On One Method of Analysis of Linear Systems with Random Stationary Coefficients

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The present paper deals with linear continuous systems with random stationary input whose coefficients are loaded with errors that are weakly stationary limited processes with arbitrary spectral densities. The paper gives an iterative method of an approximative calculation of the characteristics of the output process. Specially a "linear algebraic" algorithmus for computing output mean square value is given. The solution is made by the method of iteration on frequency domain [1, 2].

Let $(A, \mathscr{A}, P)$ be a probability space where A is a set of elementary events (that will be denoted α), $\mathscr{A}$ is a σ-algebra of subsets of A, P is a probability measure defined on $\mathscr{A}$.

Let the linear system $\mathscr{S}$ be described by the equation

$$\dot{\mathbf{y}}(t, \alpha) = \mathbf{A}\, \mathbf{y}(t, \alpha) + \mathbf{C}\, \mathbf{x}(t, \alpha) \tag{1}$$

where $\mathbf{A}$, $\mathbf{C}$ are constant $n \times n$ matrices, $\mathbf{y}$, $\mathbf{x}$ are $n \times 1$ vectors, $\mathbf{x}(t, \alpha)$ is a weakly stationary process with $\mathsf{M}\, \mathbf{x}(t, \alpha) = 0$, $\mathbf{K}_x(\tau) = \mathsf{M}[\mathbf{x}(t, \alpha)\, \mathbf{x}^*(t + \tau, \alpha)] = \mathbf{E}\, \delta(\tau)$; here M denotes the operation of the expectation, $\mathbf{E}$ is the unit matrix, $\delta(\tau)$ is the Dirac function. We shall call $\mathscr{S}$ the nominal system.

Let us further consider a "random" system $\mathscr{S}_r$ described by the equation (2) and by the following presumptions.

$$\dot{\mathbf{y}}(t, \alpha) = (\mathbf{A} + \mathbf{B}(t, \alpha))\, \mathbf{y}(t, \alpha) + \mathbf{C}\, \mathbf{x}(t, \alpha) \tag{2}$$

where

(2a) the elements $b_{ij}(t, \alpha)$ of $B(t, \alpha)$ are mutually uncorrelated weakly stationary processes independent of $\mathbf{x}(t, \alpha)$ and $P(\|\mathbf{B}(t, \alpha)\| < \delta) = 1$; $\|\cdot\|$ is Euclid norm;

(2b) $\mathbf{B}(t, \alpha) = \int e^{i\gamma t}\, d\mathbf{Z}(\gamma, \alpha)$ is the spectral decomposition of the matrix process $\mathbf{B}(t, \alpha)$;

(2c) $\mathbf{dZ}(\gamma, \alpha) = (\mathrm{d}z_{ij}(\gamma, \alpha))$,

$$\mathsf{M}(\mathrm{d}z_{ij}(\gamma, \alpha)\, \mathrm{d}z^*_{rs}(\gamma, \alpha)) = \frac{1}{2\pi} s_{ij}(\mathrm{i}\gamma)\, \mathrm{d}\gamma = \frac{1}{2\pi \mathrm{i}} s_{ij}(q)\, \mathrm{d}q$$

$(q = \mathrm{i}\gamma)$ for $(ij) = (rs)$; $\mathsf{M}(\mathrm{d}z_{ij}(\gamma, \alpha)\, \mathrm{d}z^*_{rs}(\gamma, \alpha)) = 0$ for $(ij) \neq (rs)$;

(2d) $\mathsf{M}(\mathrm{d}\mathbf{Z}(\gamma_1, \alpha)\, \mathrm{d}\mathbf{Z}^*(\gamma_2, \alpha)) = 0$ for $\gamma_1 \neq \gamma_2$;

(2e) $\mathsf{M}(\mathrm{d}\mathbf{Z}(\gamma, \alpha)) = 0$, i.e. $\mathsf{M}(\mathbf{B}(t, \alpha)) = 0$;

(2f) Let $\mathscr{S}$ be stable (i.e. all eigenvalues of $\mathbf{A}$ lie in the left halfplane); let $\Phi(t, u, \alpha)$ be the solution of (2) with $\mathbf{x}(t, \alpha) \equiv 0$, $\Phi(u, u, \alpha) = \mathbf{E}$, $\mathbf{W}(t, \tau, \alpha) = \Phi(t, t - \tau, \alpha)$. Let $\mathbf{L}_{1,2}(-\infty, \infty; K)$ be the set of $\mathbf{W}_1(t, \tau, \alpha)$ for which

1. $P(\int_0^\infty \|\mathbf{W}_1(t, \tau, \alpha)\|\, \mathrm{d}\tau < K) = 1$ (i.e. $\mathscr{S}_r$ is stable in the sense bounded input – bounded output [3]),

2. $P(\|\int_0^\infty \mathbf{W}_1(t, \tau, \alpha)\, \mathbf{W}^*_1(t, \tau, \alpha)\, \mathrm{d}\tau\| < K) = 1$; let $\mathbf{W}(t, \tau, \alpha) \in \mathbf{L}_{1,2}(-\infty, \infty; K)$.

The random weighting function $\mathbf{W}(t, \tau, \alpha)$ of the system $\mathscr{S}_r$ fulfils the adjoint equation

$$\frac{\partial}{\partial \tau} \mathbf{W}(t, \tau, \alpha) = \mathbf{W}(t, \tau, \alpha)\, [\mathbf{A} + \mathbf{B}(t - \tau, \alpha)]\,, \quad \mathbf{W}(t, 0, \alpha) = \mathbf{E}\,.$$

For the random transfer function

$$\mathbf{Y}(t, p, \alpha) = \int_0^\infty \mathbf{W}(t, \tau, \alpha)\, \mathrm{e}^{-p\tau}\, \mathrm{d}\tau$$

we get

$$\mathbf{Y}(t, p, \alpha)\,(p\mathbf{E} - \mathbf{A}) = \int_0^\infty \mathbf{W}(t, \tau, \alpha)\, \mathbf{B}(t - \tau, \alpha)\, \mathrm{e}^{-p\tau}\, \mathrm{d}\tau + \mathbf{E}\,,$$

from (2b) follows

$$\mathbf{Y}(t, p, \alpha)\,(p\mathbf{E} - \mathbf{A}) = \int \mathbf{Y}(t, p + \mathrm{i}\gamma, \alpha)\, \mathrm{e}^{\mathrm{i}\gamma t}\, \mathrm{d}\mathbf{Z}(\gamma, \alpha) + \mathbf{E}\,,$$

