there exist strong domination sequences from C to C' and from C' to C'', then also a strong domination sequence from C to C'' exists.

6. The Bargaining Prediction

In the following section, the introduced relations will be used in order to construct a prediction method for searching for the possible bargaining results in a given game. As said above, we shall limit ourselves to a game Γ without strategies choosing,

$$\Gamma = (I, \mathbf{K}, \{S_K\}_{K\in\|\mathbf{K}\|}, \{\mathscr{R}_K\}_{K\in\|\mathbf{K}\|}, \{v_i\}_{i\in I}) ,$$

where all the sets S_K, $K \in \|\mathbf{K}\|$ are one-element ones. The simplified notation, introduced at the beginning of this part, will be used even in the following proceedings.

There is the goal of this section to find a subclass, non-empty if possible, of the class $\mathscr{C}$, containing the acceptable bargaining results and, eventually, at least some rough distribution of the elements of this sub-class with respect to the measure of their acceptibility.

The following procedure, analogous to Definition 4, seems to be the most natural.

Definition 6. Supposing C, C' are two admissible configurations, we say that C *strongly predictionally avoids* C' and write

$$C \mathbf{P} C' ,$$

if:

(a) there exists a strong domination sequence from C' to C,

(b) there exists no strong domination sequence from C to C'.

If there exists, for an admissible configuration C, no $C' \in \mathscr{C}$ such that $C' \mathbf{P} C$, we shall say that C *is strongly predictionally stable.* By $\mathfrak{P}^*_\Gamma$, the set of all strongly predictionally stable configurations of the game Γ will be denoted.

Remark 6.1. The relation $\mathbf{P}$ is a partial ordering relation on the class $\mathscr{C}$; it is anti-reflexive, anti-symetric and transitive.

The strong avoiding relation, defined in this way, and the strong predictional stability property is not, and cannot be, in the same measure appropriate for all types of games. In some cases, especially if games with a larger number of the admissible coalitions, and with more complicated identification partitions are considered, the requests of Definition 6 for the predictional stability checking and for the validity of the strong predictional avoiding relation are too high. Therefore, it seems to be useful to consider the following modification of this relation.

Definition 7. Supposing $C = C(\mathscr{K}, \mathbf{D})$, $C' = C'(\mathscr{K}', \mathbf{D})$ are two admissible configurations, we say that C *weakly predictionally avoids* C' and write

$$C \mathbf{P} C',$$

if:

(a) there exists a strong domination sequence from C' to C,

(b) there exist no side-payments matrices

$$\mathbf{D}^\dagger = (D^\dagger_L)_{L \in \mathscr{K}} \in \mathscr{D}_{\mathscr{K}}\ , \quad \mathbf{D}^{\dagger\dagger}(D^{\dagger\dagger}_J)_{J \in \mathscr{K}'} \in \mathscr{D}_{\mathscr{K}}$$

and no adequate configurations

$$C^\dagger = C^\dagger(\mathscr{K}, \mathbf{D}^\dagger) \in \mathscr{C}\ , \quad C^{\dagger\dagger} = C^{\dagger\dagger}(\mathscr{K}', \mathbf{D}^{\dagger\dagger}) \in \mathscr{C}$$

satisfying the four following conditions:

(b1) there exists a strong domination sequence from C to $C^{\dagger\dagger}$,

(b2) there exists a strong domination sequence from $C^{\dagger\dagger}$ to $C^\dagger$,

(b3) $$(L, D_L) \in C_{\mathrm{rat}} \Leftrightarrow (L, D^\dagger_L) \in C^\dagger_{\mathrm{rat}}\ ,$$

(b4) $$(J, D'_J) \in C'_{\mathrm{rat}} \Leftrightarrow (J, D^{\dagger\dagger}_J) \in C^{\dagger\dagger}_{\mathrm{rat}}\ ,$$

If there exists, for a configuration $C \in \mathscr{C}$, no $C' \in \mathscr{C}$ such that $C' \mathbf{P} C$, we say that C *is weakly predictionally stable.* By $\mathfrak{P}^{**}_\Gamma$, the class of all weakly predictionally stable configurations of the game Γ will be denoted.

The weak avoiding relation enables to neglect, when the prediction stability is checked, the various highs of the side-payments in a coalition structure, supposing it does not influence the bargaining rationality and the existence of the strong domination relations for configurations. It means that actually the forming of the coalition structures is investigated, and differences among the side-payments matrices are considered only from the point of view of bargaining rationality. The weak predictional stability requests less, in such a way, for the properties of configurations and for its own checking.

Even in this case a question can arise, whether it is or is not possible to make the prediction method more simple, namely in the way we did not introduce the strong domination relation modulo a given configuration in the procedure, and we used instead it directly the weak domination relation modulo the same configuration as before.

Example 2. Let us consider the game described in Example 1, Section 1, let us denote

$$C = C(\mathscr{K}_i, \mathbf{0}), \quad C_0 = C_0(\mathscr{K}_0, \mathbf{0}).$$

Then there exists a weak domination sequence

$$\text{modulo } c(C; \{\{i\}\}) \quad \text{from} \quad C_0 \text{ to } C$$

and, at the same time, there exists a weak domination sequence

$$\text{modulo } \emptyset \text{ from } C \text{ to } C_0 .$$

The same holds, if

$$C$$

replaced by any other configuration

$$C' = C'(\mathscr{K}_i, \mathbf{D})$$

with

$$(\{j, k\}, \boldsymbol{D}_{\{j,k\}}) \in C'_{\text{rat}} .$$

If

$$(\{j, k\}, \boldsymbol{D}_{\{j,k\}}) \in C' - C'_{\text{rat}}$$

then it follows, from Definition 2, that for no

$$C'' \in \mathscr{C} \quad \text{and} \quad c \subset C'$$

he relation

$$C' \text{ dom } C'' \text{ mod } c$$

holds.

It is clear, from Example 2, that the proposed simplification would make the configuration C_0 to be weakly as well as strongly predictionally stable, what in no way corresponds with the image of an acceptable bargaining result among rational players.

Example 3. Supposing the method described by Definitions 6 and 7 is used in order to predict the bargaining, in Example 2 we obtain

$$\mathfrak{P}_{\Gamma}^{**} = \mathfrak{P}_{\Gamma}^{*} = \{C : C = C(\mathscr{K}_i, \mathbf{D}),\ \mathbf{D} = (0, \boldsymbol{D}_{\{j,k\}}),\ (\{j, k\}, \boldsymbol{D}_{\{j,k\}}) \in C_{\mathrm{rat}}\} =$$
$$= \{C : C = C(\mathscr{K}_i, \mathbf{D}),\ x_j(C) \geqq 0\,,\ x_k(C) \geqq 0\}\,.$$

The obtained result is consistent with the intuitive image on a rational bargaining and with the postulates on rationalism.

We have shown that it was useful, when complicated games considered, to moderate our requests for predictional stability. On the other hand, when games with small number of players and with simple identification partitions are considered, it may be useful to strict the stability criterion in the following way.

Definition 8. Supposing C is an admissible configuration we shall say that C *is predictionally balanced* if there exists no configuration $C' \in \mathscr{C}$ and no sub-configuration $c \subset C$ such that

$$C' \operatorname{Dom} C \bmod c\,,$$

The class of all predictionally balanced configurations in a given game Γ will be denoted by $\mathfrak{P}_{\Gamma}$.

Lemma 6.1. The classes of weakly and strongly predictionally stable configurations as well as the class of predictionally balanced configurations satisfy the inclusions:

$$\mathfrak{P}_{\Gamma}^{**} \supset \mathfrak{P}_{\Gamma}^{*} \supset \mathfrak{P}_{\Gamma}\,.$$

Proof. If $C\,\mathbf{P}\,C'$ then also $C\ \mathbf{P}\ C'$ (by Definitions 6, 7). The inclusion $\mathfrak{P}_{\Gamma}^{*} \subset \mathfrak{P}_{\Gamma}$ follows from Definitions 6, 8. □

Lemma 6.2. Every weakly predictionally stable configuration contains at least one agreement which is rational in this configuration.

Proof. If $C \in \mathscr{C}$ and $C_{\mathrm{rat}} = \emptyset$, then there exists, according to Lemma 5.3, $C' \in \mathscr{C}$, $c \subset C$ such that $C' \operatorname{dom} C \bmod c$. On the other hand, for no configuration $C^{\dagger} \in \mathscr{C}$ with $C^{\dagger}_{\mathrm{rat}} = \emptyset$ there exist $C'' \in \mathscr{C}$, $c \subset C^{\dagger}$, such that $C^{\dagger} \operatorname{dom} C'' \bmod c$ (it follows from Definition 2). Therefore, neither the relation $C^{\dagger} \operatorname{Dom} C'' \bmod c$ can hold. It follows, using Definition 7, that $C'\,\mathbf{P}\,C$, hence $C \notin \mathfrak{P}_{\Gamma}^{**}$. □

Remark 6.2. It follows from Lemma 5.3 and from Definition 8, that if $C \in \mathfrak{P}_{\Gamma}$, then $C = C_{\mathrm{rat}}$.

The proposed prediction method cannot guarantee, when games with a general finite identification considered, that the weakly or strongly predictionally stable configurations will contain only agreements, rational in this configurations. The same occurs in the following example.

Example 4. Let us consider a six-players game

$$\Gamma = (I, \mathbf{K}, \{S_K\}_{K\in\|\mathbf{K}\|}, \{\mathcal{R}_K\}_{K\in\|\mathbf{K}\|}, \{v_i\}_{i\in I})$$

which is a game without strategies choosing. Let

$$I = \{1, 2, 3, 4, 5, 6\},$$

and let

$$\mathbf{K} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5, \mathcal{K}_6, \mathcal{K}_7, \mathcal{K}_8, \mathcal{K}_9\},$$

where the admissible coalition structures and the corresponding values of pay-off functions are described by Table 1. The identification partitions of all admissible coalitions are such, that every identification set contains exactly one admissible coalition structure.

Table 1.

$\mathcal{K}$	v_1	v_2	v_3	v_4	v_5	v_6
$\mathcal{K}_1 = \{\{1, 2, 3\}, \{4, 5, 6\}\}$	1	0	1	1	2	1
$\mathcal{K}_2 = \{\{1, 2\}, \{3\}, \{4, 5, 6\}\}$	1	1	0	1	0	1
$\mathcal{K}_3 = \{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$	0	1	1	1	0	1
$\mathcal{K}_4 = \{\{1, 2, 3\}, \{4\}, \{5, 6\}\}$	1	2	1	0	0	0
$\mathcal{K}_5 = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$	1	2	1	0	0	0
$\mathcal{K}_6 = \{\{1\}, \{2, 3\}, \{4\}, \{5, 6\}\}$	1	0	0	1	0	0
$\mathcal{K}_7 = \{\{1, 2\}, \{3\}, \{4\}, \{5, 6\}\}$	0	0	1	1	0	0
$\mathcal{K}_8 = \{\{1\}, \{2, 3\}, \{4, 5\}, \{6\}\}$	1	0	0	0	0	1
$\mathcal{K}_9 = \{\{1, 2\}, \{3\}, \{4, 5\}, \{6\}\}$	0	0	1	0	0	1

The class $\mathscr{C}$ of the admissible configurations can be divided into four disjoint sub-classes:

$$\mathscr{C}_\alpha = \{C : C = C(\mathcal{K}_1, \mathbf{D}) \in \mathscr{C}\},$$
$$\mathscr{C}_\beta = \{C : C = C(\mathcal{K}, \mathbf{D}) \in \mathscr{C},\ \mathcal{K} = \mathcal{K}_2, \mathcal{K}_3\},$$
$$\mathscr{C}_\gamma = \{C : C = C(\mathcal{K}, \mathbf{D}) \in \mathscr{C},\ \mathcal{K} = \mathcal{K}_4, \mathcal{K}_5\},$$
$$\mathscr{C}_\delta = \{C : C = C(\mathcal{K}, \mathbf{D}) \in \mathscr{C},\ \mathcal{K} = \mathcal{K}_6, \mathcal{K}_7, \mathcal{K}_8, \mathcal{K}_9\}.$$

Under the given pay-off functions, the bargaining process will proceed in such a way which can be considered to be in some measure cyklic; sooner or latter, the configurations belonging to the classes

$$\mathscr{C}_\delta,\ \mathscr{C}_\gamma,\ \mathscr{C}_\alpha,\ \mathscr{C}_\beta,\ \mathscr{C}_\delta,\ \ldots$$

will be bargained.

The class of strongly or weakly predictionally stable configurations will contain configurations from all the four sub-classes (the configurations

$$C(\mathscr{K}_r, \mathbf{0}), \quad r = 1, 2, \ldots, \dot{9},$$

for example). At the same time, the sub-classes

$$\mathscr{C}_\alpha,\ \mathscr{C}_\beta$$

contain no configuration consisting only of agreements rational in this configuration.

The example just given shows to us, on the other hand, that this fact does not contradict the common idea of the bargaining. Not only the proposed prediction method but nor any other detailed analysis of the described game, insisting only in the quantitative values expressed by guaranteed pay-offs, is not able to decide to which among the four sub-classes the finally bargained configuration will belong. In order to be able to decide this question, we should have to suppose that the attitude of the players is motivated by something else than by the high of their profit, but this would contradict the postulates.

It is possible, nevertheles, to impose on the weakly or strongly predictionally stable configurations some request for a rationality, originating from the presumption that in a configuration, which is a result of a bargaining process long enough, no agreement can occur, which would not be rational at least in some among of possible configurations. Just given Example 4 illustrates this consideration. After several steps of the bargaining process, any agreement, which occurs and is not rational in the bargained configuration, may be regarded as a "rest" of the previous steps of the bargaining. This fact justifies the introduction of the following definition.

Definition 9. We shall say that an admissible configuration $C = C(\mathscr{K}, \mathbf{D})$ *is of rational character*, if it is possible to find, for every agreement $(K, \mathbf{D}_K) \in C$, some configuration $C' \in \mathscr{C}$ such that $(K, \mathbf{D}_K) \in C'_{\text{rat}}$. The class of all configurations with rational character will be denoted by $\mathscr{C}_{\text{rat}}$.

Now, we have introduced all the notions necessary to the description of the bargaining result in a given game.

Supposing Γ is a strategic coalition-game with finite identification and with side payments, without strategies choosing, then the triple

$$(\mathfrak{P}_\Gamma^{**} \cap \mathscr{C}_{\text{rat}},\ \ \mathfrak{P}_\Gamma^{*} \cap \mathscr{C}_{\text{rat}},\ \ \mathfrak{P}_\Gamma)$$

is called a *bargaining prediction characteristic in Γ*. Its components are considered to be the classes of the possible bargaining results among rational players, shaded by degrees according to their stability in the considered game.

The prediction characteristic, introduced in such a way, is rather adequate to an intuitive idea about the behaviour of rational players in an actual game of the investigated type.

Theorem 1. There exists, in every coalition-game Γ, at least one weakly predictonally stable configuration being of a rational character, i.e.

$$\mathfrak{P}_{\Gamma}^{**} \cap \mathscr{C}_{\mathrm{rat}} \neq \emptyset .$$

Proof. Let us consider a coalition structure $\mathscr{J} \in \mathbf{K}$ to which there exists no $\mathscr{K} \in \mathbf{K}$ such that $\mathscr{K}$ is a sub-partition of $\mathscr{J}$. Then $C(\mathscr{J}, \mathbf{D}) \in \mathscr{C}_{\mathrm{rat}}$ for every matrix $\mathbf{D} \in \mathscr{D}_{\mathscr{J}}$, hence $\mathscr{C}_{\mathrm{rat}} \neq \emptyset$.

Let us part the class $\mathscr{C}$ into a finite number of disjoint sub-classes

$$\mathscr{C}(\mathscr{K}) = \{C : C = C(\mathscr{K}, \mathbf{D}) \in \mathscr{C}\}, \quad \mathscr{K} \in \mathbf{K} .$$

Every of these sub-classes is parted again into $2^{\pi(\mathscr{K})}$ parts $\mathscr{C}^{(\omega)}(\mathscr{K})$, $\omega = 1, 2, \ldots, 2^{\pi(\mathscr{K})}$, such that for every pair (C, C') of configurations from the same sub-class $\mathscr{C}^{(\omega)}(\mathscr{K})$, $1 \leqq \omega \leqq 2^{\pi(\mathscr{K})}$, $\mathscr{K} \in \mathbf{K}$, and for every pair of agreements $((K, \boldsymbol{D}_K) \in C, (K, \boldsymbol{D}'_K) \in C')$ the relation

$$(K, \boldsymbol{D}_K) \in C_{\mathrm{rat}} \Leftrightarrow (K, \boldsymbol{D}'_K) \in C'_{\mathrm{rat}}$$

holds.

Now, let us choose a configuration $C_1 \in \mathscr{C}_{\mathrm{rat}}$. If $C_1 \notin \mathfrak{P}_{\Gamma}^{**}$, then there exists $C \in \mathscr{C}$ such that $C \mathbf{p} C_1$. If $C \in \mathscr{C}_{\mathrm{rat}}$, then we set $C_2 = C$. If $C \notin \mathscr{C}_{\mathrm{rat}}$, then $C_2 \in \mathscr{C}_{\mathrm{rat}}$ and a strong domination sequence from C to C_2 may be constructed, hence, there exists also a strong domination sequence from C_1 to C_2. If $C_2 \notin \mathfrak{P}_{\Gamma}^{**}$, then we can follow the same way. After a finite number of such steps either some $C_t \in \mathscr{C}_{\mathrm{rat}}$, $t \geqq 2$, will be found, such that $C_t \in \mathfrak{P}_{\Gamma}^{**}$, or a sequence $\{C_t\}_{t=1,\ldots,n}$ will be constructed such that:

$$C_t \in \mathscr{C}_{\mathrm{rat}}, \quad t = 1, \ldots, n ;$$

there exists a strong domination sequence from C_{t-1} to C_t, $t = 2, \ldots, n$;

$$n < 3.2^{\pi(I)} . \pi(\mathbf{K}) ;$$

there exists an index $t(0) < n$, such that $C_{t(0)}$ and C_n belong to the same sub-class $\mathscr{C}^{(\omega)}(\mathscr{K})$, $\mathscr{K} \in \mathbf{K}$, $1 \leqq \omega \leqq 2^{\pi(\mathscr{K})}$; and for every configuration $C \in \mathscr{C}$, for which a strong domination sequence from C_n to C exists, there exists also a configuration C' in the strong dominantion sequence from $C_{t(0)}$ to C_n, $C_{t(0)} \neq C' \neq C_n$, such that C' and C belong to the same sub-class $\mathscr{C}^{(\omega)}(\mathscr{K})$, $\mathscr{K} \in \mathbf{K}$, $1 \leqq \omega \leqq 2^{\pi(\mathscr{K})}$. It means that for no $C \in \mathscr{C}$ the relation $C \mathbf{p} C_n$ holds. □

Before closing this part, some properties of the configurations with only one admissible agreement of all players will be mentioned. Supposing this agreement is rational, the prediction procedure will be substantially simplified.

Lemma 6.3. Supposing the coalition structure $\mathscr{K}_I = \{I\}$ is admissible, $\mathscr{K}_I \in \mathbf{K}$, and $C = C(\mathscr{K}_I, \mathbf{D}) \in \mathscr{C}$ then the two following assertions hold:

(1) The following assertions are equivalent:

(a) there exists no $L \in \|\mathbf{K}\|$ and no $\boldsymbol{R}_L \in \mathscr{R}_L$ such that

$$\sum_{i \in L} v_i(L, \boldsymbol{R}_L) > \sum_{i \in L} x_i(C),$$

(b) $$C = C_{\mathrm{rat}},$$

(c) $$C \in \mathscr{C}_{\mathrm{rat}},$$

(d) $$C \in \mathfrak{P}_\Gamma,$$

(e) $$C \in \mathfrak{P}_\Gamma^*$$

(f) $$C \in \mathfrak{P}_\Gamma^{**}.$$

(2) If $C = C_{\mathrm{rat}}$, if the relation $C \,\delta\, C' \bmod \emptyset$ holds for some $C' \in \mathscr{C}$, then $C' \notin \mathfrak{P}_\Gamma^{**}$.

Proof. The both parts of the Lemma can be derived from the foregoing assertions ad definitions in the following way.

