KYBERNETIKA — VOLUME 12 (1976), NUMBER 6

On Cooperative Games Connected with Markets

MILAN MAREŠ

The presented paper deals with the model of cooperative equilibrium in markets with transferable utility, as suggested in [8] and [7]. It presents some ideas concerning the connection between such markets and cooperative games with transferable itulity. Namely, it suggests a way of construction of such games which maximally reflect the properties of given market and its cooperative equilibria.

1. INTRODUCTION

This paper is a free continuation of [7] and, especially, [8]. It presents some further properties of the $\mathcal{M}$-equilibrium defined there. The motivation of that equilibrium definition was explained in [8]. Here we are interested in the connection between equilibria and some game-theoretical solutions. These connections were already investigated and interesting results were obtained. They are introduced in [4], [6] and also in [8]. These results, usually, show that the equilibria are stronger than the game theoretical solutions in the sense that the existence of equilibrium implies the existence of some game-theoretical solution. The opposite implication may be obtained for the $\mathcal{M}$-equilibrium under very special assumptions. The main goal of this work is to suggest a new definition of game connected with market, which would better reflect the properties of market. Solution of such game, namely its core, could be more close to cooperative market equilibrium. Hence, the well-known game theoretical results could be better applicable into market theory.

2. FUNDAMENTAL CONCEPTS AND RESULTS

In this section we repeat the basic definitions and the main auxiliary results, introduced and discussed in [8] already.

If m is a positive integer number then R, R_+, R^m and R^m are the sets of real numbers, non-negative real numbers, real-valued m-dimensional vectors and of those vectors with non-negative coordinates, respectively. If $\mathscr{M}$ is a class of sets then we denote

$$(1) \qquad \langle \mathscr{M} \rangle = \{K \in \mathscr{M} : \textit{if } \mathscr{L} \subset \mathscr{M}, \mathscr{L} \textit{ is a partition of } K \textit{ then } \mathscr{L} = \{K\}\} .$$

A *cooperative market* is a quadruple

$$\boldsymbol{m} = (I, R \times R^m_+, (U_i)_{i \in I}, (\boldsymbol{a}^i)_{i \in I}) ,$$

where I is the set of all *players*, vectors $\boldsymbol{x} = (x_0, x_1, \ldots, x_m)$, $x_0 \in R$, $(x_1, \ldots, x_m) \in R^m_+$ represent x_0 units of *money* and $x_1, x_2, \ldots, x_m$ units or *regular goods*; vectors $\boldsymbol{a}^i \in R \times R^m_+$ are the *initial quantities* of money and goods owned by players, and $U_i : R^m_+ \to R$, $i \in I$, are the *utilities* of real goods for players.

We denote, further, the class $\mathscr{K}$ of all non-empty subsets of I, which are caled *coalitions*, and the mappings $u_i : R \times R^m_+ \to R$, $i \in I$, such that

$$u_i(x_0, x_1, \ldots, x_m) = x_0 + U_i(x_1, \ldots, x_m) ,$$

which are called *utility functions* of players. All goods in the market, even money, have their *prices*. We denote them by $\boldsymbol{p} = (p_0, p_1, \ldots, p_m) \in R^{m+1}_+$, where $p_j > 0$, $j = 0, 1, \ldots, m$, and the set of admissible price-vectors is denoted by $\boldsymbol{P}$. We suppose, moreover that vectors $\boldsymbol{p} \in \boldsymbol{P}$ and $\boldsymbol{x} \in R \times R^m_+$ are such that the scalar product $\boldsymbol{px}$ has sense. We denote

$$\begin{aligned}
\boldsymbol{X} &= \{X = (\boldsymbol{x}^i)_{i \in I} : \boldsymbol{x}^i \in R \times R^m_+,\ i \in I\} , \\
\boldsymbol{X}_K &= \{X \in \boldsymbol{X} : \sum_{i \in K} \boldsymbol{x}^i \leqq \sum_{i \in K} \boldsymbol{a}^i\} , \quad K \in \mathscr{K} , \\
\boldsymbol{A}_K(\boldsymbol{p}) &= \{X \in \boldsymbol{X} : \sum_{i \in K} \boldsymbol{px}^i \leqq \sum_{i \in K} \boldsymbol{pa}^i\} , \quad K \in \mathscr{K} , \quad \boldsymbol{p} \in \boldsymbol{P} .
\end{aligned}$$

The following statements were proved in [8] as Lemmas 1, 2, 3 and 4, respectively.