(3) $$\mathbf{Y}(t, p, \alpha)\,(p\mathbf{E} - \mathbf{A}) = \int \mathbf{D}^\gamma(\mathbf{Y}(t, p, \alpha)\, \mathrm{e}^{\mathrm{i}\gamma t}\, \mathrm{d}\mathbf{Z}(\gamma, \alpha) + \mathbf{E}\,,$$

where the operator $\mathbf{D}$ is described by the relation

$$\mathbf{D}(\mathbf{Y}(t, p, \alpha)) = \mathbf{Y}(t, p + \mathrm{i}, \alpha)\,.$$

Let us transcribe the equation (3)

$$\mathbf{Y}(t, p, \alpha) = (p\mathbf{E} - \mathbf{A})^{-1} + \int \mathbf{D}^\gamma(\mathbf{Y}(t, p, \alpha))\, \mathrm{e}^{\mathrm{i}\gamma t}\, \mathrm{d}\mathbf{Z}(\gamma, \alpha)\, (p\mathbf{E} - \mathbf{A})^{-1}\,,$$

or

(4) $$\mathbf{Y}(t, p, \alpha) = \mathbf{Y}_1(t, p, \alpha) + \mathbf{Y}(t, p, \alpha)\,\mathbf{H}\,,$$

where $\mathbf{Y}_1(t, p, \alpha) = \mathbf{Y}_1(p) = (p\mathbf{E} - \mathbf{A})^{-1}$ = nominal transfer function, the operator $\mathbf{H}$ is given by

$$\mathbf{H} = \int \mathbf{D}^{\gamma}(\cdot)\, e^{i\gamma t}\, d\mathbf{Z}(\gamma, \alpha)\,(p\mathbf{E} - \mathbf{A})^{-1}\,.$$

Let further $(\Omega \times A, \mathscr{B} \times \mathscr{A}, \mu \times P)$ represent measure space where Ω is the real axis, $\mathscr{B}$ is the system of Borel sets on Ω, μ is Lebesgue measure on $\mathscr{B}$. Let us denote the elements of Ω by ω, $p = i\omega$. Let $\mathscr{F}L_{1,2}(-\infty, \infty; K)$ be the set of

$$\mathscr{F}(\mathbf{f}(t, \tau, \alpha)),\ \mathbf{f} \in \mathbf{L}_{1,2}(-\infty, \infty; K),\ \mathscr{F}(\mathbf{f}(t, \tau, \alpha)) = \int_0^\infty \mathbf{f}(t, \tau, \alpha)\, e^{-i\omega\tau}\, d\tau =$$
$$= \mathbf{F}(t, i\omega, \alpha) = \mathbf{F}(t, p, \alpha)\,.$$

Let us introduce the "scalar product" and the norm (see app.) by

$$(\mathbf{f}_1, \mathbf{f}_2) = \mathsf{M} \int_0^\infty \mathbf{f}_1(t, \tau, \alpha)\, \mathbf{f}_2^*(t, \tau, \alpha)\, d\tau =$$
$$= (\mathbf{F}_1, \mathbf{F}_2) = \frac{1}{2\pi i} \int_{\Omega \times A} \mathbf{F}_1(t, p, \alpha)\, \mathbf{F}_2^*(t, p, \alpha)\, d(\mu \times P)\,,$$

$\|\mathbf{f}\| = \|\mathbf{F}\| = \sqrt{(\|(\mathbf{F}, \mathbf{F})\|)}$ (evidently a matrix analogy of the Plancherel theorem holds).

The operator $\mathbf{H}$ maps the set $\mathscr{F}\mathbf{L}_{1,2}(-\infty, \infty; \infty)$ into itself. Let us estimate its norm (see app. lemma C). For every $\mathbf{F} \in \mathscr{F}\mathbf{L}_{1,2}(-\infty, \infty; \infty)$ we get

$$\|\mathbf{FH}\|^2 = \left\| \frac{1}{2\pi i} \int_{\Omega \times A} \int_\gamma \int_\nu \mathbf{F}(t, p + i\gamma, \alpha)\, e^{i\gamma t}\, d\mathbf{Z}(\gamma, \alpha)\,(p\mathbf{E} - \mathbf{A})^{-1}\,.\right.$$
$$\left. .\,(p\mathbf{E} - \mathbf{A})^{-1*}\, d\mathbf{Z}^*(\nu, \alpha)\, e^{-i\nu t}\, \mathbf{F}^*(t, p + i\nu, \alpha)\, d(\mu \times P)\right\| \leqq$$
$$\leqq m^2 \left\| \frac{1}{2\pi i} \int_{\Omega \times A} \int_\gamma \mathbf{F}(t, p + i\gamma, \alpha)\, e^{i\gamma t}\, d\mathbf{Z}(\gamma, \alpha) \left(\int_\gamma \mathbf{F}(t, p + i\gamma, \alpha)\, e^{i\gamma t}\, d\mathbf{Z}(\gamma, \alpha) \right)^*\,.\right.$$
$$\left. .\, d(\mu \times P) \right\| = m^2 \left\| \int_A \int_{-\infty}^\infty \mathbf{f}(t, \tau, \alpha)\, \mathbf{B}(t - \tau, \alpha)\, \mathbf{B}^*(t - \tau, \alpha)\, \mathbf{f}^*(t, \tau, \alpha)\, dP\, d\tau \right\| \leqq$$
$$\leqq m^2 \delta^2 \left\| \int_A \int_{-\infty}^\infty \mathbf{f}(t, \tau, \alpha)\, \mathbf{f}^*(t, \tau, \alpha)\, d\tau\, dP \right\| =$$
$$= m^2 \delta^2 \left\| \frac{1}{2\pi i} \int_{\Omega \times A} \mathbf{F}(t, p, \alpha)\, \mathbf{F}^*(t, p, \alpha)\, d(\mu \times P) \right\| = m^2 \delta^2 \|\mathbf{F}\|^2\,;$$

here $m^2 = \max_{p \in Im} \|(p\mathbf{E} - \mathbf{A})^{-1} (p\mathbf{E} - \mathbf{A})^{-1*}\|$, hence

$$\|\mathbf{H}\| \leqq m\delta = \beta .$$

Considering (2f) the solution $\mathbf{Y}(t, p, \alpha)$ of (4) exists with $P = 1$; if $\beta < 1$ then (4) may be formally solved by iteration (see Banach fixed point theorem):

$$\mathbf{Y}_{m+1}(t, p, \alpha) = \mathbf{Y}_1(t, p, \alpha) + \mathbf{Y}_m(t, p, \alpha)\,\mathbf{H}\,, \quad \mathbf{Y}_0(t, p, \alpha) = 0$$

$$\lim_{m \to \infty} \mathbf{Y}_m(t, p, \alpha) = \mathbf{Y}(t, p, \alpha)\,;$$

it follows

(5) $$\mathbf{Y}(t, p, \alpha) = \mathbf{Y}_1(p) + \mathbf{Y}_1(p)\,\mathbf{H} + \mathbf{Y}_1(p)\,\mathbf{H}^2 + \ldots$$

If $E_m(t, p, \alpha) = \mathbf{Y}_m(t, p, \alpha) - \mathbf{Y}(t, p, \alpha)$ then

$$\|\mathbf{E}_m\| \leqq \frac{\beta}{1 - \beta} \|\mathbf{Y}_1 \mathbf{H}^{m-1}\| = \eta_m(t) .$$

Let us further examine the correlation function $\mathbf{K}_y(t_1, t_2) = \mathsf{M}[\mathbf{y}(t_1)\,\mathbf{y}^*(t_2)]$ of the output process of the system $\mathscr{S}_r$.