The equivalence of the assertions (a) and (b) follows immediately from Definition 1 and Lemma 3.1. The assertion (c) is equivalent to (b), according to Definition 9. If $C = C_{\mathrm{rat}}$, then it follows from Lemma 5.3 that there exists no $C' \in \mathscr{C}$ such that C' Dom $C \bmod \emptyset$. Hence (d) follows from (b). According to Lemma 6.1, (e) follows from (d) and (f) follows from (e). If the agreement $(I, \boldsymbol{D}_I) \in C$ is not rational in C, then, according to Lemma 6.2, $C \notin \mathfrak{P}_\Gamma^{**}$, hence (b) follows from (f). Therefore the relations

$$(a) \Leftrightarrow (b) \Leftrightarrow (c), \quad (b) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (b)$$

hold, and the first part of the Lemma is proved.

If $C = C_{\mathrm{rat}}$ and $C \,\delta\, C' \bmod \emptyset$ for some $C' \in \mathscr{C}$, then, according to Lemma 5.3, C Dom $C' \bmod \emptyset$. On the other hand, according to the same Lemma, there exists no $C'' \in \mathscr{C}$ such that C'' Dom $C \bmod \emptyset$. Hence $C \,\mathbf{p}\, C'$, according to Definitions 5 and 7. □

PART III. BARGAINING WITHOUT IDENTIFICATION

In the previous part of this paper, some general results concerning the proposed bargaining model were presented. It is possible to obtain some more detailed results, especially if we abstain from assumption of the identification possibility. This part is subjected to the model of bargaining in such coalition-games, in which no identification is possible.

It means that we suppose, in all this part, the investigated game being a game without identification as defined in section 4. The notations, introduced in the previous part of this paper, are generally kept even in this one. The slight modification of it, based on the simplification of the investigated game model, is introduced in the following section 7.

7. Special Properties of Bargaining Model

Games without identification and without strategies choosing represent one possible special example of the coalition-games. They may be interpreted as such games, in which the players have no information about the bargained behaviour of their anti-players. According to their character, these games are in the most close connection with the games investigated in literature (e.g. [1; 2; 6]).

Clearly, the impossibility of an identification simplifies, in a rather significant measure, the bargaining prediction in such games and enables, moreover, to simplify the used notation. If

$$\Gamma = (I, \mathbf{K}, \{S_K\}_{K\in\|\mathbf{K}\|}, \{\mathcal{R}_K\}_{K\in\|\mathbf{K}\|}, \{v_i\}_{i\in I})$$

is a coalition-game without identification and without strategies choosing, then for every coalition $K \in \|\mathbf{K}\|$ the set S_K and the class $\mathcal{R}_K$ are one-element sets. The pay-off functions v_i, $i \in I$, can be seen as if they were functions of only one variable K defined on the class $\{K : K \in \|\mathbf{K}\|, i \in K\}$. Instead of $v_i(K, s_K, R_K)$ or $v_i(K, R_K)$, we can write simply $v_i(K)$, i $\in K$, and $v_i(K) = v_i(\mathcal{K})$, if $i \in K \in \mathcal{K}$.

Let us denote, for every coalition $K \in \|\mathbf{K}\|$,

$$w(K) = \sum_{i\in K} v_i(K) . \tag{7.1}$$

This function w has some properties of the coalition characteristic function, known from the literature (e.g. [1; 5; 7; 10] e.t.c.).

Remark 7.1. If $C \in \mathscr{C}$, $(K, D_K) \in C$, $D_K = (d_{ij})_{i\in K, j\in K}$, if a game without an identification is considered, then $(K, D_K) \in C_{\text{rat}}$ iff (K, D_K) is rational in every configuration containing (K, D_K). This holds iff there exists no $L \in \|\mathbf{K}\|$ such that

$$w(L) > \sum_{i\in L} x_i(C) = \sum_{i\in L} v_i(K) + \sum_{i\in L} \sum_{j\in K} d_{ij} ,$$

where

$$L \subset K, K - L = \bigcup_{r=1}^{n} J_r, J_r \cap J_t = \emptyset , \quad J_r \in \|\mathbf{K}\| ,$$

$$r = 1, \ldots, n , \quad t = 1, \ldots, n , \quad r \neq t , \quad n \in N .$$

It follows that

$$\mathscr{C}_{\text{rat}} = \{C : C \in \mathscr{C} , \quad C = C_{\text{rat}}\} .$$

Instead of the phrase *"an agreement (K, D_K) is rational in a configuration C"* we can use the abbreviate form *"an agreement (K, D_K) is rational"*.

Remark 7.2. It follows from Remark 7.1 that when the games without an identification are considered, Definition 2 may be written in the following, simple, form:

If

$$C \in \mathscr{C}\,, \quad C' \in \mathscr{C}\,, \quad c \subset C \cap C'$$

then

$$C \operatorname{dom} C' \bmod c$$

if and only if

$$C\,\delta\,C' \bmod c \quad \text{and} \quad C - c \subset C_{\text{rat}}\,.$$

Remark 7.3. It follows from Remark 7.2 and from Lemma 5.3, that when the games without an identification are considered, then every weakly predictionally stable (hence, also every strongly predictionally stable) configuration C contains only rational agreements, i.e.

$$\mathscr{C}_{\text{rat}} \supset \mathfrak{P}_{\Gamma}^{**} \supset \mathfrak{P}_{\Gamma}^{*}$$

(compare with Remark 6.2). The bargaining prediction characteristic is then represented by the triple

$$(\mathfrak{P}_{\Gamma}^{**}, \mathfrak{P}_{\Gamma}^{*}, \mathfrak{P}_{\Gamma})\,.$$

There are, besides the general assertion introduced in the foregoing sections, also some other assertions which are valid when the games without an identification and without strategies choosing are considered, and which simplify the prediction procedure.

Lemma 7.1. Supposing the relation

$$C \operatorname{Dom} C' \bmod \emptyset$$

holds for admissible configurations $C, C' \in \mathscr{C}$ then also the relations $C\,\mathbf{P}\,C'$ and $C' \notin \mathfrak{P}_{\Gamma}^{*}$ hold.

Proof. The relation $C \operatorname{Dom} C' \bmod \emptyset$ implies, according to Remark 7.2, the identity $C = C_{\text{rat}}$. Suppose the relation $C\,\mathbf{P}\,C'$ does not hold. If it is the case, there exists a strong domination sequence from C to C', denoted, for example,

$$\{C^{(r)}\}_{r=1,\ldots,n}\,, \quad \{c^{(r)}\}_{r=1,\ldots,n-1}\,,$$

such that

$$C^{(1)} = C\,, \quad C^{(n)} = C'\,, \quad C^{(r)} \operatorname{Dom} C^{(r-1)} \bmod c^{(r-1)}\,, \quad r = 2, 3, \ldots, n\,.$$

Then there exists, for every $r = 1, \ldots, n-1$ a weak domination sequence modulo $c^{(r)}$ from $C^{(r)}$ to $C^{(r+1)}$ which will be denoted by

$$\{C^{(t,r)}\}_{t=1,\ldots,m(r)}\,,$$
$$C^{(1,r)} = C^{(r)}\,, \quad C^{(m(r),r)} = C^{(r+1)}\,, \quad r = 1, 2, \ldots, n-1\,.$$

Remarks 7.1 and 7.2 imply that

$$C^{(t,r)} = C^{(t,r)}_{\text{rat}} \quad \text{for all} \quad r = 1, \ldots, n\,, \quad t = 1, \ldots, m(r)\,,$$

because $C = C_{\text{rat}}$. It follows, according to Remark 5.4, that the sequence

$$\{C^{(1,1)}, \; C^{(2,1)}, \ldots, C^{(m(1)-1,1)}, \; C^{(1,2)}, \ldots, C^{(m(n-1)-1,n-1)},$$
$$C^{(m(n-1),n-1)} = C^{(n)} = C\}$$

is a weak domination sequence modulo $\emptyset$ from $C = C^{(1)}$ to $C' = C^{(n)}$. This result contradicts the assumption

$$C \text{ Dom } C' \bmod \emptyset\,.$$

Hence, the relations $C\, \mathbf{P}\, C'$ and $C' \notin \mathfrak{P}^*_\Gamma$ hold. □

Supposing Γ is a game without identification and with a strictly bounded cooperation, then the relation between the predictional avoiding relation and the strong domination relation modulo $\emptyset$ is closer.

Lemma 7.2. If C and C' are two admissible configurations in a coalition-game without identification and with strictly bounded cooperation, if $C = C_{\text{rat}}$, and if $C\, \mathbf{p}\, C'$, then the relation C Dom C' mod $\emptyset$ holds.

Proof. Supposing there exists a strong domination sequence from C' to C, and if $C = C_{\text{rat}}$ then, according to Remark 7.2 and Remark 5.4, even a weak domination relation modulo $\emptyset$ from C' to C may be constructed. On the other hand, if $C' \neq C'_{\text{rat}}$, no weak domination relation modulo $\emptyset$ from C to C' can exist.

Let us suppose that $C' = C'_{\text{rat}}$, and denote $C = C(\mathscr{K}, \mathbf{D})$, $C' = C'(\mathscr{K}', \mathbf{D}')$. As a game without an identification and with strictly bounded cooperation is considered, and as there exists a strong domination sequence from C' to C, necessarily $\mathscr{K}'$ is a sub-partition of the coalition structure $\mathscr{K}$, and $\mathscr{K}' \neq \mathscr{K}$. As $C = C_{\text{rat}}$, no $C'' = C''(\mathscr{K}'', \mathbf{D}'') \in \mathscr{C}$ can exist such that $\mathscr{K}''$ was a sub-partition of $\mathscr{K}$ and $C''\ \delta\ C \bmod \emptyset$. It means that

$$C \text{ Dom } C' \bmod \emptyset\,. \qquad \square$$

Remark 7.4. In the following the symbol $\mathfrak{Q}_\Gamma$ may be useful for the following class of configurations

$$(7.2) \quad \mathfrak{Q}_\Gamma = \{C : C \in \mathscr{C}\,, \ \textit{there exists no } C' \in \mathscr{C} \textit{ such that } \ C' \text{ Dom } C \bmod \emptyset\}\,.$$

It follows, according to Lemmas 7.1 and 7.2, that when a coalition-game without identification is considered, then $\mathfrak{P}^*_\Gamma \subset \mathfrak{Q}_\Gamma$. If the considered game, is moreover, a game with strictly bounded cooperation, then

$$\mathfrak{P}^{**}_\Gamma \supset \mathfrak{Q}_\Gamma \supset \mathfrak{P}^*_\Gamma\,.$$

Lemma 7.3. If C is a coalition-game without an identification, if $C = C(\mathscr{K}, \mathbf{D}) \in \mathscr{C}$, $C' = C'(\mathscr{L}, \mathbf{D}') \in \mathscr{C}$, if $\mathscr{L}$ is a sub-partition of $\mathscr{K}$, if $c = c(C; k)$, $c' = c'(C', k')$ so that $\|k\| = \|k'\|$, if the equality $x_i(C) = x_i(C')$ holds for all $i \in I$, then the assertions

(1) if $C \cap c \subset C_{\text{rat}}$ then $C' \cap c' \subset C'_{\text{rat}}$,

(2) supposing there exists $C_1 \in \mathscr{C}$ such that C dom C_1 mod c then there exists $C_2 \in \mathscr{C}$ such that C' **dom** C_2 mod c' and $x_i(C_1) = x_i(C_2)$ for all $i \in I$,

(3) supposing there exists $C_2 \in \mathscr{C}$ such that C_2 dom C' mod c', then there exists $C_1 \in \mathscr{C}$ such that C_1 dom C mod c, and $x_i(C_1) = x_i(C_2)$ for all $i \in I$,

hold.

Proof. The first assertion is an immediate consequence of Definition 1 and the assumptions of this Lemma. The second one follows immediately from the first one, from Definition 2 (Remark 7.2), and from the assumptions, if we choose, for a given $C_1 \in \mathscr{C}$, such a $C_2 \in \mathscr{C}$, that $C_2 \cap C' \supset c'$ and $C_1 \cap C_2 \supset C_1 - c$. The third assertion is an immediate consequence of the assumptions of this Lemma, of the relation (5.1), and of Lemma 5.2, if we choose, for a given $C_2 \in \mathscr{C}$, such a $C_1 \in \mathscr{C}$, that $C_1 \cap C \supset c$ and $C_1 \cap C_2 \supset C_2 - c'$. □

Theorem 2. Let us consider a coalition game Γ without an identification and with a strictly bounded cooperation. Denote by $\mathscr{L} \in \mathbf{K}$ the coalition structure, for which any $\mathscr{K} \in \mathbf{K}$ is a sub-partition of $\mathscr{L}$. Then every configuration $C \in \mathscr{C}$ is weakly predictionally stable if and only if it is predictionally balanced, and the bargaining prediction characteristic is defined by the relation

$$\mathfrak{P}_\Gamma^{**} = \mathfrak{P}_\Gamma^{*} = \mathfrak{P}_\Gamma = \mathfrak{Q}_\Gamma = \mathscr{Q} \cup \mathscr{Q}',$$

where $\mathfrak{Q}_\Gamma$ is defined by (7.2),

$$\mathscr{Q} = \{C : C = C(\mathscr{L}, \mathbf{D}) \in \mathscr{C},\ C = C_{\text{rat}}\},$$

$$\mathscr{Q}' = \{C' : C' = C'(\mathscr{K}, \mathbf{D}) \in \mathscr{C},\ \mathscr{K} \neq \mathscr{L},\ \textit{and there exists}$$
$$C \in \mathscr{Q} \textit{ such that } x_i(C) = x_i(C') \textit{ for all } i \in I\}.$$

Proof. If $C = C(\mathscr{L}, \mathbf{D})$, $C = C_{\text{rat}}$, then there exists no $C'' \in \mathscr{C}$ and no $c \subset C \cap C''$ so that

$$C''\ \delta\ C \bmod c.$$

Therefore $C \in \mathfrak{P}_\Gamma$ and $\mathscr{Q} \subset \mathfrak{P}_\Gamma$. If $C' \in \mathscr{Q}'$ then there exists $C \in \mathscr{Q}$ such that

$$x_i(C) = x_i(C') \quad \text{for all} \quad i \in I.$$

If there exist $C_2 \in \mathscr{C}$, $c_2 \subset C'$ such that

$$C_2\ \delta\ C' \bmod c_2,$$

then there exist, according to Lemmas 7.3 and 5.2, $C_1 \in \mathscr{C}$ and $c_1 \subset C$ such that

$$C_1 \ \delta\ C \bmod c_1 .$$

This relation contradicts the assumption $C \in \mathscr{Q}$. Hence, if $C' \in \mathscr{Q}'$, then $C' \in \mathfrak{P}_r$ and $\mathscr{Q}' \subset \mathfrak{P}_r$. So we have proved that

$$\mathfrak{P}_r \supset \mathscr{Q} \cup \mathscr{Q}' .$$

It is necessary to prove, moreover, that

$$\mathscr{Q} \cup \mathscr{Q}' \supset \mathfrak{P}_r^{**} .$$

If $C = C(\mathscr{L}, \mathbf{D}) \in \mathscr{C}$, $C \notin \mathscr{Q}$, then $C \neq C_{\text{rat}}$ and it follows, according to Remarks 7.1 and 7.3, that $C \notin \mathfrak{P}_r^{**}$. If $C' = C'(\mathscr{K}, \mathbf{D}') \in \mathscr{C}$, $\mathscr{K} \neq \mathscr{L}$, $C' \notin \mathscr{Q}'$ then either $C' \neq C'_{\text{rat}}$ and $C' \notin \mathfrak{P}_r^{**}$, or $C' = C'_{\text{rat}}$ and in such a case there always exists $C = C(\mathscr{L}, \mathscr{D}) \in \mathscr{C}$ such that

$$x_i(C) \geqq x_i(C') \quad \text{for all} \quad i \in I .$$

If for some $i \in I$ the inequality

$$x_i(C) > x_i(C')$$

holds, then $C \ \delta\ C' \bmod \emptyset$ and there exists, according to Lemma 5.2, $C'' = C''(\mathscr{K}'', \mathbf{D}'') \in \mathscr{C}$ such that C'' dom $C' \bmod \emptyset$. As $C' = C'_{\text{rat}}$, $\mathscr{K}$ is a sub-partition of $\mathscr{K}''$, $\mathscr{K} \neq \mathscr{K}''$, According to Remark 7.2, also $C'' = C''_{\text{rat}}$, therefore no weak domination sequence modulo $\emptyset$ from C'' to C can exist. It means that

$$C'' \text{ Dom } C' \bmod \emptyset .$$

It is possible to find, among all the configurations $C'' \in \mathscr{C}$, satisfying C'' Dom . . $C' \bmod \emptyset$, at least one configuration $\bar{C} = \bar{C}(\bar{\mathscr{K}}, \bar{\mathbf{D}}) \in \mathscr{C}$ such that if $C'' = C''(\mathscr{K}'', \mathbf{D}'') \in \mathscr{C}$, C'' Dom $C' \bmod \emptyset$, then $\mathscr{K}''$ is a sub-partition of $\bar{\mathscr{K}}$. At the same time $\bar{C} = \bar{C}_{\text{rat}}$ and there exist no $C_1 \in \mathscr{C}$ and no $c_1 \subset \bar{C} \cap C_1$ such that $C_1 \ \delta\ \bar{C} \bmod c_1$. It means that not only

$$\bar{C} \text{ Dom } C' \bmod \emptyset$$

but also $\bar{C}\ \mathbf{p}\ C'$ holds. Hence, $C' \notin \mathfrak{P}_r^{**}$.

Supposing the equality

$$x_i(C) = x_i(C')$$

holds for every player $i \in I$, and $C' \notin \mathscr{Q}'$, then $C \notin \mathscr{Q}$ and there exists $C_1 \in \mathscr{C}$ such that $C_1 \ \delta\ C \bmod \emptyset$. According to Lemma 5.2, there exists $C'' \in \mathscr{C}$ such that C'' dom C . . mod $\emptyset$. According to Lemma 7.3, C'' dom $C' \bmod \emptyset$ and the relation $C' \notin \mathfrak{P}_r^{**}$ can be derived in the same way as in the foregoing case.

In such a way the inclusion

$$\mathfrak{P}_r^{**} \subset \mathscr{Q} \cup \mathscr{Q}'$$

has been proved. According to the foregoing steps of this proof, to Lemma 6.1 and to Remark 7.4 the following chain of inclusions follows:

$$\mathcal{Q} \cup \mathcal{Q}' \supset \mathfrak{P}_\Gamma^{**} \supset \mathfrak{Q}_\Gamma \supset \mathfrak{P}_\Gamma^{*} \supset \mathfrak{P}_\Gamma \supset \mathcal{Q}' \cup \mathcal{Q}$$

which proves according to Theorem 1, the assertion of this Theorem. □

The following theorem deals with the prediction in another special type of coaliton-games without identification.

Theorem 3. Supposing every admissible coalition structure in a coalition-game without identification contains at most two coalitions, then every admissible configuration is strongly predictionally stable if and only if it is predictionally balanced and this holds if and only if it is not strongly dominated modulo $\emptyset$; i.e.

$$\mathfrak{P}_\Gamma^{*} = \mathfrak{P}_\Gamma = \mathfrak{Q}_\Gamma .$$

Proof. Lemma 6.1 and Remark 7.4 imply that

$$\mathfrak{Q}_\Gamma \supset \mathfrak{P}_\Gamma^{*} \supset \mathfrak{P}_\Gamma .$$

Only the inclusion

$$\mathfrak{Q}_\Gamma \subset \mathfrak{P}_\Gamma$$

remains to be proved. If $C = C(\mathcal{K}, \mathbf{D}) \in \mathcal{C}$, then either $\mathcal{K} = \{I\}$ or $\mathcal{K} = \{K, I-K\}$, $\emptyset \neq K \neq I$. If $\mathcal{K} = \{I\}$ then

$$C \in \mathfrak{Q}_\Gamma \Leftrightarrow C \in \mathfrak{P}_\Gamma \Leftrightarrow C = C_{\text{rat}} ,$$

according to Lemmas 5.3 and 6.1. If $\mathcal{K} = \{K, I-K\}$, $K \subset I$, $\emptyset \neq K \neq I$, if there exists $C' \in \mathcal{C}$, $c = C \cap C'$ such that C' Dom C mod c, then necessarily $c = \emptyset$. Hence, we obtain again

$$C \in \mathfrak{Q}_\Gamma \Leftrightarrow C \in \mathfrak{P}_\Gamma .$$

It means that

$$\mathfrak{P}_\Gamma = \mathfrak{Q}_\Gamma \supset \mathfrak{P}_\Gamma^{*} \supset \mathfrak{P}_\Gamma ,$$

which proves the assertion of the theorem. □

8. Bargaining in Three-Players Games without Identification

Three-players games represent the most simple case of the coalition-games (i.e. coalition-games without strategies choosing) to which the proposed prediction procedure can be applied. The notation introduced in Section 7 is preserved during whole this section. Moreover, the players will be denoted by the indices 1, 2, 3, i.e. $I = \{1, 2, 3\}$, the particular coalition structures will be denoted by

$$\mathcal{K}_0 = \{\{1\}, \{2\}, \{3\}\} , \quad \mathcal{K}_i = \{\{i\}, I - \{i\}\} , \quad i = 1, 2, 3 , \quad \mathcal{K}_I = \{I\} .$$

Instead of $\boldsymbol{v}_i(\{i\})$, $\boldsymbol{w}(\{i\})$, $\boldsymbol{v}_i(\{i,j\})$, $\boldsymbol{w}(\{i,j\})$, $i \in I$, $j \in I$, $i \neq j$, we shall write simply $\boldsymbol{v}_i(i)$, $\boldsymbol{w}(i)$, $\boldsymbol{v}_i(i,j)$, $\boldsymbol{w}(i,j)$, respectively.