Statement A. Let $K \in \mathscr{K}$, $\boldsymbol{p} \in \boldsymbol{P}$. Then there exist the maxima

$$\max \{ \sum_{i \in K} u_i(\boldsymbol{x}^i) : X = (\boldsymbol{x}^i)_{i \in I} \in \boldsymbol{X}_K\} \quad \text{and} \quad \max \{ \sum_{i \in K} u_i(\boldsymbol{x}^i) : X = (\boldsymbol{x}^i)_{i \in I} \in \boldsymbol{A}_K(\boldsymbol{p})\} .$$

Statement B. If $\boldsymbol{p} \in \boldsymbol{P}$, $K \in \mathscr{K}$, $\mathscr{J} \subset \mathscr{K}$, if $\mathscr{J}$ is a partition of K then for all $X = (\boldsymbol{x}^i)_{i \in I} \in \boldsymbol{A}_K(\boldsymbol{p})$ is

$$(2) \qquad \sum_{i \in K} u_i(\boldsymbol{x}^i) \leqq \sum_{J \in \mathscr{J}} \max \{ \sum_{i \in J} u_i(\boldsymbol{y}^i) : Y = (\boldsymbol{y}^i)_{i \in J} \in \boldsymbol{A}_J(\boldsymbol{p})\} .$$

Statement C. Let $\boldsymbol{p} \in \boldsymbol{P}$, $\mathscr{M} \subset \mathscr{K}$, $\langle \mathscr{M} \rangle$ be a partition of I. Let $X \in \boldsymbol{A}_J(\boldsymbol{p})$ for all $J \in \langle \mathscr{M} \rangle$. Then $X \in \boldsymbol{A}_K(\boldsymbol{p})$ for all $K \in \mathscr{M}$.

Statement D. Let $p \in \boldsymbol{P}$, $\mathcal{M} \subset \mathcal{K}$, $\langle \mathcal{M} \rangle$ be a partition of I, and let $X = (x^i)_{i \in I} \in \mathbf{X}$. If

$$\sum_{i \in J} u_i(x^i) = \max \{ \sum_{i \in J} u_i(y^i) : Y = (y^i)_{i \in I} \in \mathbf{A}_J(p) \}$$

for all $J \in \langle \mathcal{M} \rangle$ then also

$$\sum_{i \in K} u_i(x^i) = \max \{ \sum_{i \in K} u_i(y^i) : Y = (y^i)_{i \in I} \in \mathbf{A}_K(p) \}$$

for all $K \in \mathcal{M}$.

Any pair (X, p), $X \in \mathbf{X}$, $p \in \boldsymbol{P}$ is called a *market state*. If $\mathcal{M} \subset \mathcal{K}$ then the market state (X, p) is called an *$\mathcal{M}$-equilibrium* if $X \in \mathbf{X}_I$, and for any $K \in \mathcal{M}$ is $X \in \mathbf{A}_K(p)$ and

$$\sum_{i \in K} u_i(x^i) = \max \{ \sum_{i \in K} u_i(y^i) : Y = (y^i)_{i \in I} \in \mathbf{A}_K(p) \} .$$

If we denote by $\mathcal{I}$ the class of all one-element coalitions in $\mathcal{K}$, i.e.

$$\mathcal{I} = \{\{i\}\}_{i \in I} \tag{3}$$

then the classical equilibrium, defined e.g., in [4] is identical with $\mathcal{I}$-equilibrum in our terminology, and the following statement, proved in [8] as Theorem 2, is true.

Statement E. Let $\mathcal{M} \subset \mathcal{K}$, $X \in \mathbf{X}$, $p \in \boldsymbol{P}$. If (X, p) is an $\mathcal{I}$-equilibrium then it is also an $\mathcal{M}$-equilibrium.

3. GAME CONNECTED WITH A MARKET

The basic situation in cooperative market and in cooperative game is analogous in the sense that players form their coalitions, correlate their behaviour and distribute the final profit. These analogies may be expressed even in exact form. This section presents two of such expressions – the classical one, and a new one which is in some cases more adequate to the real relations between games and markets.