We have (considering (2a))

$$\mathsf{M}[\mathbf{y}(t_1)\,\mathbf{y}^*(t_2) \mid \mathbf{Y}(t, p, \alpha) = \mathbf{Y}(t, p)] =$$
$$= \frac{1}{2\pi i} \int_{Im} \mathbf{Y}(t_1, p)\,\mathbf{C}\mathbf{C}^*\mathbf{Y}^*(t_2, p)\,e^{p(t_1 - t_2)}\,dp$$

and thus ($\mathbf{Y}(t, p, \alpha)$ being evidently a random variable for fixed t, p)

$$\mathbf{K}_y(t_1, t_2) = \mathsf{M}[\mathbf{y}(t_1), \mathbf{y}^*(t_2)] =$$
$$= \mathsf{M}\left[\frac{1}{2\pi i} \int_{Im} \mathbf{Y}(t_1, p, \alpha)\,\mathbf{C}\mathbf{C}^*\mathbf{Y}^*(t_2, p, \alpha)\,e^{p(t_1 - t_2)}\,dp\right] =$$
$$= (\mathbf{Y}(t_1)\,\mathbf{C}e^{pt_1}, \mathbf{Y}(t_2)\,\mathbf{C}e^{pt_2}) .$$

Specially $\mathbf{K}_y(t, t) = (\mathbf{Y}(t)\,\mathbf{C}, \mathbf{Y}(t)\,\mathbf{C}) = \|\mathbf{Y}(t)\,\mathbf{C}\|^2 = \sigma_y^2(t)$ (here $\mathbf{Y}(t)$ represents the function given by $(p, \alpha) \mapsto \mathbf{Y}(t, p, \alpha)$).

$\mathbf{Y}_1(p)$ or $\mathbf{Y}_2(t, p, \alpha) = \mathbf{Y}_1(p) + \mathbf{Y}_1(p)\,\mathbf{H}$ may be practically used as an approximation for $\mathbf{Y}(t, p, \alpha)$. Let us denote the corresponding correlation functions by ${}_1\mathbf{K}_y(t_1, t_2)$, ${}_2\mathbf{K}_y(t_1, t_2)$ resp. Similarly we write ${}_1\sigma_y(t)$, ${}_2\sigma_y(t)$. Then referring to (2e) ($t_1 - t_2 = \tau$)

$${}_2\mathbf{K}_y(t_1, t_2) = {}_2\mathbf{K}_y(\tau) = (\mathbf{Y}_2(t_1)\,\mathbf{C}e^{pt_1}, \mathbf{Y}_2(t_2)\,\mathbf{C}e^{pt_2}) =$$
$$= (\mathbf{Y}_1\mathbf{C}e^{pt_1}, \mathbf{Y}_1\mathbf{C}e^{pt_2}) + (\mathbf{Y}_1\mathbf{H}e^{pt_1}\mathbf{C}, \mathbf{Y}_1\mathbf{H}e^{pt_2}\mathbf{C}) = {}_1\mathbf{K}_y(\tau) + \mathbf{R}(\tau) ,$$

where

$$\mathbf{R}(\tau) = (\mathbf{Y}_1 \mathbf{H} e^{pt_1} \mathbf{C}, \mathbf{Y}_1 \mathbf{H} e^{pt_2} \mathbf{C}) =$$

$$= \frac{1}{2\pi i} \int_{\Omega \times A} d(\mu \times P) \int_\gamma \int_\nu \mathbf{Y}_1(p + i\gamma) e^{i\gamma t1} d\mathbf{Z}(\gamma, \alpha) e^{pt_1} (p\mathbf{E} - \mathbf{A})^{-1} \mathbf{C} \, .$$

$$. \, \mathbf{C}^*(p\mathbf{E} - \mathbf{A})^{-1*} e^{-pt_2} d\mathbf{Z}^*(\nu, \alpha) e^{-i\nu t_2} \mathbf{Y}_1^*(p + i\nu) =$$

$$= \frac{1}{2\pi i} \int_{Im} e^{p\tau} dp \frac{1}{2\pi i} \int_{Im} ((p + q) \mathbf{E} - \mathbf{A})^{-1} \mathbf{G}(p, q) ((p + q) \mathbf{E} - \mathbf{A})^{-1*} dq \, ,$$

where

$$(6) \quad \mathbf{G}(p, q) = \begin{bmatrix} \sum_k s_{1k}(q) v_{kk}(p) & & \\ & \sum_k s_{2k}(q) v_{kk}(p) & 0 \\ & 0 & \sum_k s_{nk}(q) v_{kk}(p) \end{bmatrix}$$

$$(7) \qquad (v_{ij}(p)) = (p\mathbf{E} - \mathbf{A})^{-1} \mathbf{C}\mathbf{C}^*(p\mathbf{E} - \mathbf{A})^{-1*} \, .$$

The error of the estimate of the correlation function for $j = 1, 2$ is

$$\|\varepsilon_j(t_1, t_2)\| = \|{}_j\mathbf{K}_y(t_1, t_2) - \mathbf{K}_y(t_1, t_2)\| =$$

$$= \|(\mathbf{Y}_j(t_1) e^{pt_1} \mathbf{C}, \mathbf{Y}_j(t_2) e^{pt_2} \mathbf{C}) - (\mathbf{Y}(t_1) e^{pt_1} \mathbf{C}, \mathbf{Y}(t_2) e^{pt_2} \mathbf{C})\| =$$

$$= \|(\mathbf{Y}_j(t_1) e^{pt_1} \mathbf{C}, \mathbf{Y}_j(t_2) e^{pt_2} \mathbf{C}) - ([\mathbf{Y}_j(t_1) - \mathbf{E}_j(t_1)] e^{pt_1} \mathbf{C}, [\mathbf{Y}_j(t_2) - \mathbf{E}_j(t_2)] e^{pt_2} \mathbf{C})\| =$$

$$= \|(\mathbf{Y}_j(t_1) e^{pt_1} \mathbf{C}, \mathbf{E}_j(t_2) e^{pt_2} \mathbf{C}) + (\mathbf{E}_j(t_1) e^{pt_1} \mathbf{C}, \mathbf{Y}_j(t_2) e^{pt_2} \mathbf{C}) -$$

$$- (\mathbf{E}_j(t_1) e^{pt_1} \mathbf{C}, \mathbf{E}_j(t_2) e^{pt_2} \mathbf{C})\| \leqq 2 \, {}_j\sigma_y \eta_j \|\mathbf{C}\| + \eta_j^2 \|\mathbf{C}\|^2 \, ,$$

(${}_j\sigma_y(t)$, $\eta_j(t)$ are evidently independent of t for $j = 1, 2$).