Remark 8.1. When a three-players game without an identification and with a free cooperation is considered, then Lemmas 5.1 and 6.3, Remark 7.1 and relations (2.3) and (7.1) imply the following assertions:

(a) For any two-element coalition $\{i,j\} \subset I$, a side payments matrix $\boldsymbol{D}_{\{i,j\}}$ can be found such that the agreement $(\{i,j\}. \boldsymbol{D}_{\{i,j\}})$ was rational.

(b) If a side-payments matrix $\boldsymbol{D}_I \in \mathscr{D}_I$, for which the agreement $(I, \boldsymbol{D}_I)$ is rational, exists then

$$2 \,.\, \boldsymbol{w}(I) \geqq \sum_{\{i,j\} \subset I} \boldsymbol{w}(i,j)\,.$$

Besides the general theorems, introduced in the foregoing sections, even the following statements hold.

Lemma 8.1. If Γ is a three-players coalition-game without identification, then $\mathfrak{Q}_\Gamma \subset \mathfrak{P}_\Gamma^{**}$.

Proof. Let $C \in \mathscr{C}$, and let $C \notin \mathfrak{P}_\Gamma^{**}$. If $C \neq C_{\text{rat}}$ then there exists $C' \in \mathscr{C}$, $C' = C'_{\text{rat}}$, such that

$$C' \text{ dom } C \text{ mod } \emptyset$$

and, according to Definition 2, or Remark 7.2, there exists no $C'' \in \mathscr{C}$ such that

$$C \text{ dom } C'' \text{ mod } \emptyset\,.$$

Hence,

$$C' \text{ Dom } C \text{ mod } \emptyset$$

as well. If $C = C_{\text{rat}}$, $C = C(\mathscr{K}, \mathbf{D})$, and $C \notin \mathfrak{P}_\Gamma^{**}$, then is, in accordance with Lemma 6.3, $\mathscr{K} \neq \mathscr{K}_I$ and there exist $C' = C'(\mathscr{K}', \mathbf{D}') \in \mathscr{C}$, $C' = C'_{\text{rat}}$, and $c \subset C \cap C'$ so that

$$C' \text{ Dom } C \text{ mod } c\,.$$

If $\mathscr{K} \neq \mathscr{K}_0$ then the only c, satisfying these conditions, is $c = \emptyset$. If $\mathscr{K} = \mathscr{K}_0$ and $c \neq \emptyset$, then there exists a weak domination sequence modulo c from C to C' and, consequently, there exists $C'' = C''_{\text{rat}}$, $C'' \in \mathscr{C}$, such that

$$C'' \text{ dom } C \text{ mod } c\,.$$

The equation $C'' = C''_{\text{rat}}$ implies, according to Remark 5.4, the relation

$$C'' \text{ dom } C \text{ mod } \emptyset\,.$$

On the other hand, there is no weak domination sequence modulo $\emptyset$ from C'' to C, for $C'' = C''_{\text{rat}}$ and $\mathscr{K} = \mathscr{K}_0$. Hence,

$$C'' \text{ Dom } C \bmod \emptyset$$

and the proof is finished. □

Theorem 4. When a three-players coalition-game Γ without identification is considered, then every admissible configuration is weakly predictionally stable if and only if it is predictionally balanced, and this holds if and only if it is not strongly dominated modulo $\emptyset$. The bargaining prediction characteristic in Γ satisfies, in such a case, the identities

$$\mathfrak{P}_\Gamma^{**} = \mathfrak{P}_\Gamma^{*} = \mathfrak{P}_\Gamma = \mathfrak{Q}_\Gamma .$$

Proof. At first, the identity $\mathfrak{P}_\Gamma^{*} = \mathfrak{P}_\Gamma = \mathfrak{Q}_\Gamma$ will be proved. If $\mathscr{K}_0 \notin \mathbf{K}$, then this identity holds, according to Theorem 3. If $\mathscr{K}_0 \in \mathbf{K}$, if there exists $\mathscr{K} \neq \mathscr{K}_0$, $\mathscr{K} \in \mathbf{K}$, and $K \in \mathscr{K}$ such that

$$w(K) > \sum_{i \in K} w(i) ,$$

then there exists $C = C(\mathscr{K}, \mathscr{D}) \in \mathscr{C}$ such that $C = C_{\text{rat}}$ and C Dom $C_0 \bmod \emptyset$, where $C_0 = C_0(\mathscr{K}_0, \mathbf{0}) \in \mathscr{C}$, $\mathbf{0}$ is the zero side-payments matrix. It implies that

$$C_0 \notin \mathfrak{Q}_\Gamma \supset \mathfrak{P}_\Gamma^{*} \supset \mathfrak{P}_\Gamma .$$

Suppose that $C \neq C_0$, $C \in \mathscr{C}$. If $C \neq C_{\text{rat}}$, then there exists $C' \in \mathscr{C}$ such that C' Dom $C \bmod \emptyset$, and

$$C \notin \mathfrak{Q}_\Gamma \supset \mathfrak{P}_\Gamma^{*} \supset \mathfrak{P}_\Gamma .$$

If $C = C_{\text{rat}}$ then the relation $C_0 \,\delta\, C \bmod \emptyset$ cannot hold. If for some $C' \in \mathscr{C}$, $c \subset C$, the relation $C' \,\delta\, C \bmod c$ holds, then $c = \emptyset$, hence

$$C \in \mathfrak{P}_\Gamma \Leftrightarrow C \in \mathfrak{Q}_\Gamma$$

and the identity holds. If

$$w(K) = \sum_{i \in K} w(i)$$

for all coalitions $K \in \|\mathbf{K}\|$, then

$$\mathfrak{P}_\Gamma = \mathfrak{Q}_\Gamma = \mathfrak{P}_\Gamma^{*} = \{C : C \in \mathscr{C},\ x_i(C) = w(i)\} = \mathscr{C}_{\text{rat}} .$$

Now, the identity $\mathfrak{P}_\Gamma^{**} = \mathfrak{Q}_\Gamma$ remains to be proved. Lemma 8.1 implies that $\mathfrak{P}_\Gamma^{**} \supset \mathfrak{Q}_\Gamma$. Supposing Γ is a game with a strictly bounded cooperation, then the desired identity follows immediately from Theorem 2. If it is not the case, then the class $\mathbf{K}$ contains at least two from the three coalition structures $\mathscr{K}_i$, $i = 1, 2, 3$. Let us suppose that $\mathfrak{P}_\Gamma^{**} - \mathfrak{Q}_\Gamma \neq \emptyset$, and try to describe the configurations which may belong to the class $\mathfrak{P}_\Gamma^{**} - \mathfrak{Q}_\Gamma$. No configuration of the type $C = C(\mathscr{K}_I, \mathbf{D})$ may belong to this class. If $C \notin \mathfrak{Q}_\Gamma$, then necessary $C \neq C_{\text{rat}}$ and, according to Lemma 6.3, $C \notin \mathfrak{P}_\Gamma^{**}$. Analogously, no configuration $C = C(\mathscr{K}_0, \mathbf{0})$ belongs to the class $\mathfrak{P}_\Gamma^{**} - \mathfrak{Q}_\Gamma$. As C is the only possible configuration including the coalition structure $\mathscr{K}_0$, $C \in \mathfrak{P}_\Gamma^{**} \Leftrightarrow C \in \mathfrak{P}_\Gamma^{*}$ and, because of $\mathfrak{P}_\Gamma^{*} = \mathfrak{Q}_\Gamma$, C cannot belong to $\mathfrak{P}_\Gamma^{**} - \mathfrak{Q}_\Gamma$.

Now, the configurations of the type $C = C(\mathscr{K}_i, \mathbf{D}) \in \mathscr{C}$, $i = 1, 2, 3$, remain. Let us consider, step by step, such particular cases of games, in which the class $\mathbf{K}$ contains at least two from the coalition structures $\mathscr{K}_i$, $i = 1, 2, 3$. We limit ourselves to introducting the principal steps of the procedure, as they are sufficient in order to explain the method of this proof and because of the length of the whole detailed proof.

At first, the case, in which $\mathbf{K}$ contains exactly two from the coalition structures $\mathscr{K}_i$, $i \in I$, is considered; let us denote these two structures by $\mathscr{K}_j$, $\mathscr{K}_k$. By $\mathscr{C}_{jk}$ the class

$$\mathscr{C}_{jk} = \{C : C = C(\mathscr{K}_i, \mathbf{D}) \in \mathscr{C}, i = j, k\}$$

is denoted. It follows, from the foregoing, that:

$$\mathfrak{P}_\Gamma^{**} - \mathfrak{Q}_\Gamma = (\mathfrak{P}_\Gamma^{**} \cap \mathscr{C}_{jk}) - (\mathfrak{Q}_\Gamma \cap \mathscr{C}_{jk}).$$

If $\mathfrak{P}_\Gamma^{**} \cap \mathscr{C}_{jk} = \emptyset$, then $\mathfrak{P}_\Gamma^{**} - \mathfrak{Q}_\Gamma = \emptyset$.
If $\mathfrak{P}_\Gamma^{**} \cap \mathscr{C}_{jk} \neq \emptyset$, and if

$$w(i,j) - w(j) > w(i,k) - w(k), \quad i \in I, \quad j \neq i \neq k,$$

then

$$\mathfrak{P}_\Gamma^{**} \cap \mathscr{C}_{jk} = \mathfrak{Q}_\Gamma \cap \mathscr{C}_{jk} = \{C : C = C(\mathscr{K}_k, \mathbf{D}),$$
$$C \in \mathscr{C}_{jk}, \ x_j(C) \geqq w(j), \ x_i(C) \geqq w(i,k) - w(k)\};$$

if

$$w(i,j) - w(j) = w(i,k) - w(k), \quad i \in I, \quad j \neq i \neq k,$$

then

$$\mathfrak{P}_\Gamma^{**} \cap \mathscr{C}_{jk} = \mathfrak{Q}_\Gamma \cap \mathscr{C}_{jk} = \{C : C \in \mathscr{C}_{jk}, x_j(C) = w(j), x_k(C) =$$
$$= w(k), x_i(C) = w(i,j) - w(j)\}.$$

Hence, in every case $\mathfrak{P}_\Gamma^{**} - \mathfrak{Q}_\Gamma = \emptyset$.

Finally we shall consider the case the class $\mathbf{K}$ contains all the coalition structures $\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3$. Let us denote

$$\mathscr{C}_{123} = \{C : C = C(\mathscr{K}_i, \mathbf{D}) \in \mathscr{C}, \ i = 1, 2, 3\}.$$

If $\mathfrak{P}_\Gamma^{**} \cap \mathscr{C}_{123} = \emptyset$, the identity $\mathfrak{P}_\Gamma^{**} - \mathfrak{Q}_\Gamma = \emptyset$ holds like in the foregoing case.
If $\mathfrak{P}_\Gamma^{**} \cap \mathscr{C}_{123} \neq \emptyset$, then the following possibilities occur:

(a) There exists a coalition $\{j, k\} \in \|\mathbf{K}\|$ satisfying:

(8.1) $$w(j, k) > w(i, j) + w(i, k) - 2w(i), \quad i \in I, \quad j \neq i \neq k,$$

then $\mathfrak{P}_{\Gamma}^{**} \cap \mathscr{C}_{123} = \mathfrak{Q}_{\Gamma} \cap \mathscr{C}_{123} = \{C : C = C(\mathscr{K}_i, \mathbf{D}) \in \mathscr{C}, x_j(C) \geqq w(i, j) - w(i), x_k(C) \geqq w(i, k) - w(i)\}$,

(8.2) $$w(j, k) = w(i, j) + w(i, k) - 2w(i), \quad i \in I, \quad j \neq i \neq k,$$

and then $\mathfrak{P}_{\Gamma}^{**} \cap \mathscr{C}_{123} = \mathfrak{Q}_{\Gamma} \cap \mathscr{C}_{123} = \{C : C \in \mathscr{C}_{123}, x_i(C) = w(i), x_j(C) = w(i, j) - w(i), x_k(C) = w(i, k) - w(i)\}$.

(b) Supposing neither (8.1) nor (8.2) holds for any two-element coalition from $\|\mathbf{K}\|$ and if there exists a coalition $\{i, j\} \in \|\mathbf{K}\|$ satisfying

(8.3) $$w(i, k) - w(k) > w(i, j) - w(j), \quad w(j, k) - w(k) > w(i, j) - w(i),$$

then we can proceed at the same way as if $\mathscr{K}_k \notin \mathbf{K}$, and in the same way as in the foregoing part of the proof we can derive that

$$\mathfrak{P}_{\Gamma}^{**} \cap \mathscr{C}_{123} = \mathfrak{Q}_{\Gamma} \cap \mathscr{C}_{123}.$$

(c) Supposing no of the foregoing possibilities (a) and (b) occurs, then:

$$\mathfrak{P}_{\Gamma}^{**} \cap \mathscr{C}_{123} = \mathfrak{Q}_{\Gamma} \cap \mathscr{C}_{123} = \{C : C \in \mathscr{C}_{123}, x_i(C) \geqq w(i), i = 1. 2, 3\}.$$

It means that even in this case $\mathfrak{P}_{\Gamma}^{**} - \mathfrak{Q}_{\Gamma} = \emptyset$. Hence, this identity holds in every case. So, we have proved the identities

$$\mathfrak{P}_{\Gamma}^{*} = \mathfrak{P}_{\Gamma} = \mathfrak{Q}_{\Gamma} \quad \text{and} \quad \mathfrak{P}_{\Gamma}^{**} = \mathfrak{Q}_{\Gamma},$$

which prove the assertion of Theorem. □

Remark 8.2. The foregoing results enable us to determine, if necessary, the actual form of the bargaining prediction characteristic for particular types of the three-players game without identification. In every case

$$\mathfrak{P}_{\Gamma}^{**} \cap \mathscr{C}_{\text{rat}} = \mathfrak{P}_{\Gamma}^{**} = \mathfrak{P}_{\Gamma}^{*} = \mathfrak{P}_{\Gamma} = \mathfrak{Q}_{\Gamma}$$

and the class $\mathfrak{Q}_{\Gamma}$ is of the following form:

(α) If Γ is a game with fixed cooperation, then $\mathfrak{Q}_{\Gamma} = \mathscr{C}$.

(β) If $\mathbf{K} = \{\mathscr{K}_0, \mathscr{K}_i\}$, $i = 1, 2, 3$, if $j \in I - \{i\}$, $k \in I - \{i\}$, $j \neq k$, then $\mathfrak{Q}_{\Gamma} = \mathfrak{Q}_{\beta}$, where

(β1) $\mathfrak{Q}_{\beta} = \{C = C(\mathscr{K}_i, \mathbf{D}) : x_j(C) \geqq w(j), x_k(C) \geqq w(k)\}$, if $w(j, k) > w(j) + w(k)$;

(β2) $\mathfrak{Q}_{\beta} = \{C : C \in \mathscr{C}, x_m(C) = w(m), m = 1, 2, 3\}$, if $w(j, k) = w(j) + w(k)$.

(γ) If $\mathbf{K} = \{\mathscr{K}_0, \mathscr{K}_I\}$, then $\mathfrak{Q}_\Gamma = \mathfrak{Q}_\gamma$, where

(γ1) $\mathfrak{Q}_\gamma = \{C = C(\mathscr{K}_I, \mathbf{D}) : \boldsymbol{x}_i(C) \geqq w(i),\ i \in I\}$, if $w(I) > \sum_{i \in I} w(i)$;

(γ2) $\mathfrak{Q}_\gamma = \{C : C \in \mathscr{C}, \boldsymbol{x}_i(C) = w(i),\ i \in I\}$, if $w(I) = \sum_{i \in I} w(i)$.

(δ) If $\mathbf{K} = \{\mathscr{K}_i, \mathscr{K}_I\}$, $i = 1, 2, 3$, if $j \in I - \{i\}$, $k \in I - \{i\}$, $j \neq k$, then $\mathfrak{Q}_\Gamma = \mathfrak{Q}_\delta$, where

(δ1) $\mathfrak{Q}_\delta = \{C = C(\mathscr{K}_I, \mathbf{D}) : \boldsymbol{x}_i(C) \geqq w(i),\ \ \boldsymbol{x}_j(C) + \boldsymbol{x}_k(C) \geqq w(j, k)\}$, if $w(I) >$ $> w(i) + w(j, k)$;

(δ2) $\mathfrak{Q}_\delta = \{C : C \in \mathscr{C}, \boldsymbol{x}_i(C) = w(i)\}$, if $w(I) = w(i) + w(j, k)$.

(ε) If $\mathbf{K} = \{\mathscr{K}_0, \mathscr{K}_i, \mathscr{K}_I\}$, $i = 1, 2, 3$, $j \in I - \{i\}$, $k \in I - \{i\}$, $j \neq k$, then $\mathfrak{Q}_\Gamma =$ $= \mathfrak{Q}_\varepsilon$, where

(ε1) $\mathfrak{Q}_\varepsilon = \{C = C(\mathscr{K}_I, \mathbf{D}) : \boldsymbol{x}_m(C) \geqq w(m),\ m = 1, 2, 3,\ \boldsymbol{x}_j(C) + \boldsymbol{x}_k(C) \geqq$ $\geqq w(j, k)\}$, if $w(I) > w(j, k) + w(i)$;

(ε2) $\mathfrak{Q}_\varepsilon = \mathfrak{Q}_\beta \cup \{C = C(\mathscr{K}_I, \mathbf{D}) : C \in \mathfrak{Q}_\gamma \cap \mathfrak{Q}_\delta\}$, if $w(I) = w(j, k) + w(i)$.

(ζ) If $\mathbf{K} = \{\mathscr{K}_i, \mathscr{K}_j\}$, $i \in I$, $j \in I$, $i \neq j$, if $k \in I - \{i, j\}$, then $\mathfrak{Q}_\Gamma = \mathfrak{Q}_\zeta$, where

(ζ1) $\mathfrak{Q}_\zeta = \{C = C(\mathscr{K}_i, \mathbf{D}) : \boldsymbol{x}_k(C) \geqq w(i, k) - w(i),\ \ \boldsymbol{x}_j(C) \geqq w(j),\ \ \boldsymbol{x}_i(C) = w(i)\}$, if $w(j,k) - w(j) > w(i, k) - w(i)$;

(ζ2) $\mathfrak{Q}_\zeta = \{C = C(\mathscr{K}_m, \mathbf{D}) : m = i, j,\ \boldsymbol{x}_i(C) = w(i),\ \boldsymbol{x}_j(C) = w(j),\ \boldsymbol{x}_k(C) =$ $= w(i, k) - w(i)\}$, if $w(j, k) - w(j) = w(i, k) - w(i)$.

(η) If $\mathbf{K} = \{\mathscr{K}_0, \mathscr{K}_i, \mathscr{K}_j\}$, $i \in I$, $j \in I$, $i \neq j$, $k \in I - \{i, j\}$, then $\mathfrak{Q}_\Gamma = \mathfrak{Q}_\eta$, where

(η1) $\mathfrak{Q}_\eta = \mathfrak{Q}_\zeta$, if $w(j, k) + w(i, k) > w(i) + w(j) + 2w(k)$;

(η2) $\mathfrak{Q}_\eta = \mathfrak{Q}_\zeta \cup \{C(\mathscr{K}_0, \mathbf{0})\} = \{C = C(\mathscr{K}, \mathbf{D}) : \mathscr{K} \in \mathbf{K}, \boldsymbol{x}_m(C) = w(m),\ m = 1, 2, 3\}$, if $w(j, k) + w(i, k) = w(i) + w(j) + 2w(k)$.