Statement A (i.e. Lemma 1 from [8]) enables us to define the mapping $v : \mathcal{K} \to R$ in the following way

$$v(K) = \max \{ \sum_{J \in K} u_i(x^i) : X = (x^i)_{i \in I} \in \mathbf{X}_K \} , \quad K \in \mathcal{K} , \tag{4}$$

which has the following property proved in [8] in Lemma 5.

Statement F. The mapping v defined by (4) is superadditive, i.e.

$$v(K \cup L) \geqq v(K) + v(L) \quad \text{for any} \quad K, L \in \mathcal{K} , \; K \cap L = \emptyset .$$

We accept here the game-theoretical terminology, and we call an *imputation* any real-valued vector $\xi = (\xi^i)_{i\in I}$, $\xi^i \in R$, such that there exists $X = (x^i)_{i\in I} \in \mathbf{X}$ for which $\xi^i = u_i(x^i)$, $i \in I$. Further, we introduce the following auxiliary symbols:

$$\Xi = \{\xi = (\xi^i)_{i\in I} : \exists(X = (x^i)_{i\in I} \in \mathbf{X})\, \forall\, (i \in I),\ u_i(x^i) = \xi^i\},$$
$$\Xi_K = \{\xi \in \Xi : \exists(X = (x^i)_{i\in I} \in \mathbf{X}_K)\, \forall (i \in I),\ u_i(x^i) = \xi^i\}.$$

Then the ordered pair $\Gamma_m = (I, v)$ is called a *coalition-game connected with market* $\boldsymbol{m} = (I, R \times R^m_+, (U_i)_{i\in I}, (\boldsymbol{a}^i)_{i\in I})$, and mapping $\boldsymbol{v}$ is called its *characteristic function.*

Lemma 1. Let $\xi = (\xi^i)_{i\in I}$, $\xi^i \in R$. Then

$$\xi \in \Xi_I \Leftrightarrow \sum_{i\in I} \xi^i \leqq v(I).$$

Proof. Let $\xi \in \Xi_I$. Then there exists $X = (x^i)_{i\in I} \in \mathbf{X}_I$ such that $\xi^i = u_i(x^i)$. By (4)

$$\sum_{i\in I} \xi^i = \sum_{i\in I} u_i(x^i) \leqq v(I).$$

On the other hand, let

$$v(I) \geqq \sum_{i\in I} \xi^i.$$

Then there exists, according to Statement A, $Y = (y^i)_{i\in I} \in \mathbf{X}_I$ such that

$$v(I) = \sum_{i\in I} u_i(y^i) = \sum_{i\in I} y^i_0 + \sum_{i\in I} U_i(y^i_1, \ldots, y^i_m).$$

Let us construct $X = (x^i)_{i\in I} \in \mathbf{X}$ such that

$$x^i_0 = y^i_0 + \xi^i - u_i(y^i), \quad i \in I,$$
$$x^i_j = y^i_j, \quad i \in I, \quad j = 1, \ldots, m.$$

Then $X \in \mathbf{X}_I$, as

$$\sum_{i\in I} x^i_0 = \sum_{i\in I} y^i_0 + \sum_{i\in I} \xi^i - v(I) \leqq \sum_{i\in I} y^i_0,$$
$$\sum_{i\in I} x^i_j = \sum_{i\in I} y^i_j, \quad j = 1, \ldots, m,$$

and $Y \in \mathbf{X}_I$. Moreover, for any $i \in I$ is

$$u_i(x^i) = x^i_0 + U_i(x^i_1, \ldots, x^i_m) = y^i_0 + \xi^i - u_i(y^i) + U_i(y^i_1, \ldots, y^i_m) = \xi^i,$$

so that $\xi \in \Xi_I$.