Specially

$${}_j\sigma_y - \eta_j \|\mathbf{C}\| \leqq \sigma_y \leqq {}_j\sigma_y + \eta_j \|\mathbf{C}\| \, .$$

Let us summarize the results in the following theorem.

Theorem. Let $\mathscr{S}$ be a system described by the equation (1) (nominal system and $\mathscr{S}_r$ be a system described by (2) and by (2a) to (2f). Then for the correlation function of the output random process $\mathbf{y}(t, \alpha)$ of the system $\mathscr{S}_r$ the following relation holds:

$$\mathbf{K}_y(t_1, t_2) = {}_j\mathbf{K}_y(\tau) + \varepsilon_j(t_1, t_2) \quad (\tau = t_1 - t_2, \; j = 1, 2)$$

where

$${}_2\mathbf{K}_y(\tau) = {}_1\mathbf{K}_y(\tau) + \mathbf{R}(\tau) \, ,$$

$$\mathbf{R}(\tau) = \frac{1}{2\pi i} \int_{Im} e^{p\tau} \, dp \, \frac{1}{2\pi i} \int_{Im} ((p + q) \mathbf{E} - \mathbf{A})^{-1} \mathbf{G}(p, q) ((p + q) \mathbf{E} - \mathbf{A})^{-1*} \, dq ,$$

where ${}_1\mathbf{K}_y(\tau) = (1/2\pi i) \int_{Im} (p\mathbf{E} - \mathbf{A})^{-1} \mathbf{CC}^*(p\mathbf{E} - \mathbf{A})^{-1*} e^{p\tau} \, dp$ is the output correlation function of the nominal system, $\mathbf{G}(p, q)$ is given by (6), (7); further

$$\|\varepsilon_j(t_1, t_2)\| \leqq 2 \, {}_j\sigma_y \eta_j \|\mathbf{C}\| + \eta_j^2 \|\mathbf{C}\|^2 ;$$

$$\eta_1 = \frac{\beta}{1 - \beta} \, {}_1\sigma_{y1} , \quad \eta_2 = \frac{\beta}{1 - \beta} \sqrt{(\|\mathbf{R}_1(0)\|)} ,$$

$$\beta = m\delta , \quad m^2 = \max_{p \in Im} \|(p\mathbf{E} - \mathbf{A})^{-1} (p\mathbf{E} - \mathbf{A})^{-1*}\| ,$$

${}_j\sigma_y^2 = \|{}_j\mathbf{K}_y(0)\|$, $\mathbf{R}_1$ resp. ${}_1\sigma_{y1}$ are $\mathbf{R}$ resp. ${}_1\sigma_y$ for $\mathbf{C} = \mathbf{E}$.

Specially the following relation holds

$$|{}_j\sigma_y - \sigma_y(t)| \leqq \eta_j \|\mathbf{C}\| .$$

For an element ${}_jK_{y,ik}(t_1, t_2)$ of ${}_j\mathbf{K}_y(t_1, t_2)$ the inequality

$$|{}_jK_{y,ik}(t_1, t_2) - K_{y,ik}(t_1, t_2)| \leqq \|\varepsilon_j(t_1, t_2)\| \quad \text{holds} .$$

Note 1. When $\mathbf{A} = \mathbf{A}^*$ then $m^2 = \max_{\omega} \|(\mathbf{E}\omega^2 + \mathbf{A}^2)^{-1}\| = 1/\lambda_0$ where λ_0 is absolutely minimal eigenvalue of $\mathbf{A}$. Generally the inequality

$$m^2 \leqq \max_{\omega} \frac{\left(\omega^2 + \dfrac{\operatorname{Sp}(\mathbf{AA}^*)}{n}\right)^{n-1}}{|\Delta(i\omega)|^2} \left(\frac{n}{n - 1}\right)^{n-1} \quad \text{(see [4], p. 223)}$$

may be used. Here $\Delta(\lambda)$ is the characteristic polynomial of $\mathbf{A}$. If $|\Delta(i\omega)|^2$ is decomposed then the members of the type

$$\frac{\omega^2 + \dfrac{\operatorname{Sp}(\mathbf{AA}^*)}{n}}{(\omega - a)^2 + b^2} \quad \text{and} \quad \frac{1}{(\omega - a)^2 + b^2}$$

can be easily maximized.

${}_1\mathbf{K}_y(\tau)$ may generally be written as $e^{\mathbf{A}\tau}\mathbf{Q}$ ($\tau \geqq 0$) where $\mathbf{Q}$ is unique selfadjoined solution of $\mathbf{QA}^* + \mathbf{AQ} = -\mathbf{CC}^*$ (see section Computationaly remarks). When $\mathbf{A} = \mathbf{A}^*$, $\mathbf{C} = \mathbf{E}$ then $\mathbf{Q} = -\frac{1}{2}\mathbf{A}^{-1}$.

Note 2. Even if ${}_2\mathbf{K}_y(t_1, t_2)$ depends only on $\tau = t_1 - t_2$ resp. ${}_2\mathbf{K}_y(t, t)$ does not depend on t, if the presumptions (2a) to (2f) are fulfilled the output process $\mathbf{y}(t, \alpha)$ need not be weakly stationary.

Note 3. Theoretically the j-order approximation for $j > 2$ may be used. In this approximation there occur, however, j-order characteristics of the process $\mathbf{B}(t, \alpha)$. . ${}_j\mathbf{K}_y(t_1, t_2)$ for $j > 2$ is no more generally dependent only on τ.

Note 4. The presumption of the processes $b_{ij}(t, \alpha)$ not being correlated is not substantial. If it is not fulfilled, the formula for ${}_2\mathbf{K}_y(t_1, t_2)$ contains moreover members with cross spectral densities.