(κ) If $\mathbf{K} = \{\mathscr{K}_i, \mathscr{K}_j, \mathscr{K}_I\}$, $i \in I$, $j \in I$, $i \neq j$, $k \in I - \{i, j\}$ then $\mathfrak{Q}_\Gamma = \mathfrak{Q}_\kappa$, where

(κ1) $\mathfrak{Q}_\kappa = \{C = C(\mathscr{K}_I, \mathbf{D}) : \boldsymbol{x}_i(C) \geqq w(i),\ \ \boldsymbol{x}_j(C) \geqq w(j),\ \ \boldsymbol{x}_k(C) + \boldsymbol{x}_i(C) \geqq w(i, k),$ $\boldsymbol{x}_k(C) + \boldsymbol{x}_j(C) \geqq w(j, k)\} \cup \{C' = C'(\mathscr{K}, \mathbf{D}) : \mathscr{K} \in \mathbf{K},\ \mathscr{K} \neq \mathscr{K}_I,\ C' \in \mathfrak{Q}_\zeta,$ $\sum_{m=1}^{3} \boldsymbol{x}_m(C') = w(I)\}$, if $2w(I) \geqq w(i, k) + w(j, k) + w(i) + w(j)$;

(κ2) $\mathfrak{Q}_\kappa = \mathfrak{Q}_\zeta$, if $2w(I) < w(i, k) + w(j, k) + w(i) + w(j)$.

(λ) If $\mathbf{K} = \{\mathscr{K}_0, \mathscr{K}_i, \mathscr{K}_j, \mathscr{K}_I\}$, $i \in I, j \in I$, $i \neq j, k \in I - \{i, j\}$, then $\mathfrak{Q}_\Gamma = \mathfrak{Q}_\lambda$, where

(λ1) $\mathfrak{Q}_\lambda = \{C = C(\mathscr{K}_I, \mathbf{D}) : \boldsymbol{x}_m(C) \geqq w(m),\ \ m = 1, 2, 3,\ \boldsymbol{x}_i(C) + \boldsymbol{x}_k(C) \geqq w(i, k),$ $\boldsymbol{x}_j(C) + \boldsymbol{x}_k(C) \geqq w(j, k)\} \cup \{C' = C'(\mathscr{K}, \mathbf{D}) : \mathscr{K} \in \mathbf{K},\ \mathscr{K} \neq \mathscr{K}_I,\ C' \in \mathfrak{Q}_\eta,$ $\sum_{m=1}^{3} \boldsymbol{x}_m(C') = w(I)\}$, if $2w(I) \geqq w(i, k) + w(j, k) + w(i) + w(j)$;

(λ2) $\mathfrak{Q}_\lambda = \mathfrak{Q}_\eta$, if $2w(I) < w(i, k) + w(j, k) + w(i) + w(j)$.

(μ) If $\mathbf{K} = \{\mathscr{K}_0, \mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3\}$ (the case $\mathbf{K} = \{\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3\}$ is eliminated by condition (2.1)), then $\mathfrak{Q}_\Gamma = \mathfrak{Q}_\mu$, where

(μ1) if there exist coalition $\{i, j\} \subset I$, and $k \in I - \{i, j\}$ such that $w(i, j) > w(i, k) + w(j, k) - 2w(k)$, then $\mathfrak{Q}_\mu = \{\mathsf{C} = \mathsf{C}(\mathscr{K}_k, \mathbf{D}) : \boldsymbol{x}_i(\mathsf{C}) \geqq w(i, k) - w(k), \boldsymbol{x}_j(\mathsf{C}) \geqq w(j, k) - w(k)\}$;

(μ2) if there exist $\{i, j\} \subset I$ and $k \in I - \{i, j\}$ such that $w(i, j) = w(i, k) + w(j, k) - 2w(k)$, then $\mathfrak{Q}_\mu = \{\mathsf{C} : \mathsf{C} \in \mathscr{C}, \boldsymbol{x}_i(\mathsf{C}) = w(i, k) - w(k), \boldsymbol{x}_j(\mathsf{C}) = w(j, k) - w(k), \boldsymbol{x}_k(\mathsf{C}) = w(k)\}$;

(μ3) if there exist $\{i, j\} \subset I$ and $k \in I - \{i, j\}$ such that $w(i, k) - w(k) > w(i, j) - w(j)$, $w(j, k) - w(k) > w(i, j) - w(i)$, then $\mathfrak{Q}_\mu = \mathfrak{Q}_\eta$;

(μ4) if none of the conditions, contained in (μ1), (μ2) and (μ3), holds for any two-elements coalition $\{i, j\} \subset I$, and if there exists some coalition $\{i, j\} \subset I$ such that $w(i, j) > w(i) + w(j)$, then $\mathfrak{Q}_\mu = \{\mathsf{C} = \mathsf{C}(\mathscr{K}_m, \mathbf{D}) : m = 1, 2, 3,, \boldsymbol{x}_r(\mathsf{C}) \geqq w(r), r = 1, 2, 3\}$;

(μ5) if there is no coalition $\{i, j\} \subset I$ satisfying some of the conditions (μ1), (μ2), (μ3), (μ4), then $\mathfrak{Q}_\mu = \{\mathsf{C} : \mathsf{C} \in \mathscr{C}, \boldsymbol{x}_i(\mathsf{C}) = w(i), i = 1, 2, 3\}$.

(ν) If $\mathbf{K} = \{\mathscr{K}_0, \mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3, \mathscr{K}_I\}$ (the possibility $\mathbf{K} = \{\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_3, \mathscr{K}_I\}$ is eliminated by (2.1)), then $\mathfrak{Q}_\Gamma = \mathfrak{Q}_\nu$, where

(ν1) $\mathfrak{Q}_\nu = \{\mathsf{C} = \mathsf{C}(\mathscr{K}_I, \mathbf{D}) : \boldsymbol{x}_i(\mathsf{C}) \geqq w(i), \boldsymbol{x}_i(\mathsf{C}) + \boldsymbol{x}_j(\mathsf{C}) \geqq w(i, j),$ *for all* $i \in I, j \in I, i \neq j\} \cup \{\mathsf{C} : \mathsf{C} \in \mathfrak{Q}_\mu,$ *and* $\sum_{i \in I} \boldsymbol{x}_i(\mathsf{C}) = w(I)\}$, if $2w(I) \geqq w(1, 2) + w(1, 3) + w(2, 3)$;

(ν2) $\mathfrak{Q}_\nu = \mathfrak{Q}_\mu$, if $2w(I) < w(1, 2) + w(1, 3) + w(2, 3)$.

Remark 8.3. The three-players games with a general identification may differ from the investigated type of games without an identification only when the identification partitions of the one-element coalitions are considered; these partitions cannot contain more than two identification sets. It means that the considerations, used in this section, may be modified for the general three-players games. Even if we cannot insist in the general results of Section 7 in such a case, the obtained results will probably not differ too much from those included in Remark 8.2. Only the number of all possible cases which we shall have to consider will increase rather substantially, as the necessity will occur to distinguish the particular types of games not only with respect to the form of the class $\mathbf{K}$ and functions v_i (resp. w), but even with respect to the identification partitions of the one-element coalitions. An analysis of all such cases would request, of course, for too great space, which is also the principal reason for the limiting our investigation to the games without identification. Without any further investigation we can easily verify, that the three-players coalition-games without strategies choosing, which were described in Remark 8.2 in paragraphs

(α), (γ), (δ), (ζ) and (κ) are always games without an identification. The previous consideration concerns, consequently, to the games described in paragraphs (β), (ε), (η), (λ), (μ) and (ν) only.

PART IV. STRATEGIES IN BARGAINING.

The bargaining model suggested in the second part of this paper was defined and investigated for games without strategies choosing only. It is shown in the following sections that this bargaining model may be used also for general games defined in Section 2. The results, obtained above, remain to be true even after such modification.

Moreover, the possibility of strategies choosing enables us to study even the games with fixed cooperation in which the activity of players is realized in the strategies only.

In this part we return to the general model of a coalition-game in such a form as introduced in Section 2. Hence, we shall suppose that the players have a possibility to choose not only the coalitions to which they would like to belong, but that every coalition has to its disposal a finite set of pure strategies among which it may choose the one considered for the best from its own point of view. At the same time, according to (2.2), the set of pure strategies of a given coalition need not be only the union of the strategies sets of its members, but it may contain even some qualitatively new possibilities of behaviour, different from the possibilities of the particular players.

The possibility of choosing among the strategies necessarily influences the bargaining process and the bargaining result. From point of view of the proposed prediction method, the choosing of strategies influences, at the first time, the rationality of the agreements, depending, in this case, even on the optimality of the chosen coalition strategy.

As the prediction method, proposed in Section 5 and 6 may be modified even for the general coalition-games with a finite identification without considerable troubles, we limit ourselves, in the following, only to those steps of the proposed procedure, which bring some qualitatively new points of view or which differ from the results mentioned above. It is why the two following sections are written only on an informative and survey level, the modified forms of definitions and assertions are written in all the details only in case they differ, from those mentioned in previous parts not only by the used notations.

The last sections give then an illustration, how the choosing of strategies influences the process of bargaining when an extreme case of the strategic coalition-games is considered, namely the case of the strategic coaliton-games with a fixed cooperation (i.e. games, in which the bargaining process is limited only to a searching for appropriate strategies and to a choosing of the side-payments).

In all following part the notations introduced in sections 2 and 3 are used.

9. Extended Bargainig Model

We limit ourselves in this section only to proclaiming of the existing analogies, and to mentioning only those steps which differ when the both types of games – the coalition-games with a finite identification and the coalition-games with a finite identification and without strategies choosing – are considered.

The definitions of the auxiliary relations – the weak and strong domination relations and the definition of rationality – are the same as those introduced in the first part of this work (c.f. Definitions 1–5). They differ only as for as the formal shape of configurations is considered, because it is necessary to consider and write configurations as triples, mentioning explicitely the corresponding strategies. For example, the rationality definition will have the following form:

Definition 1.A. Let $C = C(\mathcal{K}, \mathbf{s}, \mathbf{D}) \in \mathcal{C}$. We shall say that *the agreement* $(K, s_K, \mathbf{D}_K) \in C$ *is rational in* C, if there exists no $C' \in \mathcal{C}$ satisfying $C' \,\delta\, C \bmod c(C; \mathcal{K}) - \{K\})$. The symbol C_{rat} will denote the set of all the agreements rational in C.

Also the assertions using these definitions hold mostly in analogous forms. The only exception is represented by Remark 5.3, which does not hold, as in case the games with a possibility of choosing strategies are considered, the rationality of agreements depends also on the advantage of the chosen coalition strategy. For the same reason, Lemma 5.1 does not hold; it must be modified for the general coalition-games in the following way:

Lemma 9.1. Let $(\mathcal{K}, \mathbf{s}) \in \mathbf{K}_S$, $(K, s_K) \in (\mathcal{K}, \mathbf{s})$. Let us denote

$$\mathbf{K}_S(K) = \{(\mathcal{L}, \mathbf{s}') : (\mathcal{L}, \mathbf{s}') \in \mathbf{K}_S, \ (\mathcal{L}, \mathbf{s}') \cap (\mathcal{K}, \mathbf{s}) \supset (\mathcal{K}, \mathbf{s}) - \{(K, s_K)\}\}$$

and

$$\mathbf{K}(K) = \{\mathcal{L} : \mathcal{L} \in \mathbf{K}, \mathcal{L} \cap \mathcal{K} \supset \mathcal{K} - \{K\}\}\,.$$

Let there exists a matrix $\mathbf{D} = (\mathbf{D}_L)_{L \in \mathcal{K}} \in \mathcal{D}_{\mathcal{K}}$ and a configuration $C = C(\mathcal{K}, \mathbf{s}, \mathbf{D}) \in \mathcal{C}$ such that $(K, s_K, \mathbf{D}_K) \in C_{\text{rat}}$ then for all coalition structures $\mathcal{L} \in \mathbf{K}(K)$ the inequality

$$\sum_{i \in I} v_i(\mathcal{K}, \mathbf{s}) \geqq \sum_{L \subset K, L \in \mathcal{L}} \max \{ \sum_{i \in L} v_i(\mathcal{L}, \mathbf{s}') : (\mathcal{L}, \mathbf{s}') \in \mathbf{K}_S(K)\}$$

holds.

The proof of Lemma 9.1 is analogous to that one for Lemma 5.1, supposing we included into our considerations beside the possibility of a partition of a coalition K even the possibility of choosing their common strategy.

Neither the Corollary of Lemma 5.1 holds, if we consider the games with a possibility of strategies choosing; it is necessary to subtitute the following one for it.

Corollary. If $(\mathcal{K}, \mathbf{s}) \in \mathbf{K}_S$, $(K, s_K) \in (\mathcal{K}, \mathbf{s})$, $\mathbf{s} = (s_L)_{L \in \mathcal{K}}$ then the foregoing Lemma 9.1 implies:

(a) if $K = \{i\}$, $i \in I$, then the agreement $(K, s_K, 0)$ is rational in $C(\mathscr{K}, \mathbf{s}, \mathbf{D})$, where $\mathbf{D} = (D_L)_{L\in\mathscr{K}} \in \mathscr{D}_{\mathscr{K}}$, D_K is the zero matrix, iff

$$v_i(\mathscr{K}, \mathbf{s}) = \max \{v_i(\mathscr{K}, \mathbf{s}'):\ \mathbf{s}' = (s'_L)_{L\in\mathscr{K}} \in S_{\mathscr{K}},\ s'_L = s_L \text{ for } L \in \mathscr{K},\ L \neq K\}$$

(b) if $K = \{i, j\}$, $i \in I$, $j \in I$, $i \neq j$, then there exists a matrix $\mathbf{D} = (D_L)_{L\in\mathscr{K}} \in \mathscr{D}_{\mathscr{K}}$ and a configuration $C = C(\mathscr{K}, \mathbf{s}, \mathbf{D}) \in \mathscr{C}$ such that $(K, s_K, D_K) \in C_{\text{rat}}$ iff

$$v_i(\mathscr{K}, \mathbf{s}) + v_j(\mathscr{K}, \mathbf{s}) = \max \{v_i(\mathscr{K}, \mathbf{s}') + v_j(\mathscr{K}, \mathbf{s}') : \mathbf{s}' = (s'_L)_{L\in\mathscr{K}} \in S_{\mathscr{K}},\ s'_L = s_L \text{ for } L \neq K,\ L \in \mathscr{K}\},$$

$$v_i(\mathscr{K}, \mathbf{s}) + v_j(\mathscr{K}, \mathbf{s}) \geqq \max \{v_i(\mathscr{L}, \mathbf{s}') : \mathbf{s}' \in S_{\mathscr{L}}(K)\} + \max \{v_j(\mathscr{L}, \mathbf{s}') : \mathbf{s}' \in S_{\mathscr{L}}(K)\},$$

where $\mathscr{L} \in \mathbf{K}$, $\mathscr{L} \cap \mathscr{K} = \mathscr{K} - \{K\}$, $\{i\} \in \mathscr{L}$, $\{j\} \in \mathscr{L}$ and

$$S_{\mathscr{L}}(K) = \{(\mathscr{L}, \mathbf{s}') : \mathbf{s}' = (s'_L)_{L\in\mathscr{L}},\ s'_L = s_L \text{ for } L \in \mathscr{L},\ L \cap K = \emptyset\}.$$

The other lemmas and remarks, given in Section 5, are true even for the case of coalition-games with the possibility of strategies choosing.

Also the notions of strong predictional stability and predictional balance, introduced in Section 6 (Definitions 6 and 8), will not change when the games with a strategies choosing are considered.

It is necessary to stop for a while when the definition of the weak predictional avoiding relation and the definition of the weak predictional stability, corresponding to the former one, are considered. Principally, Definition 7 may be generalized in two ways:

Definition 7A. Supposing $C = C(\mathscr{K}, \mathbf{s}, \mathbf{D}) \in \mathscr{C}$, $C' = C'(\mathscr{K}', \mathbf{s}', \mathbf{D}') \in \mathscr{C}$, then the relation $C\ \mathbf{p}\ C'$ holds, if

(a) there exists a strong domination sequence from C' to C,

(b) there exist no configurations $C^{\dagger} = C^{\dagger}(\mathscr{K}, \mathbf{s}, \mathbf{D}^{\dagger}) \in \mathscr{C}$, $C^{\dagger\dagger} = C^{\dagger\dagger}(\mathscr{K}', \mathbf{s}', \mathbf{D}^{\dagger\dagger}) \in \mathscr{C}$ satisfying the four following conditions:

  (b1) there exists a strong domination sequence from C to $C^{\dagger\dagger}$,

  (b2) there exists a strong domination sequence from $C^{\dagger\dagger}$ to $C^{\dagger}$,

  (b3) $(L, s_L, D_L) \in C_{\text{rat}} \Leftrightarrow (L, s_L, D^{\dagger}_L) \in C^{\dagger}_{\text{rat}}$,

  (b4) $(J, s'_J, D'_J) \in C'_{\text{rat}} \Leftrightarrow (J, s'_J, D^{\dagger\dagger}_J) \in C^{\dagger\dagger}_{\text{rat}}$.

If there exists, for a configuration $C \in \mathscr{C}$, no $C' \in \mathscr{C}$ such that $C'\ \mathbf{p}\ C$, we say that C is weakly predictionally stable. The class of all weakly predictionally stable configurations in a given game Γ will be denoted by $\mathfrak{P}^{**}_{\Gamma}$ (or $\mathfrak{P}^{**}_{\Gamma}(A)$).

Definition 7B. Supposing $C = C(\mathscr{K}, \mathbf{s}, \mathbf{D}) \in \mathscr{C}$, $C' = C'(\mathscr{K}', \mathbf{s}', \mathbf{D}') \in \mathscr{C}$, then the relation $C\ \mathbf{p}\ C'$ holds, if

(a) there exists a strong domination sequence from C' to C,

(b) there exist no configurations $C^{\dagger} = C^{\dagger}(\mathscr{K}, \mathbf{s}^{\dagger}, \mathbf{D}^{\dagger}) \in \mathscr{C}$, $C^{\dagger\dagger} = C^{\dagger\dagger}(\mathscr{K}', \mathbf{s}^{\dagger\dagger}, \mathbf{D}^{\dagger\dagger}) \in \in \mathscr{C}$, satisfying the four following conditions:

(b1) there exists a strong domination sequence from C to $C^{\dagger\dagger}$,

(b2) there exists a strong domination sequence from $C^{\dagger\dagger}$ to $C^{\dagger}$,

(b3) $$(L, s_L, \mathbf{D}_L) \in C_{\text{rat}} \Leftrightarrow (L, s_L^{\dagger}, \mathbf{D}_L^{\dagger}) \in C_{\text{rat}}^{\dagger},$$

(b4) $$(J, s_J', \mathbf{D}_J') \in C_{\text{rat}}' \Leftrightarrow (J, s_J^{\dagger\dagger}, \mathbf{D}_J^{\dagger\dagger}) \in C_{\text{rat}}^{\dagger\dagger}.$$

If there exists, for a configuration $C \in \mathscr{C}$, no $C' \in \mathscr{C}$ such that C' **p** C, we say that C is weakly predictionally stable. The class of all weakly predictionally stable configurations will be denoted by $\mathfrak{P}_{\Gamma}^{**}$ (or $\mathfrak{P}_{\Gamma}^{**}(B)$).

Both the definitions may be used, in order to introduce the notion of weak predictional stability. If we denote the classes of the weakly predictionally stable configurations by $\mathfrak{P}_{\Gamma}^{**}(A)$ and $\mathfrak{P}_{\Gamma}^{**}(B)$, then the relation $\mathfrak{P}_{\Gamma}^{**}(B) \cap \mathscr{C}_{\text{rat}} \neq \emptyset$ can be derived analogically as in the proof of Theorem 1 and relation $\mathfrak{P}_{\Gamma}^{**}(B) \supset \mathfrak{P}_{\Gamma}^{**}(A)$ can be also easily checked.

In the following we limit ourselves to the first modification of Definition 7 and set

$$\mathfrak{P}_{\Gamma}^{**} = \mathfrak{P}_{\Gamma}^{**}(A).$$

It means that we have decided to impose more exacting requirements to the weakly predictionally stable configurations. Such an election is logical, as this part of this paper deals not only with the investigation of the process of the coalitions forming, but even with the investigation of the process of choosing their strategies. Definition 7B abstracts from the problem of strategies choosing as well as from the problem of side payments choosing, and deals with the chosen strategies only under the condition they do not affect the rationality of the agreements.

For the class $\mathfrak{P}_{\Gamma}^{**} = \mathfrak{P}_{\Gamma}^{**}(A)$ the following analogy of Theorem 1 holds.