Now, we may formulate the following definition of a game-theoretical solution. We say that vector $\xi = (\xi^i)_{i\in I}$ is *$\mathscr{M}$-stable* in game Γ_m, where $\mathscr{M} \subset \mathscr{K}$, if

$$\xi \in \Xi_I \quad \text{and} \quad \sum_{i\in K} \xi^i \geqq v(K) \quad \text{for all} \quad K \in \mathscr{M} .$$

According to Lemma 1, ξ is $\mathscr{M}$-stable iff

$$\sum_{i\in I} \xi^i \leqq v(I) \quad \text{and} \quad \sum_{i\in K} \xi^i \geqq v(K) , \quad \text{for all} \quad K \in \mathscr{M} .$$

The reasons for presenting the game-theoretical solution in such form were discussed in [8]. Here we note, only, that the $\mathscr{M}$-stability is a generalization of the concept of core, as any $\xi \in \Xi$ is $\mathscr{K}$-stable iff it is an element of core in game Γ_m.

There are some interesting results concerning mutual connections between equilibrium and core, given in literature. Their analogies, formulated in terms of our cooperative market model and game model, were proved in [8], in Theorem 3 and Corollary 4. It simplifies the further references if we repeat here Theorem 3 from [8] as the following statement.

Statement G. Let $\boldsymbol{m}$ be a market and let Γ_m be the coalition-game connected with $\boldsymbol{m}$. Let $\mathscr{M} \subset \mathscr{K}$, $X = (x^i)_{i\in I} \in \mathbf{X}$, $p \in P$, $\xi = (\xi^i)_{i\in I} \in \Xi$, and let $\xi^i = u_i(x^i)$ for all $i \in I$. If (X, p) is an $\mathscr{M}$-equilibrium then the imputation ξ is $\mathscr{M}$-stable.

Theorem 3 and Corollary 4 in [8] are formulated as implications, where the property of $\mathscr{M}$-equilibrium or $\mathscr{I}$-equilibrium, according to (3), for (X, p) implies the $\mathscr{M}$-stability or $\mathscr{K}$-stability of $\xi \in \Xi$ where $\xi^i = u_i(x^i)$, $i \in I$. The form of implication is essential for statements of this type. It may be substituted by equivalence only under very strong assumptions, as it was done in Theorem 5 in [8]. The main problem of the presented work is to find another definition of game connected with a market, which would better reflect the existing analogies, and which would enable us to formulate some analogical statements in stronger form with equivalence instead of implication.

4. MARKET CHARACTERISTIC FUNCTION

The topic of this section is to introduce a modified form of coalition-game connected with a market, and to prove the main results concerning mutual connections between $\mathscr{M}$-stability in such game and $\mathscr{M}$-equilibrium in the considered market. Statement A, i.e. Lemma 1 from [8], enables us to introduce the following mapping $w : \mathscr{K} \times P \to R$ such that

$$w(K, p) = \max \left\{ \sum_{i\in K} u_i(x^i) : X = (x^i)_{i\in I} \in A_K(p) \right\} , \quad K \in \mathscr{K}, p \in P , \tag{5}$$

which we call a *characteristic function of market* $\boldsymbol{m}$.

Remark 1. For $K \in \mathscr{K}$, $p \in P$ is $w(K, p) \geqq v(K)$ as $\mathbf{X}_K \subset \mathbf{A}_K(p)$.

Remark 2. Let $K \in \mathscr{K}$, $p \in P$, $X = (x^i)_{i \in I} \in \mathbf{X}_I \cap \mathbf{A}_K(p)$. Then (X, p) is a $\{K\}$-equilibrium iff

$$\sum_{i \in K} u_i(x^i) = w(K, p),$$

and, according to Statement A, for any $K \in \mathscr{K}$, $p \in P$, there exists $X \in \mathbf{X}$ such that (X, p) is a $\{K\}$-equilibrium.

Mapping $w(., p) : \mathscr{K} \to R$ for fixed $p \in P$ is not generally superadditive, so that it is not convenient for definition of a new coalition-game $\Gamma = (I, w(., p))$ which could describe the cooperation in the considered market. Moreover, the following statement can be proved.

Lemma 2. The mapping $w(., p)$ is subadditive for any $p \in P$, i.e.

$$w(L, p) + w(M, p) \geqq w(L \cup M, p), \quad L, M \in \mathscr{K}, \quad L \cap M = \emptyset.$$

Proof. Lemma follows immediately from Stetement B, i.e. from Lemma 2, [8]; if we put $K = L \cup M$ and $\mathscr{J} = \{L, M\}$. Then (2) represents the desired inequality.