Note 5. If $b_{ij}(t, \alpha)$ are high frequency noises and if $\mathbf{Y}_1(p)$ is "great" only for small p then it is easily seen that $\mathbf{R}(\tau)$ is small. In such case the estimate of the correlation function by means of the correlation function of the nominal output is loaded with a small error.

Note 6. It may be assumed that under certain special presumptions the estimate of the error may be substantially improved. These problems are objects of further investigations.

COMPUTATIONAL REMARKS

It is seen (with respect to the form of $\mathbf{G}(p, q)$) that $\mathbf{R}(\tau)$ may be written as

$$\mathbf{R}(\tau) = \sum_{i,k} \mathbf{R}^{ik}(\tau),$$

$$\mathbf{R}^{ik}(\tau) =$$

$$= \frac{1}{2\pi i} \int_{Im} e^{p\tau} v_{kk}(p) \, dp \, \frac{1}{2\pi i} \int_{Im} ((p + q) \mathbf{E} - \mathbf{A})^{-1} \mathbf{E}^{ii}((-p - q) \mathbf{E} - \mathbf{A}^*)^{-1} s_{ik}(q) \, dq ;$$

($\mathbf{R}^{ik}(\tau)$ is the component of $\mathbf{R}(\tau)$ corresponding to the noise $b_{ik}(t, \alpha)$) where

$$(8) \qquad \mathbf{E}^{rs} = (e_{ij}) , \quad e_{rs} = 1 , \quad e_{ij} = 0 \quad \text{for} \quad (i, j) \neq (r, s) ;$$

$v_{kk}(p)$ resp. $s_{ik}(q)$ may be written (in the case of single roots) in the form of linear combination of members of the form

$$\frac{1}{(p + \gamma^*)(-p + \gamma)} \quad \text{resp.} \quad \frac{1}{(q + \beta^*)(-q + \beta)} ;$$

β, γ lie in the right half plane. Then for one component of inner integral of $\mathbf{R}^{ik}(\tau)$ we get

$$\frac{1}{2\pi i} \int_{Im} ((p + q) \mathbf{E} - \mathbf{A})^{-1} \mathbf{E}^{ii}((-p - q) \mathbf{E} - \mathbf{A}^*)^{-1} \frac{1}{(q + \beta^*)(-q + \beta)} \, dq =$$

$$= \frac{1}{2\pi i} \int_{Im} [((p + q) \mathbf{E} - \mathbf{A})^{-1} \mathbf{X} + \mathbf{X}((-p - q) \mathbf{E} - \mathbf{A}^*)^{-1}] \,.$$

$$\cdot \frac{1}{(q + \beta^*)(-q + \beta)} \, dq =$$

$$= [(p - (\mathbf{A} - \beta \mathbf{E}))^{-1} \mathbf{X} + \mathbf{X}(-p - (\mathbf{A}^* - \beta^* \mathbf{E}))^{-1}] \frac{1}{2 \operatorname{Re} \beta},$$

$\mathbf{X}$ is unique (selfadjoint) solution of the equation $\mathbf{AX} + \mathbf{XA}^* = -\mathbf{E}^{ii}$ (see below).

Finally $\mathbf{R}^{ik}(\tau)$ may be computed by means of residua, where $\mathbf{A} - \beta\mathbf{E}$ may be formally considered as a pole in the left half plane. γ is, however, a eigenvalue of $\mathbf{A}$. It may happen that γ is also eigenvalue of $(\mathbf{A} - \beta\mathbf{E})$. In this case, even if multiple eigenvalues occur, we get the integrals of the type

$$\mathbf{J} = \frac{1}{2\pi i} \int_{Im} \frac{f(p)}{(p + \gamma^*)^k} (p\mathbf{E} - \mathbf{B})^{-1} \, dp$$

where $-\gamma^*$ is an eigenvalue of $\mathbf{B}$ (in the left half plane). It holds

$$\mathbf{J} = g(\mathbf{B}) \,,$$

where

$$g(\lambda) =$$

$$= \frac{f(\lambda) - [f(-\gamma^*) + (\lambda + \gamma^*) f'(-\gamma^*) + \ldots + \frac{1}{(k-1)!} (\lambda + \gamma^*)^{k-1} f^{(k-1)}(-\gamma^*)]}{(\lambda + \gamma^*)^k} \,.$$

For practice $\mathbf{R}(0)$ is important; for one component of $\mathbf{R}^{ik}(0)$ we get a simple formula:

$$\mathbf{L} + \mathbf{L}^* \,,$$

where

$$\mathbf{L} = \frac{1}{4 \operatorname{Re} \beta \operatorname{Re} \gamma} ((\beta + \gamma) \mathbf{E} - \mathbf{A})^{-1} \mathbf{X} \,. \tag{9}$$

When we have to compute $\mathbf{R}(0)$ only, it is possible to avoid the solving of characteristic equation of $\mathbf{A}$. In this section we give a "linear algebraic" algorithmus for computing $\mathbf{R}(0)$. We begin with a note taking the solution of the following matrix equation:

$$\mathbf{AX} + \mathbf{XB}^* = \mathbf{C} \quad (\mathbf{A}, \mathbf{B}, \mathbf{X}, \mathbf{C} \text{ are matrices } n \times n) \,. \tag{10}$$

Let us denote

$$(11) \qquad {}^{\sim}\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A} & & 0 \\ & \boldsymbol{A} & \\ 0 & \ddots & \\ & & \boldsymbol{A} \end{bmatrix} \left.\begin{matrix} 1 \\ 2 \\ \vdots \\ n \end{matrix}\right\}, \qquad {}^{\wedge}\boldsymbol{B} = \begin{bmatrix} b_{11}\mathbf{E}, & b_{12}\mathbf{E}, & \dots & \\ b_{21}\mathbf{E}, & b_{22}\mathbf{E}, & \dots & \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & b_{nn}\mathbf{E} \end{bmatrix}$$

($\mathbf{E}$ is the unit matrix $n \times n$).