Theorem 1A. Supposing Γ is a coalition-game with a finite identification then there exists at least one weakly predictionally stable configuration having rational character, i.e.

$$\mathfrak{P}_{\Gamma}^{**} \cap \mathscr{C}_{\text{rat}} \neq \emptyset.$$

Proof. It is necessary to realize, firstly, that $\mathscr{C}_{\text{rat}} \neq \emptyset$. If $\mathscr{K} \in \mathbf{K}$ is such a configuration that for no $\mathscr{J} \in \mathbf{K}$ $\mathscr{J}$ is a subpartition of $\mathscr{K}$, if we choose $\mathbf{s} = (s_K)_{K \in \mathscr{K}} \in \mathbf{S}_{\mathscr{K}}$ in such a way that for every $K \in \mathscr{K}$ the identification set $\mathbf{R}_K \in \mathscr{R}_K$ existed, satisfying

$$\sum_{i \in K} v_i(K, s_K, \mathbf{R}_K) \geqq \sum_{i \in K} v_i(K, s_K', \mathbf{R}_K)$$

for all $s_K' \in \mathbf{S}_K$ (it is possible, according to Remark 3.1), then $C(\mathscr{K}, \mathbf{s}, \mathbf{D}) \in \mathscr{C}_{\text{rat}}$ for every matrix $\mathbf{D} \in \mathscr{D}_{\mathscr{K}}$.

We construct, for every coalition $K \in \|\mathbf{K}\|$ and for every identification set $\boldsymbol{R}_K \in \mathscr{R}_K$, the set

$$S_K(\boldsymbol{R}_K) = \{s_K : s_K \in S_K , \quad \sum_{i \in K} \boldsymbol{v}_i(K, s_K, \boldsymbol{R}_K) \geqq \sum_{i \in K} \boldsymbol{v}_i(K, s'_K, \boldsymbol{R}_K) \quad \text{for all } s'_K \in S_K\} .$$

Remark 3.1 implies that $S_K(\boldsymbol{R}_K) \neq \emptyset$.

Let us denote by $\mathbf{K}_K$ the class

$$\mathbf{K}_K = \{\mathscr{K} : \mathscr{K} \in \mathbf{K}, K \in \mathscr{K}\} .$$

Then there exists a finite set Ψ_K of indices and a partition of the set S_K into subsets $S_K(\psi)$, $\psi \in \Psi_K$, such that for every pair s_K, s'_K of strategies, belonging to the same set $S_K(\psi)$, the following holds:

If $\quad \mathscr{L} \in \mathbf{K}_K, (\mathscr{L}, \mathbf{s}) \in \mathbf{K}_S, (\mathscr{L}, \mathbf{s}') \in \mathbf{K}_S, \mathbf{s} = (s_L)_{L \in \mathscr{L}}, \mathbf{s}' = (s'_L)_{L \in \mathscr{L}}, s'_L = s_L$

for $L \in \mathscr{L}$, $L \neq K$, then for all coalitions $L \in \mathscr{L}$ the strategic structures $(\mathscr{L}, \mathbf{s}')$ belong to the same identification ser $\boldsymbol{R}_L \in \mathscr{R}_L$. This means that the pay-offs $\boldsymbol{v}_i$ of players $i \in I - K$ will not change, if the coalition K changes its strategy inside the set $S_K(\psi)$.

If

$$K \in \|\mathbf{K}\| , \quad s_K \in S_K - \bigcup_{\boldsymbol{R}_K \in \mathscr{R}_K} S_K(\boldsymbol{R}_K) ,$$

then for no matrix $\boldsymbol{D}_K \in \mathscr{D}_K$ and for no configuartion $C \in \mathscr{C}$ is $(K, s_K, \boldsymbol{D}_K) \in C_{\text{rat}}$. Hence, if $(K, s_K, \boldsymbol{D}_K) \in C$, then $C \in \mathscr{C} - \mathscr{C}_{\text{rat}}$.

The class $\mathbf{K}_S$ may be divided into a finite number of disjoint subsets

$$\eta_0, \eta_1, \eta_2, \ldots, \eta_\tau ,$$

where

$$\tau \leqq \pi(\mathbf{K}) \big(\sum_{K \in \|\mathbf{K}\|} \pi(\mathscr{R}_K) \big) \big(\sum_{K \in \|\mathbf{K}\|} \pi(\Psi_K) \big) ,$$

so that

$$\eta_0 = \{(\mathscr{K}, \mathbf{s}) : (\mathscr{K}, \mathbf{s}) \in \mathbf{K}_S,\ \mathbf{s} = (s_L)_{L \in \mathscr{K}},\ \textit{and there exists } K \in \mathscr{K} \textit{ such that}$$
$$s_K \in S_K - \bigcup_{\boldsymbol{R}_K \in \mathscr{R}_K} S_K(\boldsymbol{R}_K))\} ,$$

and the sets η_t, $1 \leqq t \leqq \tau$, satisfy:

If $(\mathscr{K}, \mathbf{s}) \in \eta_t$, $(\mathscr{K}', \mathbf{s}') \in \eta_t$, then $\mathscr{K} = \mathscr{K}'$, $\mathbf{s} = (s_K)_{K \in \mathscr{K}}$, $\mathbf{s}' = (s'_K)_{K \in \mathscr{K}}$ and for all $K \in \mathscr{K}$ the strategies s_K and s'_K belong to the same set

$$S_K(\boldsymbol{R}_K) \cap S_K(\psi) , \quad \boldsymbol{R}_K \in \mathscr{R}_K , \quad \psi \in \Psi_K .$$

We know, from the foregoing, that if $(\mathscr{K}, \mathbf{s}) \in \eta_0$, then for every matrix $\mathbf{D} \in \mathscr{D}_{\mathscr{K}}$ the configuration $C(\mathscr{K}, \mathbf{s}, \mathbf{D}) \in \mathscr{C} - \mathscr{C}_{\text{rat}}$. If $(\mathscr{K}, \mathbf{s})$ and $(\mathscr{K}, \mathbf{s}')$ belong to the same set η_t, $1 \leqq t \leqq \tau$, then for all coalitions $K \in \mathscr{K}$

$$\sum_{i \in K} \boldsymbol{v}_i(K, s_K, \boldsymbol{R}_K) = \sum_{i \in K} \boldsymbol{v}_i(K, s'_K, \boldsymbol{R}_K) ,$$

where $(\mathscr{K}, \mathbf{s}) \in \boldsymbol{R}_K$, $(\mathscr{K}, \mathbf{s}') \in \boldsymbol{R}_K \in \mathscr{R}_K$. Hence, there exists, for every matrix $\mathbf{D} \in \mathscr{D}_{\mathscr{K}}$, such a $\mathbf{D}' \in \mathscr{D}_{\mathscr{K}}$ that for $C = C(\mathscr{K}, \mathbf{s}, \mathscr{D}) \in \mathscr{C}$, $C' = C'(\mathscr{K}, \mathbf{s}', \mathbf{D}') \in \mathscr{C}$ the relation $\boldsymbol{x}_i(C) = \boldsymbol{x}_i(C')$ holds for every $i \in I$. It follows that $(K, s_K, \boldsymbol{D}_K) \in C_{\text{rat}}$ iff $(K, s'_K, \boldsymbol{D}'_K) \in C_{\text{rat}}$.

The following procedure is analogous to the procedure, by which Theorem 1 was proved. We choose $C'' \in \mathscr{C}_{\text{rat}}$ and a strong domination sequence

$$\{C^{(r)}\}_{r=1,\ldots,n}, \quad \{c^{(r)}\}_{r=1,\ldots,n-1}$$

from $C^{(1)} = C$ to a configuration $C^{(n)} \in \mathscr{C}$ such that for $r = 2, 3, \ldots, n-1$ either

$$C^{(r+1)} \mathbf{p}\, C^{(r)},$$

holds (if $C^{(r)} \in \mathscr{C}_{\text{rat}}$), or

$$C^{(r+1)} \operatorname{Dom} C^{(r)} \bmod c^{(r)} \quad \text{and} \quad C^{(r)} \mathbf{p}\, C^{(r-1)}$$

hold (if $C^{(r)} \in \mathscr{C} - \mathscr{C}_{\text{rat}}$). After a finite number of steps such a $C^{(n)} \in \mathscr{C}_{\text{rat}}$ may be constructed that $C^{(n)} = C^{(n)}(\mathscr{K}, \mathbf{s}, \mathbf{D})$ and there exist an index w, $1 \leqq w < n$ and a configuration $C^{(w)} = C^{(w)}(\mathscr{K}, \mathbf{s}^{(w)}, \mathbf{D}^{(w)})$ such that $(\mathscr{K}, \mathbf{s})$ and $(\mathscr{K}, \mathbf{s}^{(w)})$ belong to the same set η_t, $1 \leqq t \leqq \tau$; to every configuration $C' = C'(\mathscr{K}', \mathbf{s}', \mathbf{D}')$ for which there exists a strong domination sequence from $C^{(n)}$ to C' such an index v, $w < v < n$ and such a configuration $C^{(v)} = C^{(v)}(\mathscr{K}', \mathbf{s}^{(v)}, \mathbf{D}^{(v)})$ may be found that $(\mathscr{K}', \mathbf{s}^{(v)})$ and $(\mathscr{K}', \mathbf{s}')$ belong to the same set η_t, if $1 \leqq t \leqq \tau$, or they both belong to the set

$$\eta_0 \cap \{(\mathscr{K}', \mathbf{s}'') : \mathbf{s}'' = (s''_K)_{K \in \mathscr{K}'}, \; s''_K \in S_K(\psi_K) \; \textit{for all } K \in \mathscr{K}'\}$$

for some vector of indices

$$(\psi_K)_{K \in \mathscr{K}'} \in \bigtimes_{K \in \mathscr{K}'} \Psi_K,$$

if $(\mathscr{K}', \mathbf{s}') \in \eta_0$. It means that no $C' \in \mathscr{C}$ exists such that $C' \mathbf{p}\, C^{(n)}$, hence $C^{(n)} \in \mathfrak{P}^{**}_{\Gamma}$. □

The other assertions, introduced in section 6, hold, when the coalition-games with a finite identification are considered, in the form analogous to that in the case of the games without strategies choosing.

10. Strategies in Games without Identification

The games without identification are the limit case of the games with a finite identification and everything mentioned in the foregoing section holds even in this case.

Some special properties of the games without identification imply that the process of bargaining is influenced by strategies, in such a case, less then in the case of more general games with a finite identification.

We denote, similarly as in relation (7.1),

$$(10.1) \quad w(K) = \max \{ \sum_{i \in K} v_i(K, s_K, R_K) : s_K \in S_K \} = \max \{ \sum_{i \in K} v_i(\mathscr{K}, s) : s \in S_{\mathscr{K}} \} ,$$

where $K \in \mathscr{K} \in \mathbf{K}$, $R_K \in \mathscr{R}_K$.

It can be easily checked that all the assertions of sections 7 and 8 hold in analogous form even in the case of the general coalition-games without identification. For every admissible coalition $K \in \|\mathbf{K}\|$ is $\mathscr{R}_K = \{R_K\}$ and there exists a non-empty set $\bar{S}_K \subset S_K$ of strategies,

$$(10.2) \qquad \bar{S}_K = \{s_K : s_K \in S_K, \sum_{i \in K} v_i(K, s_K, R_K) = w(K)\} .$$

If $s_K \in S_K - \bar{S}_K$ then there exists neither a matrix $D_K \in \mathscr{D}_K$, nor a configuration $C \in \mathscr{C}$ such that $(K, s_K, D_K) \in C_{\text{rat}}$. On the other hand, if $s_K \in \bar{S}_K$, $s'_K \in \bar{S}_K$, then the relation

$$\sum_{i \in K} v_i(K, s_K, R_K) = \sum_{i \in K} v_i(K, s'_K, R_K) = w(K)$$

holds as well as the assertion of Remark 3.4.

At the same time, an analogy of Remark 7.1 is satisfied when the rationality of agreements considered; two agreements (K, s_K, D_K) and (K, s'_K, D'_K) such that

$$s_K \in \bar{S}_K,\ s'_K \in \bar{S}_K,\ D_K = (d_{ij})_{i \in K, j \in K},\ D'_K = (d'_{ij})_{i \in K, j \in K} ,$$

and

$$v_i(K, s_K, R_K) + \sum_{j \in K} d_{ij} = v_i(K, s'_K, R_K) + \sum_{j \in K} d'_{ij}$$

for all $i \in I$, have the property that both of them are rational or none of them is rational.

It follows from the foregoing, that when a coalition-game without identification is considered, the strategies s_K of the coalitions. satisfying $s_K \in S_K - \bar{S}_K$, do not occur in the configurations, belonging to the bargaining prediction characteristic. On the other hand, the strategies of the coalitions, belonging to the same set $\bar{S}_K$ are equivalent from the point of view of their influence to the bargaining process and from the point of view of their occurrence in the result.

The foregoing consideration can be formulated in the following theorem.

Theorem 5. Let $(\mathscr{K}, s) \in \mathbf{K}_S$, $(\mathscr{K}, s') \in \mathbf{K}_S$ be two admissible strategic structures differing only in the vector of strategies. Let for all coalitions $K \in \mathscr{K}$ the relation

$$\sum_{i \in K} v_i(\mathscr{K}, s) = \sum_{i \in K} v_i(\mathscr{K}, s')$$

holds, Then the following holds:

If there exists a coalition $K \in \mathscr{K}$ satisfying

$$\sum_{i \in K} v_i(\mathscr{K}, s) < w(K) ,$$

then for any side payments matrices $\mathbf{D}$, $\mathbf{D}' \in \mathcal{D}_{\mathcal{K}}$ the configutations $C = C(\mathcal{K}, \mathbf{s}, \mathbf{D})$, $C' = C'(\mathcal{K}, \mathbf{s}', \mathbf{D}')$ neither are weakly predictionally stable nor have rational character. On the other hand, if the equality

$$\sum_{i \in K} v_i(\mathcal{K}, \mathbf{s}) = w(K)$$

holds for all coalitions $K \in \mathcal{K}$ then there exists, for every matrix $\mathbf{D} \in \mathcal{D}_{\mathcal{K}}$ and every configuration $C = C(\mathcal{K}, \mathbf{s}, \mathbf{D}) \in \mathcal{C}$, a matrix $\mathbf{D}' \in \mathcal{D}_{\mathcal{K}}$ and a configuration $C' = C'(\mathcal{K}, \mathbf{s}, \mathbf{D}') \in \mathcal{C}$ such that

$$x_i(C) = x_i(C')$$

for all $i \in I$, and

$$C \in \mathcal{C}_{\text{rat}} \Leftrightarrow C' \in \mathcal{C}_{\text{rat}}\,, \quad C \in \mathfrak{P}^{**}_{\Gamma} \Leftrightarrow C' \in \mathfrak{P}^{**}_{\Gamma}\,,$$
$$C \in \mathfrak{P}^{*}_{\Gamma} \Leftrightarrow C' \in \mathfrak{P}^{*}_{\Gamma}\,, \quad C \in \mathfrak{P}_{\Gamma} \Leftrightarrow C' \in \mathfrak{P}_{\Gamma}\,,$$
$$C \in \mathfrak{Q}_{\Gamma} \Leftrightarrow C' \in \mathfrak{Q}_{\Gamma}\,,$$

where

$$\mathfrak{Q}_{\Gamma} = \{C : C \in \mathcal{C} \textit{ and there exists no } C' \in \mathcal{C} \textit{ such that } C' \operatorname{Dom} C \bmod \emptyset\}\,.$$

Proof. Definition 1 and Lemma 3.1 imply that if $s_K \in S_K - \bar{S}_K$, then the agreement (K, s_K, D_K) is not rational for any matrix $D_K \in \mathcal{D}_K$. According to the fact that Remarks 7.1 and 7.3 hold even in the case of the general coalition-games, the first assertion of Theorem 3 holds as well. The second assertion follows from Remark 3.4 and from the definitions. □

Theorem 5 means that every coalition-game Γ without identification may be considered, from the point of view of the proposed prediction method, as if it was a coalition game $\bar{\Gamma}$ without choosing strategies, in which every coalition K has prescribed a fixed strategy from the set $\bar{S}_K$, The procedures and results of sections 7 and 9 may be applied to the game $\bar{\Gamma}$ and the classes of configurations $\mathcal{C}_{\text{rat}}(\bar{\Gamma})$, $\mathfrak{P}^{**}_{\bar{\Gamma}}$, $\mathfrak{P}^{*}_{\bar{\Gamma}}$, $\mathfrak{P}_{\bar{\Gamma}}$ and $\mathfrak{Q}_{\bar{\Gamma}}$ may be obtained. The analogous sets for the game Γ follow from them according to the schema:

$$\mathfrak{Z}_{\Gamma} = \mathfrak{Z}_{\bar{\Gamma}} \cup \{C' : C' = C'(\mathcal{K}, \mathbf{s}', \mathbf{D}') \in \mathcal{C}, \textit{ and there exists } C = C(\mathcal{K}, \mathbf{s}, \mathbf{D}) \in \mathfrak{Z}_{\bar{\Gamma}} \textit{ such that } x_i(C') = x_i(C) \textit{ for all } i \in I\}\,,$$

where we substitute for $\mathfrak{Z}_{\bar{\Gamma}}$ and $\mathfrak{Z}_{\Gamma}$ the pairs $(\mathcal{C}_{\text{rat}}(\bar{\Gamma}), \mathcal{C}_{\text{rat}}(\Gamma))$, $(\mathfrak{P}^{**}_{\bar{\Gamma}}, \mathfrak{P}^{**}_{\Gamma})$, $(\mathfrak{P}^{*}_{\bar{\Gamma}}, \mathfrak{P}^{*}_{\Gamma})$, $(\mathfrak{P}_{\bar{\Gamma}}, \mathfrak{P}_{\Gamma})$ and $(\mathfrak{Q}_{\bar{\Gamma}}, \mathfrak{Q}_{\Gamma})$ step by step.

In the case of the general three-players games Remark 8.3 does not hold. According to the possibility of choosing strategies the complexity of the identification partitions of the one-element and two-element coalitions can considerably increase and the bargaining prediction becomes, in such a case, incomparably more complicated than for the three-players games without identification.

11. Games with a Fixed Cooperation

As for as we did not consider the possibility of strategies choosing, the bargaining prediction in the games with a fixed cooperation had no actual raison. Such a case was a degenerated one, when the players were not given any possibility of an active intervention into the game, and all the admissible configurations were at the same measure possible.

The situation rather changes in case we consider a possibility of choosing strategies. Then the game with a fixed cooperation represents a model of conflict situation in which the influence of the process of coalition forming is avoided and the bargaining prediction is concentrated to the prediction of the chosen strategies.

As it will be shown later, even in this case the weak and strong predictional stability, as well as the predictional balance of the configurations, does not depend on the value of side payments, what follows from the fact that the members of a coalition are not given a possibility to force a change of the side payments through threats.

In this way, from the point of view of strategies choosing, every coalition may be considered for a particular player, only except the fact that, when applying our proposed prediction method, we must respect the information about the value of the guaranteed profit particulary for every single member of coalition.

The necessity to respect, when particular steps of the prediction applied, every particular player will not influence, however, the prediction result.

As a game with a fixed cooperation is considered, where $\mathbf{K} = \{\mathscr{K}\}$, we shall simplify the used notation in such a way that instead of $(\mathscr{K}, \mathbf{s})$, $C(\mathscr{K}, \mathbf{s}, \mathbf{D})$ and $v_i(\mathscr{K}, \mathbf{s})$ we shall write abbreviately $\mathbf{s}$, $C(\mathbf{s}, \mathbf{D})$, $v_i(\mathbf{s})$.. In whole the following section we consider a given coalition-game

$$\Gamma = (I, \mathbf{K}, \{S_K\}_{K \in \|\mathbf{K}\|}, \{\mathscr{R}_K\}_{K \in \|\mathbf{K}\|}, \{v_i\}_{i \in I}),$$

where $\mathbf{K} = \{\mathscr{K}\}$.

Firstly, we show that the bargaining result is independent on the level of the side payments. As the results are quite objective and their proofs are mostly descriptive ones (but, when all the steps written in all details, rather lenghty) we limit ourselves, for the sake of lucidity, to mentioning of the substantial steps of the particular proofs.