Lemma 3. Let $X = (x^i)_{i \in I} \in \mathbf{X}_I$, $p \in P$, $\mathscr{M} \subset \mathscr{K}$, let $\mathscr{M}$ be a partition of I, and let

$$\sum_{i \in K} u_i(x^i) \geqq w(K, p) \quad \text{for all} \quad K \in \mathscr{M}.$$

Then $X \in \mathbf{A}_K(p)$ for all $K \in \mathscr{M}$.

Proof. Let us suppose that there exists $L \in \mathscr{M}$ such that $X \notin \mathbf{A}_L(p)$, i.e.

$$\sum_{i \in L} p x^i > \sum_{i \in L} p a^i. \tag{6}$$

As $X \in \mathbf{X}_I$, the inequality

$$\sum_{i \in I} p x^i \leqq \sum_{i \in J} p a^i$$

holds, and there exists $J \in \mathscr{M}$ such that

$$\sum_{i \in J} p x^i < \sum_{i \in J} p a^i.$$

Let us construct $Y = (y^i)_{i \in I} \in \mathbf{X}$ such that

$$y_0^i = x_0^i + (p a^i - p x^i)/p_0,$$
$$y_j^i = x_j^i, \quad j = 1, \ldots, m, \quad i \in J.$$

This $\boldsymbol{Y}$ belongs to $\mathbf{A}_J(\boldsymbol{p})$, as

$$\sum_{i \in J} \boldsymbol{p}\boldsymbol{y}^i = \sum_{i \in J} \boldsymbol{p}\boldsymbol{a}^i ,$$

and

$$\sum_{i \in J} u_i(\boldsymbol{y}^i) = \sum_{i \in J} u_i(\boldsymbol{x}^i) + \frac{1}{p_0} \left(\sum_{i \in J} \boldsymbol{p}\boldsymbol{a}^i - \sum_{i \in J} \boldsymbol{p}\boldsymbol{x}^i \right) > \sum_{i \in J} u_i(\boldsymbol{x}^i) = w(J, \boldsymbol{p}) ,$$

which is a contradiction with (5). Consequently, (6) can not be true for any $L \in \mathcal{M}$ and $X \in \mathbf{A}_K(\boldsymbol{p})$ for all $K \in \mathcal{M}$.

Remark 3. If $\mathcal{M} \subset \mathcal{K}$ and $\langle \mathcal{M} \rangle$ is defined by (1) then there is no set of coalitions $K_1, \ldots, K_t \in \langle \mathcal{M} \rangle$, such that $K_r \cap K_s = \emptyset, r \neq s, r, s = 1, \ldots, t, K_1 \cup \ldots \cup K_t \in \langle \mathcal{M} \rangle$. It means that any real-valued set function defined on $\langle \mathcal{M} \rangle$ may be considered to be superadditive on $\langle \mathcal{M} \rangle$.

The last Remark implies that the triple $\Gamma_m(\boldsymbol{p}) = (I, \langle \mathcal{M} \rangle, w(., \boldsymbol{p}))$ forms a coalition-game in usual sense, e.g. in the form used in [2] and in other papers. The set I is set of players, $\langle \mathcal{M} \rangle$ is the class of admissible coalitions and $w(., \boldsymbol{p})$, where $\boldsymbol{p} \in \boldsymbol{P}$ is fixed, is the characteristic function of the game $\Gamma_m(\boldsymbol{p})$. The core of that game is identical with the class of all $\langle \mathcal{M} \rangle$-stable real-valued vectors $\xi = (\xi^i)_{i \in I} \in \Xi$, where Ξ is the set of imputations in the game $\Gamma_m(\boldsymbol{p})$. For the game defined in such way we can prove the following statement.

Theorem 1. Let $X = (\boldsymbol{x}^i)_{i \in I} \in \boldsymbol{X}_I$, $\boldsymbol{p} \in \boldsymbol{P}$, $\mathcal{M} \subset \mathcal{K}$, let $\langle \mathcal{M} \rangle$ be a partition of I, and let $\xi = (\xi^i)_{i \in I} \in \Xi_I$ be such that $\xi^i = u_i(\boldsymbol{x}^i)$ for $i \in I$. Then $(X, \boldsymbol{p})$ is an $\mathcal{M}$-equilibrium if and only if ξ is $\langle \mathcal{M} \rangle$-stable in the game $\Gamma_m(\boldsymbol{p})$, i.e. iff ξ is an element in the core of $\Gamma_m(\boldsymbol{p})$.