We define the operator $\mathscr{V}$ and the constant vector $\boldsymbol{v}$

$$(12) \qquad \mathscr{V}(\mathbf{C}) = \begin{bmatrix} c_{11} \\ \cdot \\ \cdot \\ c_{n1} \\ \dots \\ c_{12} \\ \cdot \\ \cdot \\ c_{n2} \\ \dots \\ c_{1n} \\ \cdot \\ \cdot \\ c_{nn} \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ \dots \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ \dots \\ \cdot \\ \cdot \\ \dots \\ 0 \\ \cdot \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \\ \\ \} n \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix}$$

It is seen that

$$(13) \qquad \mathscr{V}(\mathbf{C}) = {}^{\sim}\boldsymbol{C}\boldsymbol{v}\,.$$

Further there evidently holds ${}^{\sim}(\boldsymbol{A} + \boldsymbol{B}) = {}^{\sim}\boldsymbol{A} + {}^{\sim}\boldsymbol{B}$, ${}^{\sim}(\boldsymbol{AB}) = {}^{\sim}\boldsymbol{A}\,{}^{\sim}\boldsymbol{B}$, ${}^{\sim}[f(\boldsymbol{A})] = f({}^{\sim}\boldsymbol{A})$, where $f(\lambda)$ is a polynomial; $\boldsymbol{A}\,\mathscr{V}^{-1}(\boldsymbol{r}) = \mathscr{V}^{-1}({}^{\sim}\boldsymbol{A}\boldsymbol{r})$, $\boldsymbol{r}$ is a n^2-column vector. It may easily be verified that $\mathscr{V}(\boldsymbol{AX}) = {}^{\sim}\boldsymbol{A}\,\mathscr{V}(\mathbf{X}) = {}^{\sim}\boldsymbol{A}\,{}^{\sim}\boldsymbol{X}\boldsymbol{v}$; $\mathscr{V}(\boldsymbol{XB}^*) = {}^{\wedge}\boldsymbol{B}\,\mathscr{V}(\mathbf{X}) = {}^{\wedge}\boldsymbol{B}\boldsymbol{X}\boldsymbol{v}$, thus we get from (10)

$$\mathscr{V}(\boldsymbol{AX} + \boldsymbol{XB}^*) = \mathscr{V}(\mathbf{C})\,, \quad ({}^{\sim}\boldsymbol{A} + {}^{\wedge}\mathbf{B})\,\mathscr{V}(\mathbf{X}) = \mathscr{V}(\mathbf{C}) = {}^{\sim}\boldsymbol{C}\boldsymbol{v}\,,$$

$$\boldsymbol{X} = \mathscr{V}^{-1}[({}^{\sim}\boldsymbol{A} + {}^{\wedge}\mathbf{B})^{-1}\,{}^{\sim}\boldsymbol{C}\boldsymbol{v}]\,.$$

Now we return to $\mathbf{R}^{ik}(0)$ which may be written in another form

$$\mathbf{R}^{ik}(0) =$$

$$= \frac{1}{2\pi i}\int_{Im} dp \, \frac{1}{2\pi i}\int_{Im} ((p + q)\,\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{E}^{ik}(p\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{C}\mathbf{C}^*(-p\mathbf{E} - \mathbf{A}^*)^{-1}\,\mathbf{E}^{ki}\,.$$

$$.\,((-p - q)\,\mathbf{E} - \mathbf{A}^*)^{-1}\,\frac{1}{(q + \beta^*)(-q + \beta)}\,dq\,;$$

here

$$s_{ik}(q) = \frac{1}{(q + \beta^*)(-q + \beta)}\,;$$

for the inner integral we get

$$\frac{1}{2\pi i}\int_{Im} [((p + q)\,\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{D}(p) + \mathbf{D}(p)\,((-p - q)\,\mathbf{E} - \mathbf{A}^*)^{-1}]\,.$$

$$.\,\frac{1}{(q + \beta^*)(-q + \beta)}\,dq =$$

$$= \frac{1}{2\,\mathrm{Re}\,\beta}\,[((p + \beta)\,\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{D}(p) + \mathbf{D}(p)\,((-p + \beta^*)\,\mathbf{E} - \mathbf{A}^*)^{-1}]$$

where $\mathbf{D}(p)$ satisfies the equation

$$-\mathbf{A}\mathbf{D}(p) - \mathbf{D}(p)\,\mathbf{A}^* = \mathbf{E}^{ik}(p\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{C}\mathbf{C}^*(-p\mathbf{E} - \mathbf{A}^*)^{-1}\,\mathbf{E}^{ki}\,,$$

thus

$$\mathbf{D}(p) = \mathscr{V}^{-1}[\mathbf{G}^{-1}{}^{\sim}\mathbf{E}^{ik}(p\mathbf{E} - {}^{\sim}\mathbf{A})^{-1}\,{}^{\sim}\mathbf{C}{}^{\sim}\mathbf{C}^*(-p\mathbf{E} - {}^{\sim}\mathbf{A}^*)^{-1}\,{}^{\sim}\mathbf{E}^{ki}\mathbf{v}] =$$

$$= \mathscr{V}^{-1}[\mathbf{G}^{-1}{}^{\sim}\mathbf{E}^{ik}((p\mathbf{E} - {}^{\sim}\mathbf{A})^{-1}\,{}^{\sim}\mathbf{Q} + {}^{\sim}\mathbf{Q}(-p\mathbf{E} - {}^{\sim}\mathbf{A}^*)^{-1})\,{}^{\sim}\mathbf{E}^{ki}\mathbf{v}]\,;$$

$$\mathbf{G} = -({}^{\sim}\mathbf{A} + {}^{\wedge}\mathbf{A})\,, \quad \mathbf{Q} = \mathscr{V}^{-1}(\mathbf{G}^{-1}{}^{\sim}\mathbf{C}{}^{\sim}\mathbf{C}^*\mathbf{v})$$

($\mathbf{Q}$ satisfies the equation $-\mathbf{A}\mathbf{Q} - \mathbf{Q}\mathbf{A}^* = \mathbf{C}\mathbf{C}^*$) then

$$\mathbf{P}(p) = ((p + \beta)\,\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{D}(p) =$$

$$= \mathscr{V}^{-1}[((p + \beta)\,\mathbf{E} - {}^{\sim}\mathbf{A})^{-1}\,\mathbf{G}^{-1}{}^{\sim}\mathbf{E}^{ik}(p\mathbf{E} - {}^{\sim}\mathbf{A})^{-1}\,{}^{\sim}\mathbf{Q}{}^{\sim}\mathbf{E}^{ki}\mathbf{v} +$$

$$+ ((p + \beta)\,\mathbf{E} - {}^{\sim}\mathbf{A})^{-1}\,\mathbf{G}^{-1}{}^{\sim}\mathbf{E}^{ik}{}^{\sim}\mathbf{Q}(-p\mathbf{E} - {}^{\sim}\mathbf{A}^*)^{-1}\,{}^{\sim}\mathbf{E}^{ki}\mathbf{v}]\,.$$