Remark 11.1. Supposing $C = C(\mathbf{s}, \mathbf{D}) \in \mathscr{C}$, $K \in \mathscr{K}$, $(\mathscr{K}, \mathbf{s}) \in R_K$, $R_K \in \mathscr{R}_K$, then $(K, s_K, D_K) \in C_{\text{rat}}$ iff

$$\sum_{i \in K} v_i(\mathbf{s}) = \max \{ \sum_{i \in K} v_i(\mathbf{s}') : \mathbf{s}' \in S_{\mathscr{K}} \ (\mathscr{K}, \mathbf{s}') \in R_K \} .$$

Lemma 11.1. Supposing $C^{(1)} = C^{(1)}(\mathbf{s}, \mathbf{D}^{(1)}) \in \mathscr{C}$, $C^{(2)} = C^{(2)}(\mathbf{s}, \mathbf{D}^{(2)}) \in \mathscr{C}$, $k \subset \mathscr{K}$, $k \neq \mathscr{K}$, then the following holds:

There exist $c^{(1)} = c^{(1)}(C^{(1)}; k)$ and $C' = C'(\mathbf{s}', \mathbf{D}') \in \mathscr{C}$ satisfying at least one among the relations

(1a) C' dom $C^{(1)}$ mod $c^{(1)}$, or

(2a) $C^{(1)}$ dom C' mod $c^{(1)}$, or

(3a) C' Dom $C^{(1)}$ mod $c^{(1)}$, or

(4a) $C^{(1)}$ Dom C' mod $c^{(1)}$,

if and only if there exist $c^{(2)} = c^{(2)}(C^{(2)}; k)$ and $C'' = C''(\mathbf{s}', \mathbf{D}'') \in \mathscr{C}$, satisfying the corresponding relation among the following ones

(1b) C'' dom $C^{(2)}$ mod $c^{(2)}$, or

(2b) $C^{(2)}$ dom C'' mod $c^{(2)}$, or

(3b) C'' Dom $C^{(2)}$ mod $c^{(2)}$, or

(4b) $C^{(2)}$ Dom C'' mod $c^{(2)}$, respectively.

The side payments matrices $\mathbf{D}' \in \mathscr{D}_{\mathscr{X}}$, $\mathbf{D}'' \in \mathscr{D}_{\mathscr{X}}$ can be chosen in such a way that the equality

$$x_i(C') - x_i(C'') = x_i(C^{(1)}) - x_i(C^{(2)})$$

holds for all $i \in I$.

Proof. If

$$C^{(1)} = C^{(1)}(\mathbf{s}, \mathbf{D}^{(1)}), \quad C^{(2)} = C^{(2)}(\mathbf{s}, \mathbf{D}^{(2)}), \quad C' = C'(\mathbf{s}', \mathbf{D}'),$$
$$\mathbf{D}^{(1)} = (d_{ij}^{(1)})_{i \in I, j \in I}, \quad \mathbf{D}^{(2)} = (d_{ij}^{(2)})_{i \in I, j \in I}, \quad \mathbf{D}' = (d'_{ij})_{i \in I, j \in I}$$

we construct

$$\mathbf{D}'' \in \mathscr{D}_{\mathscr{X}}, \quad C'' = C''(\mathbf{s}', \mathbf{D}'') \in \mathscr{C}$$

in such a way that

$$\mathbf{D}'' = (d''_{ij})_{i \in I, j \in I}, \quad d''_{ij} = d_{ij} + d_{ij}^{(2)} - d_{ij}^{(1)}, \quad i \in I, j \in I, \tag{11.1}$$

then

$$x_i(C'') - x_i(C^{(2)}) = x_i(C') - x_i(C^{(1)}) \tag{11.2}$$

and, according to Definition 2 and Remark 11.1 the relation (1a) holds iff (1b) holds, (2a) holds iff (2b) holds. Supposing there exist weak domination sequences modulo $c^{(1)}$ from $C^{(1)}$ to C' or from C' to $C^{(1)}$, even the corresponding weak domination sequences modulo $c^{(2)}$ from $C^{(2)}$ to C'' or from C'' to $C^{(2)}$ can be constructed in such a way that we construct the corresponding side payments matrices sequences, satisfying equalities of the type (11.1). Between the configurations from the both sequences, then some relations, analogous to (11.2), hold. This implies that the relations (3a) and (4a) hold iff (3b) and (4b) hold, respectively. □

Lemma 11.2. If $C = C(\mathbf{s}, \mathbf{D}) \in \mathscr{C}$, $C' = C'(\mathbf{s}, \mathbf{D}') \in \mathscr{C}$, $c = c(C; \mathbf{k})$, $c \subset C \cap C'$, if there exists a weak domination sequence modulo c from C to C', then there exists also a weak domination sequence modulo c from C' to C.

Proof. Let us denote by $\{C^{(r)}\}_{r=1,\ldots,u}$ a weak domination sequence modulo c from C to C',

$$C^{(r)} = C^{(r)}(\mathbf{s}^{(r)}, \mathbf{D}^{(r)}) \in \mathscr{C}\,, \quad \mathbf{D}^{(r)} = (\mathbf{D}_L^{(r)})_{L\in\mathscr{K}} = (d_{ij}^{(r)})_{i\in I, j\in I}, \quad r = 1, \ldots, u\,,$$

(11.3) $$C^{(1)} = C\,, \quad C^{(u)} = C'\,, \quad c = c(C; \mathbf{k})\,,$$

(11.4) $$C^{(r)} \operatorname{dom} C^{(r-1)} \bmod c\,, \quad r = 2, \ldots, u\,.$$

Now, we construct a new sequence $\{{}^1C^{(r)}\}_{r=1,\ldots,u}$, where

$${}^1C^{(r)} = {}^1C^{(r)}(\mathbf{s}^{(r)}, {}^1\mathbf{D}^{(r)})\,, \quad {}^1\mathbf{D}^{(r)} = ({}^1\mathbf{D}_L^{(r)})_{L\in\mathscr{K}} = ({}^1d_{ij}^{(r)})_{i\in I, j\in I}\,, \quad r = 1, \ldots, u\,,$$

so that

(11.5) $${}^1\mathbf{D}_J^{(r)} = \mathbf{D}_J^{(r)} \quad \text{for} \quad J \in \mathbf{k}\,, \quad r = 1, \ldots, u\,,$$

(11.6) $${}^1\mathbf{D}^{(1)} = \mathbf{D}^{(u)} = \mathbf{D}'\,,$$

(11.7) $${}^1d_{ij}^{(r)} = (\pi(K))^{-1} \,.\, (\mathbf{x}_i({}^1C^{(r-1)}) - \mathbf{v}_i(\mathbf{s}^{(r)}) - \mathbf{x}_j({}^1C^{(r-1)}) + \mathbf{v}_j(\mathbf{s}^{(r)}))$$
$$\text{for } i \in K,\ j \in K,\ K \in \mathscr{K} - \mathbf{k},\ r = 2, 3, \ldots, u - 1.$$

It means that ${}^1C^{(r)} \operatorname{dom} {}^1C^{(r-1)} \bmod c$ for $r = 2, \ldots, u - 1$.

(11.8) If $K \in \mathscr{K} - \mathbf{k}$, and if the agreement $(K, \mathbf{s}_K^{(u)}, \mathbf{D}_K^{(u)}) \in C^{(u)}$ is not active (c.f. Definition 2) in the relation $C^{(u)} \operatorname{dom} C^{(u-1)} \bmod c$, then we choose ${}^1\mathbf{D}_K^{(u)} = \mathbf{D}_K^{(1)} = \mathbf{D}_K$.

(11.9) If the agreement $(K, \mathbf{s}_K^{(u)}, \mathbf{D}_K^{(u)}) \in C^{(u)}$ is active in the relation $C^{(u)} \operatorname{dom} C^{(u-1)} \bmod c$, then we construct the matrix ${}^1\mathbf{D}_K^{(u)}$ in the following way:

For every player $i \in K$ we define numbers

$$\alpha_i^{(1)} = \mathbf{x}_i({}^1C^{(u-1)}) - \mathbf{v}_i(\mathbf{s}^{(u)})\,,$$
$$\beta_i^{(1)} = \max\{0,\ \mathbf{x}_i(C^{(1)}) - \mathbf{x}_i({}^1C^{(u-1)})\}\,,$$
$$\gamma_i^{(1)} = B_K^{(1)} \,.\, \beta_i^{(1)}\,,$$

where

$$B_K^{(1)} = \frac{\sum_{j\in K} (\mathbf{v}_j(\mathbf{s}^{(u)}) - \mathbf{v}_j(\mathbf{s}^{(u-1)}) - \beta_j^{(1)})}{\sum_{j\in K} \beta_j^{(1)}}\,,$$

is a real number,

$$-1 < B_{K}^{(1)} \leqq 0\,,$$

and $B_K^{(1)} = 0$ iff $x_i(C^{(1)}) \geqq x_i({}^1C^{(u-1)})$ for all $i \in K$. Then we set

$${}^1d_{ij}^{(u)} = (\pi(K))^{-1} \cdot (\alpha_i^{(1)} + \beta_i^{(1)} + \gamma_i^{(1)} - \alpha_j^{(1)} - \beta_j^{(1)} - \gamma_j^{(1)}), \quad i \in K, j \in K.$$

In such a case

$$\sum_{j \in I} {}^1d_{ij}^{(u)} = x_i({}^1C^{(u-1)}) - v_i(\mathbf{s}^{(u)}) + \beta_i^{(1)} \cdot (1 + B_K^{(1)}),$$

hence,

$$x_i({}^1C^{(u)}) = x_i({}^1C^{(u-1)}) + \beta_i^{(1)} \cdot (1 + B_K^{(1)}),$$

so that

$$x_i({}^1C^{(u)}) = x_i({}^1C^{(u-1)}), \quad \text{if} \quad \beta_i^{(1)} = 0, \tag{11.10}$$

$$x_i(C^{(1)}) \geqq x_i({}^1C^{(u)}) > x_i({}^1C^{(u-1)}), \quad \text{if} \quad \alpha_i^{(1)} < 0. \tag{11.11}$$

Relations (11.10) and (11.11), along with Remark 11.1, follow that the sequence $\{{}^1C^{(r)}\}_{r=1,\dots,u}$ is a weak domination sequence modulo c from $C^{(u)}$ to ${}^1C^{(u)}$, and that the agreement $(K, s_K^{(u)}, {}^1D_K^{(u)})$ is active in the relation

$${}^1C^{(u)} \text{ dom } {}^1C^{(u-1)} \bmod c,$$

if the agreement $(K, s_K^{(u)}, D_K^{(u)})$ is active in the relation

$$C^{(u)} \text{ dom } C^{(u-1)} \bmod c.$$

Relation (11.11) and the properties of $B_K^{(1)}$ imply, that if for all players $i \in K$ is

$$x_i(C^{(1)}) \geqq x_i({}^1C^{(u-1)}),$$

then

$$x_i({}^1C^{(u)}) = x_i(C^{(1)}) \quad \text{for all} \quad i \in K.$$

It means: if for all agreements $(K, s_K^{(u)}, D_K^{(u)})$, being active in the relation $C^{(u)}$ dom $C^{(u-1)}$ mod c, is $B_K^{(1)} = 0$, then ${}^1C^{(u)} = C^{(1)} = C$, and the wanted weak domination sequence modulo c from C' to C is found.

If the last condition is not valid, it means, if for some $K \in \mathscr{K}$ and some $i \in K$ the inequality

$$x_i(C^{(u)}) > x_i({}^1C^{(u)}) = x_i({}^1C^{(u-1)}) > x_i(C^{(1)})$$

holds, then a new sequence $\{{}^2C^{(r)}\}_{r=1,\dots,u}$ can be constructed in the similar way, which is a weak domination sequence modulo c from ${}^1C^{(u)} = {}^2C^{(1)}$ to ${}^2C^{(u)}$, so that:

$${}^2C^{(r)} = {}^2C^{(r)}(\mathbf{s}^{(r)}, {}^2\mathbf{D}^{(r)}), \quad r = 1, \dots, u;$$

if for all $i \in J$, $J \in \mathscr{K}$, is

$$x_i({}^1C^{(u)}) = x_i(C^{(1)}) = x_i(C),$$

then ${}^2\boldsymbol{D}_J^{(r)} = {}^1\boldsymbol{D}_J^{(r)}$ for $r = 2, \ldots, u$; if for some $K \in \mathscr{K}$, $j \in K$ is

$$\boldsymbol{x}_j({}^1C^{(u)}) > \boldsymbol{x}_j(C^{(1)}),$$

then we set for all $i \in K$:

$$\alpha_i^{(2)} = \boldsymbol{x}_i({}^2C^{(u-1)}) - \boldsymbol{v}_i(\boldsymbol{s}^{(u)}),$$

$$\beta_i^{(2)} = \max\{0, \boldsymbol{x}_i(C^{(1)}) - \boldsymbol{x}_i({}^2C^{(u-1)})\},$$

$$\gamma_i^{(2)} = B_K^{(2)} \, . \, \beta_i^{(2)},$$

where

$$B_K^{(2)} = \frac{\sum_{j \in K} (\boldsymbol{v}_j(\boldsymbol{s}^{(u)}) - \boldsymbol{v}_j(\boldsymbol{s}^{(u-1)}) - \beta_j^{(2)})}{\sum_{j \in K} \beta_j^{(2)}},$$

$${}^2d_{ij}^{(u)} = (\pi(K))^{-1} \, . \, (\alpha_i^{(2)} + \beta_i^{(2)} + \gamma_i^{(2)} - \alpha_j^{(2)} - \beta_j^{(2)} - \gamma_j^{(2)}),$$

so that, like in the previous case,

$$\boldsymbol{x}_i({}^2C^{(u)}) = \boldsymbol{x}_i({}^2C^{(u-1)}) + \gamma_i^{(2)} + \beta_i^{(2)} = \boldsymbol{x}_i({}^2C^{(u-1)}) + \beta_i^{(2)} \, . \, (1 + B_K^{(2)}).$$

Then for such a coliation K the following relations hold.

(11.12) if $\boldsymbol{x}_i({}^2C^{(u-1)}) \leqq \boldsymbol{x}_i(C^{(1)})$ for all $i \in K$, then

$$\boldsymbol{x}_i({}^2C^{(u)}) = \boldsymbol{x}_i(C^{(1)}) = \boldsymbol{x}_i(C)$$

for all $i \in K$,

(11.13) if for some $i \in K$ is $\boldsymbol{x}_i({}^2C^{(u-1)}) > \boldsymbol{x}_i(C^{(1)})$, then

$$\boldsymbol{x}_i({}^2C^{(u)}) = \boldsymbol{x}_i({}^2C^{(u-1)}) = \boldsymbol{x}_i({}^1C^{(u)}) - (\pi(K))^{-1} \, . \, (\sum_{j \in K} (\boldsymbol{v}_j(\boldsymbol{s}^{(u)}) - \boldsymbol{v}_j(\boldsymbol{s}^{(u-1)}))) < \boldsymbol{x}_i({}^1C^{(u)}),$$

and for all the players $j \in K$, for which is

$$\boldsymbol{x}_j({}^2C^{(u-1)}) \leqq \boldsymbol{x}_j(C^{(1)}),$$

the inequality

$$\boldsymbol{x}_j({}^1C^{(u-1)}) \leqq \boldsymbol{x}_j({}^2C^{(u)}) \leqq \boldsymbol{x}_j(C^{(1)})$$

holds.

These relations imply that after some finite number of such steps, which number is limited by $m(K)$,

$$m(K) = \left[\frac{\max_{i \in K} \{\boldsymbol{x}_i(C^{(u)}) - \boldsymbol{x}_i(C^{(1)})\} \, \pi(K)}{\sum_{i \in K} (\boldsymbol{v}_i(\boldsymbol{s}^{(u)}) - \boldsymbol{v}_i(\boldsymbol{s}^{(u-1)}))} \right] + 1,$$

a new sequence of configurations, $\{C' = C^{(u)} = {}^1C^{(1)}, \ldots, {}^1C^{(u)} = {}^2C^{(1)}, \ldots, {}^2C^{(u)} = = {}^3C^{(1)}, \ldots, {}^{n(K)}C^{(u-1)}, {}^{n(K)}C^{(u)}\}$, $n(K) \leqq m(K)$, which is a weak domination sequence modulo c from $C^{(u)}$ to ${}^{n(K)}C^{(u)}$, will be constructed, so that $B_K^{(n)K))} = 0$ and, consequently,

$$(K, s_K^{(u)}, {}^{n(K)}D_K^{(u)}) = (K, s_K^{(1)}, D_K^{(1)}) = (K, s_K, D_K).$$

Hence, if the described procedure will be repeated n-times, where n can be chosen such that

$$n \leqq \max\{m(K) : K \in \mathscr{K} - k, \textit{ and the agreement } (K, s_K^{(u)}, D_K^{(u)}) \textit{ is active in the relation } C^{(u)} \operatorname{dom} C^{(u-1)} \bmod c\},$$

then ${}^nC^{(u)} = C^{(1)} = C$, and the weak domination sequence modulo c from $C' = = C^{(u)} = {}^1C^{(1)}$ to ${}^nC^{(u)} = C^{(1)} = C$ is definitively found. □

Lemma 11.3. If $C = C(\mathbf{s}, \mathbf{D}) \in \mathscr{C}$ and $C' = C'(\mathbf{s}, \mathbf{D}') \in \mathscr{C}$ are admissible configurations, and if there exists a strong domination sequence from C to C', then there exists also a strong domination sequence from C' to C.

Proof. Lemmas 11.1 and 11.2 imply, that if $C = C(\mathbf{s}, \mathbf{D})$, $C^\dagger = C^\dagger(\mathbf{s}, \mathbf{D}^\dagger)$, $C^{(1)} = = C^{(1)}(\mathbf{s}^{(1)}, \mathbf{D}^{(1)})$ and $C^{\dagger(1)} = C^{\dagger(1)}(\mathbf{s}^{(1)}, \mathbf{D}^{\dagger(1)})$ are admissible configurations, if $c = c(C; k) \subset C \cap C^{(1)}$ and $c^\dagger = c^\dagger(C^\dagger; k) \subset C^\dagger \cap C^{\dagger(1)}$ are their sub-configurations, if the relation $C \operatorname{Dom} C^{(1)} \bmod c$ holds, and if there exists a weak domination sequence modulo $c^\dagger$ from $C^{\dagger(1)}$ to $C^\dagger$, then also $C^\dagger \operatorname{Dom} C^{\dagger(1)} \bmod c^\dagger$. If it was not true, then there exists a weak domination sequence modulo $c^\dagger$ from $C^\dagger$ to $C^{\dagger(1)}$ and, hence, according to Lemma 11.1, there exists a configuration $C' = C'(\mathbf{s}, \mathbf{D}') \in \mathscr{C}$ and a weak domination sequence modulo c from $C^{(1)}$ to C'. Then, according to Lemma 11.2, there must exist a weak domination sequence modulo c from C' to C. This is, according to Remark 5.5, a contradiction with the supposed relation $C \operatorname{Dom} C^{(1)} \bmod c$, and, consequently, the stated auxiliary assertion is true.

The aim of the following steps of the proof is to form a strong domination sequence from C' to C. For the idea of this proof is the same as the idea of the proof of Lemma 11.2, we introduce only the main steps of the described procedure, with the reference to the analogous steps of the previous proof.