Proof. If $(X, \boldsymbol{p})$ is an $\mathcal{M}$-equilibrium then

$$\sum_{i \in K} \xi^i \geqq w(K, \boldsymbol{p}) \tag{7}$$

for all $K \in \mathcal{M}$ and, consequently, ξ is $\langle \mathcal{M} \rangle$-stable in $\Gamma_m(\boldsymbol{p})$, as $\langle \mathcal{M} \rangle \subset \mathcal{M}$. On the other hand, let (7) be true for all $K \in \langle \mathcal{M} \rangle$. Then, according to Lemma 3, $X \in \mathbf{A}_K(\boldsymbol{p})$ for all $K \in \langle \mathcal{M} \rangle$. As $X \in \boldsymbol{X}_I$, $(X, \boldsymbol{p})$ is an $\langle \mathcal{M} \rangle$-equilibrium. Lemmas 3 and 4 from [8], i.e. Stetements C and D from Section 2, imply that it is also an $\mathcal{M}$-equilibrium.

Corollary 1. Under assumptions of Theorem 1, if ξ is $\langle \mathcal{M} \rangle$-stable in game $\Gamma_m(\boldsymbol{p})$ then ξ is $\mathcal{M}$-stable in Γ_m, as follows from Theorem 1 and from Statement G, i.e. from Theorem 3 in [8].

Theorem 2. Let $p \in \boldsymbol{P}$, $\mathcal{M} \subset \mathcal{K}$, and let $\langle \mathcal{M} \rangle$ be a partition of I. Then there exists $X \in \boldsymbol{X}$ such that (X, p) is an $\mathcal{M}$-equilibrium if and only if

$$\sum_{K \in \langle \mathcal{M} \rangle} w(K, p) = w(I, p) .$$

Proof. If (X, p) is an $\mathcal{M}$-equilibrium then for all $K \in \langle \mathcal{M} \rangle$

$$\sum_{i \in K} \xi^i \geqq w(K, p) ,$$

it means that

$$\sum_{i \in I} \xi^i = \sum_{K \in \langle \mathcal{M} \rangle} \sum_{i \in K} \xi^i \geqq \sum_{K \in \langle \mathcal{M} \rangle} w(K, p)$$

and

$$\sum_{i \in I} \xi^i \leqq v(I) \leqq w(I, p) ,$$

as $X \in \boldsymbol{X}_I$ and Remark 1 holds. Consequently, according to Lemma 2, the desired equality holds. Let, now.

$$w(I, p) < \sum_{K \in \langle \mathcal{M} \rangle} w(K, p) , \tag{8}$$

and let there exists $X \in \boldsymbol{X}_I$, $X = (x^i)_{i \in I}$, such that

$$\sum_{i \in I} u_i(x^i) = \sum_{i \in K} \xi^i \geqq w(K, p) ,$$

for all $K \in \mathcal{M}$, where $\xi^i = u^i(x^i)$, $i \in I$. Then

$$\sum_{i \in I} \xi^i \geqq \sum_{K \in \langle \mathcal{M} \rangle} w(K, p) > w(I, p) \geqq v(I) ,$$

according to Remark 1. It follows from Lemma 1 that this result contradicts to the assumption $X \in \boldsymbol{X}_I$. So, (X, p) can not be an $\mathcal{M}$-equilibrium for any $X \in \boldsymbol{X}$, if (8) is fulfilled.