The integral of the first member (i.e. the summ of residua in the left half plane) is evidently zero; let us decompose the second member:

$$((p + \beta)\,\mathbf{E} - {}^{\sim}\mathbf{A})^{-1}\,\mathbf{G}^{-1}{}^{\sim}\mathbf{E}^{ik}{}^{\sim}\mathbf{Q}(-p\mathbf{E} - {}^{\sim}\mathbf{A}^*)^{-1}\,{}^{\sim}\mathbf{E}^{ki}\mathbf{v} =$$

$$= [((p + \beta)\,\mathbf{E} - {}^{\sim}\mathbf{A})^{-1}\,\mathbf{U} + \mathbf{U}(-p\mathbf{E} - {}^{\sim}\mathbf{A}^*)^{-1}]\,{}^{\sim}\mathbf{E}^{ki}\mathbf{v}\,,$$

where

$$-(\tilde{}A - \beta E)\, U - U\tilde{}A^* = G^{-1}\tilde{}E^{ik}\tilde{}Q = N:$$

N, U are, however, the matrices $n^2 \times n^2$. Let us decompose N and U into blocs:

$$N = \begin{bmatrix} N_{11}, & N_{12}, & \dots \\ \cdot & \cdot & \\ \cdot & & \cdot\ N_{nn} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11}, & U_{12}, & \dots \\ \cdot & \cdot & \\ \cdot & & \cdot\ U_{nn} \end{bmatrix}$$

then we get

$$-(A - \beta E)\, U_{ij} - U_{ij} A^* = N_{ij}$$

$$U_{ij} = \mathscr{V}^{-1}[-(\tilde{}A - \beta E + \hat{}A)^{-1}\, \tilde{}N_{ij} v]$$

and thus

$$R^{ik}(0) = \frac{1}{2 \operatorname{Re} \beta} (\mathscr{V}^{-1}(U\tilde{}E^{ki} v) + [\mathscr{V}^{-1}(U\tilde{}E^{ki} v)]^*) .$$

But it is easily seen that $U\tilde{}E^{ki} v$ = i-th column of k-th bloc column of U i.e. $((k-1)\, n + i)$th column of U and

$$\mathscr{V}^{-1}(U\tilde{}E^{ki} v) = G_{i.}^{-1}(\beta)\, (\mathscr{V}(N_{1k}), \mathscr{V}(N_{2k}), \dots, \mathscr{V}(N_{nk}))$$

where $G^{-1}(\beta) = -(\tilde{}A - \beta E + \hat{}A)^{-1}$, $G_{i.}^{-1}(\beta)$ is the i-th bloc row of $G^{-1}(\beta)$, $(\mathscr{V}(N_{1k}), \mathscr{V}(N_{2k}), \dots, \mathscr{V}(N_{nk}))$ is the matrix formed by the column vectors $\mathscr{V}(N_{1k})$, $\mathscr{V}(N_{2k}), \dots, \mathscr{V}(N_{nk})$.

We summarize this results in the following

Corollary. For computing $R^{ik}(0)$ (i.e. component of $R(0)$ due to noise $b_{ik}(t, \alpha)$ with spectral density $1/[(q + \beta^*)(-q + \beta)]$) the following algorithmus is given (see (10), ..., (13)):

$$G(0) = -(\tilde{}A + \hat{}A)\,; \quad G(\beta) = G(0) + \beta E\,,$$

$$Q = \mathscr{V}^{-1}(G^{-1}(0)\tilde{}C\tilde{}C^* v)\,,$$

$$N = G^{-1}(0)\tilde{}E^{ik}\tilde{}Q = \begin{bmatrix} N_{11}, & N_{12}, & \dots \\ \cdot & & \cdot \\ \cdot & & \cdot\ N_{nn} \end{bmatrix}$$

$$G^{-1}(\beta) = \begin{bmatrix} G_{11}^{-1}(\beta), & G_{12}^{-1}(\beta), & \dots \\ \cdot & \cdot & \\ \cdot & & \cdot\ G_{nn}^{-1}(\beta) \end{bmatrix} \quad \text{(decomposition into blocs)}\,,$$

$$G_i^{-1}(\beta) = (G_{i1}^{-1}(\beta), G_{i2}^{-1}(\beta), \dots, G_{in}^{-1}) = i\text{-th bloc row}\,,$$

$$\mathbf{L}_1^{ik} = \mathbf{G}_i^{-1}(\beta)\left(\mathscr{V}(\mathbf{N}_{1k}), \mathscr{V}(\mathbf{N}_{2k}), \ldots, \mathscr{V}(\mathbf{N}_{nk})\right),$$

$$\mathbf{R}^{ik}(0) = \frac{1}{2\,\mathrm{Re}\,\beta}\left(\mathbf{L}_1^{ik} + (\mathbf{L}_1^{ik})^*\right).$$

EXAMPLE

$$\dot{z} + (3 + b(t, \alpha))\, z = u(t, \alpha), \quad \text{where}$$
$$\dot{u} + 2u = x(t, \alpha), \quad x(t, \alpha) \quad \text{is a white noise}.$$

Thus we get a system

$$\dot{z} = (-3 + b(t, \alpha))\, z + u,$$
$$\dot{u} = \qquad\qquad - 2u + x(t, \alpha) \quad \text{or}$$

$$\dot{\mathbf{y}} = \begin{bmatrix} \dot{z} \\ \dot{u} \end{bmatrix} = (\mathbf{A} + \mathbf{B}(t, \alpha)) \begin{bmatrix} z \\ u \end{bmatrix} + \mathbf{C}\,\mathbf{x}(t, \alpha), \quad \text{where}$$

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B}(t, \alpha) = \begin{bmatrix} b(t, \alpha) & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $K_b(\tau) = \sigma^2 \mathrm{e}^{-|\tau|} \cos \omega_0 \tau$, $\delta = 3\sigma$; thus

$$s_{11}(q) = \sigma^2 \left(\frac{1}{(q + 1 + q_0)(-q + 1 - q_0)} + \frac{1}{(q + 1 - q_0)(-q + 1 + q_0)} \right),$$
$$q_0 = \mathrm{i}\omega_0, \quad \beta = 1 - q_0.$$

The solution of $\mathbf{AQ} + \mathbf{QA}^* = -\mathbf{CC}^*$ is

$$\mathbf{Q} = \frac{1}{60} \begin{bmatrix} 1 & 3 \\ 3 & 15 \end{bmatrix}, \quad \text{so that} \quad (\tau > 0)$$

$$_1\mathbf{K}_y(\tau) = \mathrm{e}^{\mathbf{A}\tau}\mathbf{Q} = \left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \mathrm{e}^{-3\tau} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \mathrm{e}^{-2\tau} \right) \frac{1}{60} \begin{bmatrix} 1 & 3 \\ 3 & 15 \end{bmatrix}.$$