Let

$$\{C^{(r)}\}_{r=1,\ldots,u}, \quad \{c^{(r)}\}_{r=1,\ldots,u-1}$$

be the existing strong domination sequence from C to C',

$$C^{(r)} = C^{(r)}(\mathbf{s}^{(r)}, \mathbf{D}^{(r)}), \quad c^{(r)} = c^{(r)}(C^{(r)}; k^{(r)}),$$
$$C^{(1)} = C, \quad C^{(u)} = C'.$$

There exists a weak domination sequence modulo $c^{(r)}$ from $C^{(r)}$ to $C^{(r+1)}$, for every index $r = 1, \ldots, u - 1$. Let us denote this sequence

$$\{C^{(t,r)}\}_{t=1,\ldots,m(r)},$$

$$C^{(1,1)} = C^{(1)} = C, \quad C^{(1,r)} = C^{(r)}, \quad C^{(m(r),r)} = C^{(r+1)}, \quad \text{for}$$

$$r = 1, \ldots, u - 1, \quad C^{(m(u-1),u-1)} = C^{(u)} = C',$$

$$C^{(t+1,r)} \operatorname{dom} C^{(t,r)} \bmod c^{(r)}, \quad \text{for} \quad t = 1, \ldots, m(r) - 1, \quad r = 1, \ldots, u - 1.$$

Let $C^{(t,r)} = C^{(t,r)}(\mathbf{s}^{(t,r)}, \mathbf{D}^{(t,r)})$, and let the side payments matrices are

$$\mathbf{D}^{(r)} = (D^{(r)})_{L\in\mathscr{K}} = (d_{ij}^{(r)})_{i\in I, j\in I} \in \mathscr{D}_{\mathscr{K}},$$

$$\mathbf{D}^{(t,r)} = (D_L^{(t,r)})_{L\in\mathscr{K}} = (d_{ij}^{(t,r)})_{i\in I, j\in I} \in \mathscr{D}_{\mathscr{K}},$$

for

$$t = 1, \ldots, m(r), \quad r = 1, \ldots, u, \quad (t, r) \neq (t, u), \quad t = 1, \ldots, m(r).$$

Now, we can introduce new sequences of configurations and their sub-configurations,

$$\{{}^1C^{(r)}\}_{r=1,\ldots,u}, \quad \{{}^1c^{(r)}\}_{r=1,\ldots,u-1}, \quad \{{}^1C^{(t,r)}\}_{t=1,\ldots,m(r)}, \quad r = 1, \ldots, u = 1,$$

such that

$${}^1C^{(r)} = {}^1C^{(r)}(\mathbf{s}^{(r)}, {}^1\mathbf{D}^{(r)}), \quad {}^1C^{(t,r)} = {}^1C^{(t,r)}(\mathbf{s}^{(t,r)}, {}^1\mathbf{D}^{(t,r)}),$$

$${}^1c^{(r)} = {}^1c^{(r)}({}^1C^{(r)}; k^{(r)}), \quad t = 1, \ldots, m(r), \quad r = 1, \ldots, u;$$

$${}^1C^{(m(r),r)} = {}^1C^{(r+1)}, \quad r = 1, \ldots, u = 1,$$

$${}^1C^{(1,1)} = {}^1C^{(1)} = C^{(u)}.$$

The side payments matrices

$${}^1\mathbf{D}^{(r)} = ({}^1D_L^{(r)})_{L\in\mathscr{K}} = ({}^1d_{ij}^{(r)})_{i\in I, j\in I} \in \mathscr{D}_{\mathscr{K}}, \quad r = 1, \ldots, u,$$

$${}^1\mathbf{D}^{(t,r)} = ({}^1D_L^{(t,r)})_{L\in\mathscr{K}} = ({}^1d_{ij}^{(t,r)})_{i\in I, j\in I} \in \mathscr{D}_{\mathscr{K}},$$

$$t = 1, \ldots, m(r), \quad r = 1, \ldots, u - 1,$$

are defined in the following way:

If

$$K \in \bigcap_{r=1}^{u-1} k^{(r)}$$

then

$${}^1D_K^{(t,r)} = D_K^{(t,r)} = D'_K = D_K.$$

If

$$K \in \mathscr{K} - \bigcap_{r=1}^{u-1} k^{(r)}$$

then we find out the highest value of the index r, $1 \leqq r \leqq u - 1$, for which $K \in \mathscr{K} - k^{(r)}$ and denote it by $\varrho(K)$. For all the pairs of indices (t, r) such that $1 \leqq r \leqq \varrho(K)$, $1 < t \leqq m(r)$, $t \neq m(\varrho(K))$, $K \in \mathscr{K} - k^{(r)}$, we set

$$
{}^1d_{ij}^{(t,r)} = (\pi(K))^{-1} \cdot (x_i({}^1C^{(t-1,r)}) - v_i(s^{(t,r)}) - x_i({}^1C^{(t-1,r)}) + v_j(s^{(t,\ )})), \quad i \in K, \quad j \in K .
$$

If $r = \varrho(K)$, and if the agreement

$$
(K, s_K^{(m(r),r)}, D_K^{(m(r),r)}) = (K, s_K^{(1,r+1)}, D_K^{(1,r+1)}) = (K, s_K^{(r+1)}, D_K^{(r+1)}) = (K, s_K^{(u)}, D_K^{(u)})
$$

is not active in the relation

$$
C^{(m(r),r)} \text{ dom } C^{(m(r)-1,r)} \bmod c^{(r)} ,
$$

then we set

$$
{}^1D_K^{(m(r),r)} = D_K^{(1)} .
$$

If $r = \varrho(K)$, and if the agreement $(K, s_K^{(m(r),r)}, D_K^{(m(r),r)})$ is active in the relation

$$
C^{(m(r),r)} \text{ dom } C^{(m(r)-1,r)} \bmod c^{(r)} ,
$$

then we construct the matrix

$$
{}^1D_K^{(m(r),r)} = {}^1D_K^{(1,r+1)} = {}^1D_K^{(q)} , \quad q = r + 1, \ldots, u ,
$$

in the following way:

For every player $i \in K$ we define the numbers

$$
\begin{aligned}
\alpha_i^{(1)} &= x_i({}^1C^{(m(r)-1,r)}) - v_i(s^{(m(r),r)}) , \\
\beta_i^{(1)} &= \max \{0, v_i(s^{(m(r),r)}) + \sum_{j\in K} d_{ij}^{(1)} - x_i({}^1C^{(m(r)-1,r)})\} , \\
\gamma_i^{(1)} &= B_K^{(1)} \cdot \beta_i^{(1)} ,
\end{aligned}
$$

where

$$
B_K^{(1)} = \frac{\sum_{j\in K} (v_j(s^{(m(r),r)}) - v_j(s^{(m(r)-1,r)} - \beta_j^{(1)})}{\sum_{j\in K} \beta_j^{(1)}} ,
$$

and then we set

$$
{}^1d_{ij}^{(m(r),r)} = {}^1d_{ij}^{(1,r+1)} = (\pi(K))^{-1} \cdot (\alpha_i^{(1)} + \beta_i^{(1)} + \gamma_i^{(1)} - \alpha_j^{(1)} - \beta_j^{(1)} - \gamma_j^{(1)}) = {}^1d_{ij}^{(q)} = {}^1d_{ij}^{(p,q)} = {}^1d_{ij}^{(u)} , \quad p = 1, \ldots, m(q) , \quad q = 1, \ldots, u - 1
$$

The auxiliary assertion, introduced in the first part of this proof, implies that the, in such a way constructed, sequence

$$
\{{}^1C^{(r)}\}_{r=1,\ldots,u} , \quad \{{}^1c^{(r)}\}_{r=1,\ldots u-1} ,
$$

is a strong domination sequence from ${}^1C^{(1)} = C^{(u)}$ to ${}^1C^{(u)}$.

The following advance of the proof is similar to the analogous steps of the proof of Lemma 11.2. After a finite and limited number of repetitions of the described procedure, we construct the configuration ${}^{n}C^{(m(u-1),u-1)} = {}^{n}C^{(u)} \in \mathscr{C}$, and a strong domination sequence from ${}^{1}C^{(1,1)} = C'$ to ${}^{n}C^{(m(u-1),u-1)}$ such that ${}^{n}C^{(m(u-1),u-1)} = C^{(1)} = C$. Consequently, the desired sequence is constructed. □

Theorem 6. Provided Γ is a coalition-game with fixed cooperation and with finite identification, any admissible configuration in Γ is weakly predictionally stable, if and only if it is strongly predictionally stable; i.e.

$$\mathfrak{P}_{\Gamma}^{**} = \mathfrak{P}_{\Gamma}^{*}.$$

Proof. According to Lemma 6.1, $\mathfrak{P}_{\Gamma}^{**} \supset \mathfrak{P}_{\Gamma}^{*}$. Let $C' = C'(\mathbf{s}', \mathbf{D}') \in \mathfrak{P}_{\Gamma}^{**}$, $C = C(\mathbf{s}, \mathbf{D}) \in \mathscr{C}$ and let there exists a strong domination sequence from C' to C. Because $C' \in \mathfrak{P}_{\Gamma}^{**}$, there must exist some $C'' = C''(\mathbf{s}', \mathbf{D}'') \in \mathscr{C}$ and a strong domination sequence from C to C''. According to Lemma 4 and Remark 5.7, there exists also a strong domination sequence from C to C' and, consequently, the relation $C \mathbf{p} C'$ does not hold. It means that

$$\mathfrak{P}_{\Gamma}^{**} \subset \mathfrak{P}_{\Gamma}^{*},$$

and the equality is proved. □

Corollary. Theorems 1 and 6 imply that $\mathfrak{P}_{\Gamma}^{*} \cap \mathscr{C}_{\mathrm{rat}} \neq \emptyset$, whenever Γ is a game with fixed cooperation.

Remark 11.1, Lemmas 11.2 and 11.3, and Theorem 6 imply that if $C = C(\mathbf{s}, \mathbf{D}) \in \mathscr{C}$ and $C' = C'(\mathbf{s}, \mathbf{D}') \in \mathscr{C}$ are admissible configurations with the same vector of strategies, and if Γ is a game with fixed cooperation, then the following equivalences hold:

$$C \in \mathscr{C}_{\mathrm{rat}} \Leftrightarrow C' \in \mathscr{C}_{\mathrm{rat}},$$

$$C \in \mathfrak{P}_{\Gamma}^{**} = \mathfrak{P}_{\Gamma}^{*} \Leftrightarrow C' \in \mathfrak{P}_{\Gamma}^{**} = \mathfrak{P}_{\Gamma}^{*},$$

$$C \in \mathfrak{P}_{\Gamma} \Leftrightarrow C' \in \mathfrak{P}_{\Gamma},$$

Remark 11.2. If Γ is a coalition-game with fixed cooperation and without identification, then the statements of Theorems 2 and 5 may be applied, and the following equations:

$$\mathfrak{P}_{\Gamma}^{**} = \mathfrak{P}_{\Gamma}^{*} = \mathfrak{P}_{\Gamma} = \{C : C = C(\mathbf{s}, \mathbf{D}) \in \mathscr{C},$$

$$\mathbf{s} = (s_K)_{K \in \mathscr{K}}, \ \sum_{i \in K} v_i(\mathbf{s}) = \max \{ \sum_{i \in K} v_i(\mathbf{s}') : \mathbf{s}' \in S_{\mathscr{K}} \},$$

$$\textit{for all} \quad K \in \mathscr{K}\} = \mathscr{C}_{\mathrm{rat}},$$

hold.

12. Two-Players Games

The task of this section is to illustrate the proposed prediction method and its results on the most simple (and in the game theory best investigated) type of conflict situations.

It is advantageous, for the purposes of this section, to use the notations of the previous section 11, with some further simplifications.

In the whole section we suppose, that

$$\Gamma = (I, \mathbf{K}, \{S_K\}_{K \in \|\mathbf{K}\|}, \{\mathcal{R}_K\}_{K \in \|\mathbf{K}\|}, \{v_i\}_{i \in I})$$

is a coalition-game with finite identification and with fixed cooperation, and that $I = \{1, 2\}$; $\mathbf{K} = \{\mathcal{K}\}$, $\mathcal{K} = \{\{1\}, \{2\}\}$. Instead of $S_{\mathcal{K}}$ we write simply S, instead of $S_{\{i\}}$, $s_{\{i\}}$, $\mathcal{R}_{\{i\}}$ and $R_{\{i\}}$ we write S_i, s_i, $\mathcal{R}_i$, R_i, respectively. If $\mathbf{s} = (s_1, s_2) \in S$, then we write, instead of $(\mathcal{K}, \mathbf{s})$, $v_i(\mathcal{K}, \mathbf{s})$, $C(\mathcal{K}, \mathbf{s}, \mathbf{D})$ and $(\{i\}, s_{\{i\}}, D_{\{i\}})$ (where the side payments matrices $\mathbf{D}$ and $D_{\{i\}}$ are necessarily the zero-metrices), also briefly $\mathbf{s}$ or (s_1, s_2), $v_i(\mathbf{s})$ or $v_i(s_1, s_2)$, $C(\mathbf{s})$ or $C(s_1, s_2)$, and $(i, s_i, 0)$, respectively.

In order to confront, reliably enough, the results of the proposed prediction method with the classical ones, we introduce some notions, usual in the two-players game theory, and show their coherence with the notions, introduced in this paper.

Let us suppose that, for any player $i = 1, 2$ a real-valued bounded function h_i, defined on the set S, is given, and that:

(12.1) if A_1, A_2 are finite, non-empty sets such that $S_i = S(A_i)$, $i = 1, 2$, (c.f. (1.3)), then for $\mathbf{s} = (s_1, s_2) \in S$ the equation

$$h_i(\mathbf{s}) = \sum_{a = (a_1, a_2) \in A_1 \times A_2} s_1(a_1) \, . \, s_2(a_2) \, . \, h_i(\mathbf{s}^{[a]}) \, ,$$

where

$$\mathbf{s}^{[a]} = (s_1^{[a]}, s_2^{[a]}) \in S \, , \quad s_i^{[a]}(a_i) = 1 \, , \quad s_i^{[a]}(a_i') = 0 \, , \quad a_i' \in A_i \, , \quad a_i' \neq a_i \, ,$$

(c.f. (3.1), (1.5)), holds.

Functions h_i will be called, in the following section, classical pay-off functions. We suppose, that the following relation between classical pay-off functions h_i and our pay-off functions v_i holds:

(12.2) if $\mathbf{s} = (s_1, s_2) \in S$, and if for $i \in I$ is $(\mathcal{K}, \mathbf{s}) \in R_i \in \mathcal{R}_i$, then

$$h_i(\mathbf{s}) \geqq v_i(\mathbf{s}) = \inf \{h_i(\mathbf{s}') : \mathbf{s}' = (s_1', s_2'), \ s_i' = s_i, \ (\mathcal{K}, \mathbf{s}') \in R_i\} \, ,$$

(compare with Remark 3.3 and with (3.2)).

If H is a real number, then Γ is said to be a *constant-sum game*, if for all vectors $\mathbf{s} \in S$ the equation $h_1(\mathbf{s}) + h_2(\mathbf{s}) = H$ holds.

If Γ is a constant-sum game then the vector of strategies $\mathbf{s}_* = (s_1^*, s_2^*) \in \mathbf{S}$ is said to be a *maximin-vector*, if

$$h_1(\mathbf{s}_*) = \max_{s_1 \in S_1} \min_{s_2 \in S_2} h_1(s_1, s_2) ,$$

and

$$h_2(\mathbf{s}_*) = \max_{s_2 \in S_2} \min_{s_1 \in S_1} h_2(s_1, s_2) .$$

Lemma 12.1. If $\mathbf{s}_* \in \mathbf{S}$ is a maximin-vector of strategies, then

$$v_1(\mathbf{s}_*) = \max_{s_1 \in S_1} \min_{s_2 \in S_2} h_1(s_1, s_2) , \tag{12.3}$$

$$v_2(\mathbf{s}_*) = \max_{s_2 \in S_2} \min_{s_1 \in S_1} h_2(s_1, s_2) , \tag{12.4}$$

$$v_i(\mathbf{s}_*) = h_i(\mathbf{s}_*) , \quad i = 1, 2 . \tag{12.5}$$

Proof. According to Remark 10.3 there exists to any $s_1 \in S_1$ such a strategy $s_2^*(s_1) \in S_2$ that

$$h_1(s_1, s_2^*(s_1)) = \min \{h_1(s_1, s_2) : s_2 \in S_2\} .$$

Hence, according to (12.2),

$$h_1(s_1, s_2^*(s_1)) \geqq v_1(s_1, s_2^*(s_1)) \geqq \min \{v_1(s_1, s_2) : s_2 \in S_2\} = \\ = \min \{h_1(s_1, s_2) : s_2 \in S_2\} = h_1(s_1, s_2^*(s_1)) .$$

This implies that

$$v_1(s_1, s_2^*(s_1)) = \min \{v_1(s_1, s_2) : s_2 \in S_2\} = h_1(s_1, s_2^*(s_1)) . \tag{12.6}$$

Now, we choose the strategy $s_1^* \in S_1$, for which

$$h_1(s_1^*, s_2^*(s_*^1)) = \max \{h_1(s_1, s_2^*(s_1)) : s_1 \in S_1\} = \\ = \max \{\min \{h_1(s_1, s_2) : s_2 \in S_2\} : s_1 \in S_1\} . \tag{12.7}$$

Relation (12.6) holds even for s_1^*, so that, consequently,

$$h_1(s_1^*, s_2^*(s_1^*)) = v_1(s_1^*, s_2^*(s_1^*)) . \tag{12.8}$$

If there exists $s' \in S_1$ such that

$$v_1(s'_1, s_2^*(s'_1)) > v_1(s_1^*, s_2^*(s_1^*)) ,$$

then

$$h_1(s'_1, s_2^*(s'_1)) > h_1(s_1^*, s_2^*(s_1^*)_j .$$

This inequality is a contradiction with (12.7). It means, that

$$v_1(s_1^*, s_2^*(s_1^*)) = \max\{\min\{v_1(s_1, s_2) : s_2 \in S_2\} : s_1 \in S_1\}.$$

This proves the equation (12.3) and one of the equations (12.5); equation (12.4) and the latter of equations (12.5) can be proved analogously. □

In the following statements, the results of the proposed bargaining prediction method (in this case it is a prediction of the strategies choosing) is confronted with the results of the classical game theory. We shall verify, how the proposed prediction method corresponds with the deep-rooted idea of a two-players rational behaviour in a non-cooperative game.

Theorem 7. If Γ is a constant-sum coalition-game with two players and with fixed cooperation $\mathscr{K} = \{\{1\}, \{2\}\}$, then:

(1) The class of all predictionally balanced configurations $\mathfrak{P}_\Gamma$ is non-empty and it contains exactly all the configurations with the maximin-vector of strategies; i.e.

$$\mathfrak{P}_\Gamma = \{C : C = C(\mathbf{s}_*), \mathbf{s}_* \textit{ is a maximin-vector}\} \neq \emptyset.$$

(2) Any weakly predictionally stable configuration is strongly predictionally stable; i.e.

$$\mathfrak{P}_\Gamma^{**} = \mathfrak{P}_\Gamma^{*}.$$

(3) If $\mathbf{s}_* = (s_1^*, s_2^*) \in S$ is a maximin-vector, and if the strategies vectors $\mathbf{s}' = (s_1^*, s_2) \in S$ and $\mathbf{s}'' = (s_1, s_2^*) \in S$ are not maximin-vectors, then the admissible configurations $C' = C'(\mathbf{s}') \in \mathscr{C}$ and $C'' = C''(\mathbf{s}'') \in \mathscr{C}$ are not weakly predictionally stable.

(4) If the game Γ is, moreover, the game without an identification, the any configuration is weakly predictionally stable if and only if it is predictionally balanced; i.e.

$$\mathfrak{P}_\Gamma^{**} = \mathfrak{P}_\Gamma^{*} = \mathfrak{P}_\Gamma.$$

Proof. If $\mathbf{s}_* = (s_1^*, s_2^*) \in S$ is a maximin-vector, and if $C^* = C^*(\mathbf{s}_*) \in \mathscr{C}$, then, according to Lemma 12.1, is $C^* = C_{\text{rat}}$. Hence, there is no $C \in \mathscr{C}$ and $i \in I$ such that

$$C \operatorname{dom} C^* \bmod c(C^*; \{\{i\}\}).$$

If there exists $C \in \mathscr{C}$ and a weak domination sequence modulo $\emptyset$ from C^* to C, then:

either for some $i \in I$ is

$$x_i(C^*) \neq x_i(C),$$

and then

$$C^* \operatorname{dom} C \bmod \emptyset,$$

as follows from Lemma 12.1, or

$$x_i(C^*) = x_i(C), \quad i = 1, 2,$$

and then the same sequence of configurations is a weak domination sequence modulo $\emptyset$ from C to C^*. It means, that the relation

$$C \text{ Dom } C^* \bmod \emptyset$$

is not true for any $C \in \mathscr{C}$, and, consequently, $C^* \in \mathfrak{P}_\Gamma$.

According to the well-known classical results (introduced, for example, in [5], [7], and [10]) there exists at least one maximin-vector of strategies in any two-players constant-sum game. It means that $\mathfrak{P}_\Gamma \neq \emptyset$.

If $C = C(\mathbf{s}) \in \mathscr{C}$, where $\mathbf{s} = (s_1, s_2)$ is not a maximin-vector, and if $\mathbf{s}_* = (s_1^*, s_2^*) \in \mathbf{S}$ is a maximin-vector, then there exists a player in I – without loss of generality we may assume that it is the player 1 – for which

$$h_1(\mathbf{s}) < h_1(\mathbf{s}^*) \leqq h_1(s_1^*, s_2)$$

and, according to Lemma 12.1,

$$v_1(\mathbf{s}) < v_1(\mathbf{s}_*) = h_1(\mathbf{s}_*) \leqq v_1(s_1^*, s_2).$$

It means that $C \neq C_{\text{rat}}$, according to Lemma 11.1, and, consequently, $C \notin \mathfrak{P}_\Gamma$, according to Remark 6.2. The first part of this Theorem is proved.

The second statement of Theorem 7 is an immediate consequence of Theorem 6.

Preserving the notations of the third statement of Theorem 7, the relations

$$C^* \text{ Dom } C' \bmod c(C', \{\{1\}\}) \quad \text{and} \quad C^* \text{ Dom } C'' \bmod c(C'', \{\{2\}\})$$

hold.