Corollary 2. Under the assumptions of Theorem 2, if $\mathcal{I} \subset \mathcal{M}$, then $\mathcal{I} = \langle \mathcal{M} \rangle$, and there exists $X \in \boldsymbol{X}$ such that (X, p) is an $\mathcal{M}$-equilibrium if and only if

$$\sum_{i \in I} w(\{i\}, p) = w(I, p) .$$

Corollary 3. Let $p \in \boldsymbol{P}$, $\mathcal{M} \subset \mathcal{K}$, let $\mathcal{L} \subset \mathcal{M}$ be a partition of I. If there exists $X \in \boldsymbol{X}$ such that (X, p) is an $\mathcal{M}$-equilibrium then

$$\sum_{L \in \mathcal{L}} w(L, p) = w(I, p) .$$

Theorem 4. Let $p \in P$, $\mathscr{M} \subset \mathscr{K}$ and let $\langle \mathscr{M} \rangle$ be a partition of I. Then there exists $X \in \mathbf{X}$ such that (X, p) is an $\mathscr{M}$-equilibrium if and only if the mapping $w(., p)$ is additive on the class of coalitions $\mathscr{M} \cup \{I\}$.

Proof. According to Theorem 3, it is sufficient to prove that

$$\sum_{K \in \langle \mathscr{M} \rangle} w(K, p) = w(I, p) \tag{9}$$

if and only if $w(., p)$ is additive on the class $\mathscr{M} \cup \{I\}$. According to the assumption that $\langle \mathscr{M} \rangle$ is a partition of I, Lemma 2 implies that for any $K \in \mathscr{M}$ is

$$\sum_{J \in \langle \mathscr{M} \rangle} w(J, p) \geqq \sum_{J \in \langle \mathscr{M} \rangle, J \subset I - K} w(J, p) + w(K, p) \geqq w(I, p),$$

and (9) implies that

$$w(L, p) = \sum_{J \in \langle \mathscr{M} \rangle,\ J \subset L} w(J, p)$$

for all $L \in \mathscr{M}$. This equation means the additivity of $w(\cdot, p)$ on $\mathscr{M} \cup \{I\}$. On the other hand, if

$$w(K, p) + w(L, p) = w(K \cup L, p)$$

for all K, L, $K \cup L \in \mathscr{M} \cup \{I\}$ then also

$$\sum_{K \in \mathscr{L}} w(K, p) = w(I, p)$$

for all $\mathscr{L} \subset \mathscr{M}$ such that $\mathscr{L}$ is a partition of I, and, consequently (9) holds.

5. ADDITIVE CHARACTERISTIC FUNCTIONS

The last statements of the previous section show the importance of additivity of $w(\cdot, p)$ for existence of equilibrium. Also Theorem 4 in [8] concerns the same subject and some other results dealing with that topic are introduced in this conclusive section of the presented paper. They all represent auxiliary tools for solving some special problems connected with the existence of $\mathscr{M}$-equilibria in cooperative markets.

Lemma 4. If $p \in P$ then the mappings $v(.)$ and $w(., p)$ are equal on $\mathscr{K}$ if and only if they are equal on $\mathscr{I}$. In such a case both $v(.)$ and $w(., p)$ are additive, i.e.

$$v(K \cup L) = v(K) + v(L), \quad w(K \cup L, p) = w(K, p) + w(L, p),$$

$$K, L \in \mathscr{K}, K \cap L = \emptyset.$$

Proof. If $v(\{i\}) = w(\{i\}, p)$ for all $\{i\} \in \mathscr{I}$ then Lemma 5 from [8] (i.e. Statement F in Section 3) and Lemma 2 from Section 4 imply for any $K \in \mathscr{K}$

$$w(K, p) \leqq \sum_{i \in K} w(\{i\}, p) = \sum_{i \in K} v(\{i\}) \leqq v(K) .$$

On the other hand, $\mathbf{A}_K(p) \supset \mathbf{X}_K$ for any $K \in \mathscr{K}$, $p \in \mathbf{P}$, so that $w(K, p) \geqq v(K)$, and the equation is proved. The opposite implication follows from $\mathscr{I} \subset \mathscr{K}$ immediately. As $v(.)$ is superadditive and $w(., p)$ is subadditive, and as they are equal, they must be additive.

Theorem 5. Let $p \in \mathbf{P}$, $\mathscr{M} \subset \mathscr{K}$, let $\mathscr{M}$ be a partition of I, and let the mapping $w(., p)$ be additive. If there exists $X \in \mathbf{X}$ such that (X, p) is an $\mathscr{M}$-equilibrium then there exists $Y \in \mathbf{X}$ such that (Y, p) is an $\mathscr{N}$-equilibrium for all $\mathscr{N} \subset \mathscr{K}$.