Because $C^* \in \mathfrak{P}_\Gamma$, there is no strong domination sequence from C^* to any $C \in \mathscr{C}$, and, consequently, $C^* \mathbf{p}\, C'$ and $C^* \mathbf{p}\, C''$.

The last statement of the theorem is an immedite consequence of Theorem 2 and Remark 11.2. □

Even for the more general two-players games with non-constant sum, some assertions may be stated.

Remark 12.1. If $C \in \mathscr{C}$ is an admissible configuration in a two-players game Γ with fixed cooperation $\mathscr{K} = \{\{1\}, \{2\}\}$, if $C = C(\mathbf{s})$, $C = C_{\text{rat}}$, and if

$$v_1(\mathbf{s}) + v_2(\mathbf{s}) \geqq v_1(\mathbf{s}') + v_2(\mathbf{s}')$$

for all $\mathbf{s}' \in \mathbf{S}$, then C is a predictionally balanced configuration; i.e. $C \in \mathfrak{P}_\Gamma$.

Theorem 8. If Γ is a two-players strategic coalition-game with finite identification and with fixed cooperation $\mathscr{K} = \{\{1\}, \{2\}\}$, then:

(1) Any admissible configuration is weakly predictionally stable if and only if it is strongly predictionally stable, and there exists at least one strongly predictionally stable configuration, being of the rational character.

(2) If $\mathbf{s}_* = (s_1^*, s_2^*) \in \mathbf{S}$, $\mathbf{s}' = (s_1^*, s_2) \in \mathbf{S}$, $\mathbf{s}'' = (s_1, s_2^*) \in \mathbf{S}$ are vectors of strategies, if $C^* = C^*(\mathbf{s}_*) \in \mathscr{C}$, $C' = C'(\mathbf{s}') \in \mathscr{C}$, $C'' = C''(\mathbf{s}'') \in \mathscr{C}$ are admissible configurations, and if $C^* = C^*_{\text{rat}}$, $(1, s_1, \mathit{0}) \in C'' - C''_{\text{rat}}$, $(2, s_2, \mathit{0}) \in C' - C'_{\text{rat}}$, then the configurations C' and C'' are not (either strongly or weakly) predictionally stable; i.e.

$$C' \notin \mathfrak{P}^{**}_\Gamma = \mathfrak{P}^*_\Gamma\,, \quad C'' \notin \mathfrak{P}^{**}_\Gamma = \mathfrak{P}^*_\Gamma\,.$$

(3) If Γ is, moreover, a game without identification, then any admissible configuration is predictionally balanced if and only if it is of rational character; the class of all configurations with rational character is given by the equations

$$\mathscr{C}_{\text{rat}} \doteq \mathfrak{P}^{**}_\Gamma = \mathfrak{P}^*_\Gamma = \mathfrak{P}_\Gamma =$$
$$= \{C : C = C(\mathbf{s}) \in \mathscr{C},\ \mathbf{v}_i(\mathbf{s}) \geqq \mathbf{v}_i(\mathbf{s}')\ \textit{for all } \mathbf{s}' \in \mathbf{S},\ i = 1, 2\}\,.$$

Proof. The first and the third assertions of Theorem are immediate consequences of Theorems 2 and 6 and Remark 11.2.

We prove tue second assertion of Theorem for the configuration C' only. The proof for C'' is quite similar. First of all, we construct, for every player $i = 1, 2$, and for every identification set $\mathbf{R}_i \in \mathscr{R}_i$, the non-empty set of strategies $\mathbf{S}_i(\mathbf{R}_i) \subset \mathbf{S}_i$ using the formulas

$$\mathbf{S}_1(\mathbf{R}_1) = \{s_1 : s_1 \in \mathbf{S}_1,\ \mathbf{v}_1(s_1, s_2) \geqq \mathbf{v}_1(s_1', s_2')$$
$$\textit{for all strategic structures } (\mathscr{K}, (s_1', s_2')) \in \mathbf{R}_1\}\,, \quad \mathbf{R}_1 \in \mathscr{R}_1\,,$$
$$\mathbf{S}_2(\mathbf{R}_2) = \{s_2 : s_2 \in \mathbf{S}_2,\ \mathbf{v}_2(s_1, s_2) \geqq \mathbf{v}_2(s_1', s_2')$$
$$\textit{for all strategic structures } (\mathscr{K}, (s_1', s_2')) \in \mathbf{R}_2\}\,, \quad \mathbf{R}_2 \in \mathscr{R}_2\,.$$

If the strategic structures $(\mathscr{K}, \mathbf{s})$ and $(\mathscr{K}, \mathbf{s}')$ belong into the same intersection $\mathbf{R}_1 \cap \mathbf{R}_2$, $\mathbf{R}_1 \in \mathscr{R}_1$, $\mathbf{R}_2 \in \mathscr{R}_2$, and if $\mathbf{s}$ and $\mathbf{s}'$ belong into the same Cartesian product

$$\mathbf{S}_1(\mathbf{R}_1) \times \mathbf{S}_2(\mathbf{R}_2)\,,$$

then necessarily

$$\mathbf{v}_i(\mathbf{s}) = \mathbf{v}_i(\mathbf{s}')\,, \quad i = 1, 2\,,$$

and for the configurations $C_1 = C_1(\mathbf{s})$ and $C_2 = C_2(\mathbf{s}')$ the equations

$$\mathbf{x}_i(C_1) = \mathbf{x}_i(C_2)\,, \quad i = 1, 2\,,$$

hold. Hence, the relation

$$C_1 \text{ Dom } C_2 \bmod \emptyset$$

is not true.

Let us return to the configurations C^* and C', described in the second statement of Theorem. According to Lemma 5.3, the relations

$$C^* \text{ dom } C' \bmod c(C^*; \{\{1\}\}), \quad C^* \text{ dom } C' \bmod \emptyset, \quad \text{and} \quad C^* \text{ Dom } C \bmod c(C^*, \{\{1\}\}), \tag{12.9}$$

hold. Let us assume, now, that $C' \in \mathfrak{P}^*_T$. Then there exists a strong domination sequence from C^* to C', which will be denoted

$$\{C^{(r)}\}_{r=1,\ldots,n}, \quad \{c^{(r)}\}_{r=1,\ldots,n-1},$$

$C^{(1)} = C^*$, $C^{(n)} = C'$. The assumption $C^* = C_{\text{rat}}$ implies that

$$c^{(1)} = \emptyset, \quad \text{and} \quad C^{(2)} \text{ dom } C^* \bmod \emptyset, \tag{12.10}$$

so that there exists a weak domination sequence modulo $\emptyset$ from C^* to $C^{(2)}$.

Let a weak domination sequence modulo $\emptyset$ from C^* to $C^{(r)}$, for an r, $1 < r < n$, is constructed. Relations (12.9) and (12.10), together with the assumption $C' \in \mathfrak{P}^*_T$, imply that there exists an index u, $1 < u \leqq n$, such that $c^{(r)} = \emptyset$ for all $r < u$, and $c^{(u)} \neq \emptyset$. It means that a weak domination sequence modulo $c^{(u)}$ from $C^{(u)}$ to $C^{(u+1)}$ consists of exactly two elements, i.e.

$$C^{(u+1)} \text{ dom } C^{(u)} \bmod c^{(u)}.$$

As the relation

$$C^{(u+1)} \text{ dom } C^{(u)} \bmod \emptyset$$

does not hold, then $C^{(u+1)} \neq C^{(u+1)}_{\text{rat}}$, and there must exist such a ${}^{(1)}C^* \in \mathscr{C}$, for which

$${}^{(1)}C^* \cap C^{(u+1)} = C^{(u+1)} - c^{(u)}, \quad {}^{(1)}C^* = {}^{(1)}C^*_{\text{rat}}$$

and

$${}^{(1)}C^* \text{ dom } C^{(u)} \bmod \emptyset,$$

in accordance with Definition 2.

Let us consider, now, the properties of ${}^{(1)}C^*$. There exists a weak domination sequence modulo $\emptyset$ from $C^{(2)}$ to ${}^{(1)}C^*$ and, according to (12.10), there exists a weak domination sequence modulo $\emptyset$ from C^* to ${}^{(1)}C^*$. The relation

$$C^{(2)} \text{ Dom } C^* \bmod \emptyset$$

implies

$$^{(1)}C^* \text{ Dom } C^* \bmod \emptyset .$$

It means that there exists a strong domination sequence from C' to $^{(1)}C^*$, it is the sequence

$$\{C', C^*, {}^{(1)}C^*\}, \quad \{c(C'; (1, s_1^*, 0)), \emptyset\} .$$

The assumption $C' \in \mathfrak{P}_\Gamma^*$ implies that there must exist a strong domination sequence from $^{(1)}C^*$ to C'. In the case of this sequence we can advance in the same way as in the case of the foregoing one, and we can construct, gradually, the configurations

$$^{(0)}C^* = C^*, \ {}^{(1)}C^*, \ {}^{(2)}C^*, \ \ldots, \ {}^{(m)}C^* ,$$

which satisfy the following conditions:

(12.11) $^{(t)}C^* = {}^{(t)}C^*_{\text{rat}}, \quad t = 0, 1, \ldots, m ,$

(12.12) $^{(t)}C^* \text{ Dom } {}^{(r)}C^* \bmod \emptyset , \quad r = 0, \ldots, t - 1, \ t = 1, \ldots, m ,$

(12.13) if we denote $^{(t)}C^* = {}^{(t)}C^*(\mathbf{s}_*^{(t)})$, $t = 0, 1, \ldots, m$, then (12.12) and the properties of the sets $S_i(R_i)$, introduced in the first steps of the proof, imply that no pair of strategies vectors $\mathbf{s}_*^{(t)}, \mathbf{s}_*^{(r)}$, $t = 0, 1, \ldots, m$, $r = 0, 1, \ldots, m$, $r \neq t$, can belong to the same set $S_1(R_1) \times S_2(R_2)$, $R_1 \in \mathscr{R}_1$, $R_2 \in \mathscr{R}_2$.

(12.14) The assumption $C' \in \mathfrak{P}_\Gamma^*$ means that we suppose that a strong domination sequence from $^{(t)}C^*$ to C' (and, consequently, to $^{(0)}C^*$), for $t = 1, \ldots, m$, exists.

The sequence $^{(0)}C^*, \ldots, {}^{(m)}C^*$ can be, by repetition of the described procedure, constructed so that for every configuration $^{(m+1)}C^* = {}^{(m+1)}C^*(\mathbf{s}_*^{(m+1)}) \in \mathscr{C}$, $^{(m+1)}C^* = {}^{(m+1)}C^*_{\text{rat}}$, for which there exists a weak domination sequence modulo $\emptyset$ from $^{(m)}C^*$ to $^{(m+1)}C^*$, an index t, $0 \leqq t \leqq m$, exists, such that $\mathbf{s}_*^{(m+1)}$ and $\mathbf{s}_*^{(t)}$ belong to the same set

$$S_1(R_1) \times S_2(R_2) \subset S, \quad R_1 \in \mathscr{R}_1, \quad R_2 \in \mathscr{R}_2 .$$

But, this is a contradiction with (12.12), and, consequently, there exists no $^{(m+1)}C^* \in \mathscr{C}$, $^{(m+1)}C^* = {}^{(m+1)}C^*_{\text{rat}}$, and no weak domination sequence modulo $\emptyset$ from $^{(m)}C^*$ to $^{(m+1)}C^*$. It means, together with the equation $^{(m)}C^* = {}^{(m)}C^*_{\text{rat}}$, that there exists no $C''' \in \mathscr{C}$ and no $c \subset {}^{(m+1)}C^*$ such that

$$C''' \text{ dom}^{(m)}C^* \bmod c .$$

Hence, the requirement (12.14) cannot be satisfied for $^{(m)}C^*$. This implies that $^{(m)}C^* \mathbf{P} C'$ and $C' \notin \mathfrak{P}_\Gamma^*$. □

Remark 12.2. If we leave the assumption, stated in the introductory paragraphs of this section, that Γ is a game with the fixed cooperation $\mathscr{K} = \{\{1\}, \{2\}\}$, and if we suppose, now, that Γ is a game with free cooperation, with the classical pay -off functions $\boldsymbol{h}_i$, $i \in I$, defined on the set

$$\bigcup_{\mathscr{K} \in \mathbf{K}} \boldsymbol{S}_{\mathscr{K}}$$

according to Remark 3.3 and (3.1), and with functions $\boldsymbol{v}_i$, $i \in I$, given by (3.2), then we may use Lemma 6.3 and, sometimes, even Theorems 2 and 3 (if Γ is a game without identification). In such a case, the following statement holds:

If $\mathscr{K}_0 = \{\{1\}, \{2\}\}$, $\mathscr{K}_I = \{I\}$, and if

$$\max \{ \sum_{i=1}^{2} \boldsymbol{v}_i(\mathscr{K}_I, \mathbf{s}) : \mathbf{s} \in \boldsymbol{S}_{\mathscr{K}_I} \} \geqq \sum_{i=1}^{2} \max \{ \boldsymbol{v}_i(\mathscr{K}_0, \mathbf{s}) : \in \boldsymbol{S}_{\mathscr{K}_0} \} , \tag{12.15}$$

then

$$\mathfrak{P}_\Gamma^{**} = \mathfrak{P}_\Gamma^{*} = \mathfrak{P}_\Gamma = \{ C : C = C(\mathscr{K}_I, \mathbf{s}, \mathbf{D}),\ C = C_{\text{rat}} \} \cup$$

$$\cup \{ C' : C' = C'(\mathscr{K}_0, \mathbf{s}, \boldsymbol{0}),\ \sum_{i=1}^{2} \boldsymbol{x}_i(C') = \max \{ \sum_{i=1}^{2} \boldsymbol{v}_i(\mathscr{K}_I, \mathbf{s}_*) : \mathbf{s}_* \in \boldsymbol{S}_{\mathscr{K}_I} \} \} .$$

If, instead of (12.15), the opposite, strong, inequality holds, then the bargaining prediction characteristic in Γ is the same as if Γ were a game with fixed cooperation $\mathbf{K} = \{\mathscr{K}_0\}$, with the same values of pay-off functions $\boldsymbol{h}_i$ and $\boldsymbol{v}_i$ for all vectors $\mathbf{s} \in \boldsymbol{S}_{\mathscr{K}_0}$, and with the same identification partitions of one-element coalitions.

Remark 12.3. Provided a game Γ with fixed cooperation $\mathbf{K} = \{\mathscr{K}_I\}$, $\mathscr{K}_I = \{I\}$, is considered, then the bargaining prediction characteristic is obvious. Such a game may be regarded as a game without identification, and, according to Theorems 2 and 3 and Remark 11.2, is

$$\mathscr{C}_{\text{rat}} = \mathfrak{P}_\Gamma^{**} = \mathfrak{P}_\Gamma^{*} = \mathfrak{P}_\Gamma = \{ C : C = C(\mathscr{K}_I, \mathbf{s}, \mathbf{D}) \in \mathscr{C}, \sum_{i \in I} \boldsymbol{v}_i(\mathscr{K}_I, \mathbf{s}) \geqq$$

$$\geqq \sum_{i \in I} \boldsymbol{v}_i(\mathscr{K}_I, \mathbf{s}') \textit{ for all strategies } \mathbf{s}' \in \boldsymbol{S}_{\mathscr{K}_I} \} .$$

It is unmistakable, according to Theorems 7 and 8 and to the previous Remarks, that the bargaining results, obtained by the proposed prediction method, are in the considered case of two-players games comparable with the well-known results of the classical game theory, as well as with the general idea of the rational behaviour of participants of such a game.

A FEW REMARKS ON BARGAINING MODELS

The following, conclusive, section of the presented paper contains some remarks, concerning the connections between this work and the well-known literature on the same subject.

(1) Many of the works, dealing with the bargaining prediction in coalition-games, are engaged in games without side payments, which games need a qualitatively different approach to the investigated situation.

(2) Among those works, which investigate the games with side payments, the most interesting one (from the our point of view) is the Auman's and Maschler's work [1]. This work deals with a game model, which is comparable with the game without identification and without strategies choosing, introduced in the presented paper. The notion of the rationality is, in [1], analogous to the our concept of rationality. The bargaining prediction, as such, is , owing to the simple type of investigated games, quite easy to survey, its results, however, do not quite correspond, in some cases, with the common concept of the players' rationality. The method, described in [1], does not eliminate the bargaining results, which we could name „improfitably small cooperation", i.e. the configurations, which contain a major number of agreements, with the possibility to increase significantly the guaranted profit of their players, if these agreements were united.

The method of work [1] is, on the other hand, very objective, and it is, to a certain extent, based on similar idea as the presented work.

Work [1] deals, among others, with the actual bargaining prediction in three-players games. The comparison with section 7 of the presented paper illustrates the similarity and the dissimilarity, of the both approaches to the investigated situation.

(3) A rather different approach to the described questions is presented in Harshanyi's work [2]. Its author considers games, similar to games with a free cooperation and without identification, introduced in the presented paper.

Work [2] predicts the resultant imputations (i.e. the resultant distribution of the profit), however, it does not solve the problem of coalitions forming, at all. It means, that work [2] deals with quite another problems than the presented paper, and, consequently, even the used methods are quite dissimilar.

(4) The further connection concerns the function $\boldsymbol{w}$, on the class $\|\mathbf{K}\|$, defined in section 7. This function has the properties of *characteristic function* of the coalition-game, which is introduced, for example, in von Neumann's and Morgenstern's fundamental work [7]. This is an immediate consequence of the fact, that the coalition-game without identification is close to the games, investigated usually in the literature, for which the characteristic function was originally defined, and for which it has a real sense. More general games, with more complex identification partitions, require even more complicated pay-off functions. It would be possible to define, even for such games, some function on the set of all pairs $(K, \boldsymbol{R}_K)$, $K \in \|\mathbf{K}\|$, $\boldsymbol{R}_K \in \mathscr{R}_K$ (or of all triples $(K, s_K, \boldsymbol{R}_K)$, $K \in \|\mathbf{K}\|$, $s_K \in S_K$, $\boldsymbol{R}_K \in \mathscr{R}_K$), as a total of pay-off functions $\boldsymbol{v}_i$ for $i \in K$. But, such a function would not dispose some useful properties of the characteristic function, and, consequently, its introduction would not result in any substantial advantage.

(5) There are, in [7] (in paragraphs 14 and 51), introduced the notions of the *solution* and *kernel* of a game, for games with characteristic functions. The classes of predictionally stable (strongly or weakly) and predictionally balanced configurations have some (generally not all) properties of the solution and kernel of the game, respectively. There may occur, consequently, the apprehensions, if the introduction of the class $\mathfrak{P}_r$ of all balanced configurations, gives some new information on the game, if the class $\mathfrak{P}_r$ is not always either empty, or identical with the class $\mathfrak{P}_r^*$ of all strongly predictionally stable configurations. It is possible, still, to construct an example of five-players general coalition-game, or seven-players coalition-game without strategies choosing, in which $\mathfrak{P}_r^* \neq \mathfrak{P}_r \neq \emptyset$.

(6) It was already stated in Section 4 that it is possible to introduce even games with an *infinite identification*, i.e. with infinitely many identification sets in identification partitions. The assumption of the finite identification is necessary in the proofs of Theorems 1 and 8, and in the proofs of some their consequences (some statements of Theorem 2. Corollary of Theorem 6, e.t.c.). Nevertheless, many assertions can be, without any remarkable difficulties, extended even for the games with infinite identification.

If we consider a two-players game with the fixed cooperation $\mathscr{K} = \{\{1\}, \{2\}\}$ and with all the identification sets $\boldsymbol{R}_i \in \mathscr{R}_i$, i = 1, 2, given by the formula

$$\boldsymbol{R}_i = \{(\mathscr{K}, \mathbf{s}) : \mathbf{s} = (s_i, s_j),\ s_i \in \boldsymbol{S}_i,\ s_j \textit{ is fixed}\} ,$$

i.e. every player fully identifies the behaviour and intention of his anti-player, then the works [3], [4], [8] imply, that there exists at least one configuration $C = C_{\text{rat}}$. It is significant, as soon as we want to state an analogy of Remark 12.1, or, an analogy of the seccond part of Theorem 8. In the last case, we may replace the assumption of the finite identification (and, consequently, the existence of only finite number of the sets $\boldsymbol{S}_i(\boldsymbol{R}_i)$) in the proof by the fact that, according to [4], the set of vectors

$$\{\mathbf{x} : \mathbf{x} = (x_i)_{i=1,2},\ \textit{there exists}\ C \in \mathscr{C},\ C = C_{\text{rat}}\,,\ \textit{such that}\ x_i = \boldsymbol{x}_i(C)\ \textit{for}\ i = 1, 2\}$$

is finite.

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