Proof. Let us consider $X = (x^i)_{i \in I} \in \mathbf{X}_I$ such that (X, p) is an $\mathscr{M}$-equilibrium. It means that for all $K \in \mathscr{M}$ is $X \in \mathbf{A}_K(p)$, and

$$\sum_{i \in K} u_i(x^i) = w(K, p) . \tag{10}$$

The additivity of $w(\cdot, p)$ and relation (10) imply

$$\sum_{i \in K} u_i(x^i) = \sum_{i \in K} w(\{i\}, p) \quad \text{for all} \quad K \in \mathscr{M} ,$$

and, as $\mathscr{M}$ is a partition of I,

$$\sum_{i \in I} u_i(x^i) = \sum_{i \in I} w(\{i\}, p) .$$

Let $Y = (y^i)_{i \in I} \in \mathbf{X}$, be such that for all $i \in I$

$$y_0^i = x_0^i + w(\{i\}, p) - u_i(x^i) ,$$
$$y_j^i = x_j^i , \quad j = 1, \ldots, m .$$

This Y is an element of $\mathbf{X}_I$, as

$$\sum_{i \in I} y^i = \sum_{i \in I} x^i \leqq \sum_{i \in I} a^i ,$$

and

$$u_i(y^i) = w(\{i\}, p) \quad \text{for all} \quad i \in I . \tag{11}$$

According to Lemma 3, $Y \in \mathbf{A}_{\{i\}}(p)$ for all $i \in I$. This fact, together with (11), implies that (Y, p) is an $\mathscr{I}$-equilibrium and, in accordance with Theorem 2 from [8] (i.e. Statement E in Section 2), (Y, p) is an $\mathscr{N}$-equilibrium for $\mathscr{N} \subset \mathscr{K}$.

Theorem 6. If $p \in P$, and if the mapping $w(., p)$ is additive then there always exists $X \in \mathbf{X}$ such that (X, p) is an $\mathscr{M}$-equilibrium for any $\mathscr{M} \subset \mathscr{K}$.

Proof. According to Remark 2 and according to definition of $\mathscr{M}$-equilibrium, there exists $Y \in \mathbf{X}$ such that (Y, p) is an $\{I\}$-equilibrium. As $\{I\}$ is also a partition of I, Theorem 5 may be applied. Consequently, there exists $X \in \mathbf{X}$ such that (X, p) is an $\mathscr{M}$-equilibrium for any $\mathscr{M} \subset \mathscr{K}$.

Corollary 4. If $p \in P$ and if $v(\{i\}) = w(\{i\}, p)$ for all $i \in I$ then there always exists $X \in \mathbf{X}$ such that (X, p) is an $\mathscr{M}$-equlibrium for all $\mathscr{M} \subset \mathscr{K}$.

(Recived May 3, 1976.)

REFERENCES

[1] R. J. Aumann, M. Maschler: The Bargaining Set for Cooperative Games. Annals of Math. Studies, No 52 (1964), 443–476.

[2] M. Mareš: Stability of Coalition Structures and Imputations in Coalition-Games. Kybernetika *10* (1974), 6, 461–490.

[3] J. von Neumann, O. Morgenstern: Theory of Games and Economic Behaviour. Princeton Univ. Press, Princeton 1944.

[4] J. Rosenmüller: Kooperative Spiele und Märkte. Springer Verlag, Heidelberg—Berlin—New York 1971.

[5] H. Scarf: The Core of an *n*-person Game. Econometrica *35* (1967), No. 1.

[6] L. S. Shapley, M. Shubik: On Market Games. Journal of Econ. Theory, Vol. *1*, No 1, June 1969.

[7] M. Mareš: Cooperative Equilibrium in a Market with Limited Cooperation. Transactions of the 7th Prague Conference on Information Theory etc., 1974. In print.

[8] M. Mareš: Suggestion of a Cooperative Market Model. Kybernetika *12* (1976), 6, 439–450.

[9] W. Hildenbrand: Core and Equilibria of a Large Economy. Princeton Univ. Press, Princeton 1974.

RNDr Milan Mareš, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 180 76 Praha 8. Czechoslovakia.