The element v_{11} of the matrix $(p\mathbf{E} - \mathbf{A})^{-1}\, \mathbf{CC}^*(-p\mathbf{E} - \mathbf{A}^*)^{-1}$ is

$$\frac{1}{(p + 3)(p + 2)(-p + 3)(-p + 2)} = \frac{1}{5} \left(\frac{1}{(p + 2)(-p + 2)} - \frac{1}{(p + 3)(-p + 3)} \right).$$

The solution of the equation $\mathbf{AX} + \mathbf{XA}^* = -\mathbf{E}^{11}$ is

$$\mathbf{X} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 0 \end{bmatrix} = \tfrac{1}{6}\mathbf{E}^{11}.$$

With respect to $s_{11}(q)$ we get $(\tau > 0)$

$$\mathbf{R}^{11}(\tau) = \sigma^2 2 \operatorname{Re} \frac{1}{2\pi i} \frac{1}{5} \int_{Im} e^{p\tau} \left(\frac{1}{(p+2)(-p+2)} - \frac{1}{(p+3)(-p+3)} \right) \cdot$$

$$\cdot \frac{1}{2 \operatorname{Re} \beta} [(p\mathbf{E} - (\mathbf{A} - \beta\mathbf{E}))^{-1} \mathbf{X} + \mathbf{X}(-p\mathbf{E} - (\mathbf{A}^* - \beta^*\mathbf{E}))^{-1}] \, dp =$$

$$= \frac{\sigma^2}{5 \operatorname{Re} \beta} \operatorname{Re} \{ e^{-2\tau} \tfrac{1}{4} [(-2\mathbf{E} - \mathbf{A} + \beta\mathbf{E})^{-1} \mathbf{X} + \mathbf{X} (2\mathbf{E} - \mathbf{A}^* + \beta^*\mathbf{E})^{-1}] +$$

$$+ e^{(\mathbf{A} - \beta\mathbf{E})\tau} (\mathbf{A} - \beta\mathbf{E} + 2\mathbf{E})^{-1} (-\mathbf{A} + \beta\mathbf{E} + 2\mathbf{E})^{-1} \mathbf{X} -$$

$$- e^{-3\tau} \tfrac{1}{6} [(-3\mathbf{E} - \mathbf{A} + \beta\mathbf{E})^{-1} \mathbf{X} + \mathbf{X} (3\mathbf{E} - \mathbf{A}^* + \beta^*\mathbf{E})^{-1}] -$$

$$- e^{(\mathbf{A} - \beta\mathbf{E})\tau} (\mathbf{A} - \beta\mathbf{E} + 3\mathbf{E})^{-1} (-\mathbf{A} + \beta\mathbf{E} + 3\mathbf{E})^{-1} \mathbf{X} \} ;$$

but $f(\mathbf{A}) \mathbf{X} = \frac{1}{6} f(-3) \mathbf{E}^{11}$, thus

$$\mathbf{R}^{11}(\tau) = \mathbf{E}^{11} \frac{\sigma^2}{6.5 \operatorname{Re} \beta} \operatorname{Re} \left\{ \frac{e^{-2\tau}}{4} \left(\frac{1}{-2+3+\beta} + \frac{1}{2+3+\beta^*} \right) + \right.$$

$$+ e^{(-3-\beta)\tau} \frac{1}{(-3-\beta+2)} \frac{1}{(3+\beta+2)} - \frac{e^{-3\tau}}{6} \left(\frac{1}{-3+3+\beta} + \frac{1}{3+3+\beta^*} \right) -$$

$$\left. - e^{-(3-\beta)\tau} \frac{1}{(-3-\beta+3)} \frac{1}{(3+\beta+3)} \right\} =$$

$$(15) \quad = \mathbf{E}^{11} \frac{\sigma^2}{30} \left\{ \frac{e^{-2\tau}}{4} \left(\frac{2}{4+\omega_0^2} + \frac{6}{36+\omega_0^2} \right) - \frac{e^{-3\tau}}{6} \left(\frac{1}{1+\omega_0^2} + \frac{7}{49+\omega_0^2} \right) + \right.$$

$$+ e^{-4\tau} \cos \omega_0 \tau \left[\frac{1}{4} \left(\frac{-2}{4+\omega_0^2} + \frac{6}{36+\omega_0^2} \right) - \frac{1}{6} \left(\frac{-1}{1+\omega_0^2} + \frac{7}{49+\omega_0^2} \right) \right] +$$

$$\left. + e^{-4\tau} \omega_0 \sin \omega_0 \tau \left[\frac{1}{4} \left(\frac{1}{4+\omega_0^2} - \frac{1}{36+\omega_0^2} \right) - \frac{1}{6} \left(\frac{1}{1+\omega_0^2} - \frac{1}{49+\omega_0^2} \right) \right] \right\}$$

When we want to compute $\mathbf{R}(0)$ only we get according to (9), (14)

$$\mathbf{L} = \frac{1}{5} \left(\frac{1}{4.2} ((\beta + 2) \mathbf{E} - \mathbf{A})^{-1} \mathbf{X} - \frac{1}{4.3} ((\beta + 3) \mathbf{E} - \mathbf{A})^{-1} \mathbf{X} = \right.$$

$$= \mathbf{E}^{11} \frac{1}{6} \frac{1}{5} \left(\frac{1}{4.2} \frac{1}{6 - q_0} - \frac{1}{4.3} \frac{1}{7 - q_0} \right)$$

so that we get with respect to the given $s_{11}(q)$

$$\mathbf{R}^{11}(0) = \mathbf{R}(0) = 2\sigma^2(\mathbf{L} + \mathbf{L}^*) =$$

$$= \mathbf{E}^{11} \sigma^2 \left(\frac{1}{10} \frac{1}{36 + \omega_0^2} - \frac{7}{90} \frac{1}{49 + \omega_0^2} \right).$$

Now we use the "linear algebraic" algorithmus for computing $\mathbf{R}^{11}(0)$ given by the Corollary $((i, k) = (1, 1))$:

$$\mathbf{G}(\beta) = -\left[\begin{array}{cc|cc} -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ \hline 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right] - \left[\begin{array}{cc|cc} -3 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ \hline 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right] + \beta \left[\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array}\right],$$

$$\mathbf{G}^{-1}(\beta) = \left[\begin{array}{cc|cc} \frac{1}{6+\beta}, & \frac{1}{(6+\beta)(5+\beta)} & \frac{1}{(5+\beta)(6+\beta)}, & \frac{2}{(6+\beta)(5+\beta)(4+\beta)} \\ 0, & \frac{1}{5+\beta} & 0 \quad , & \frac{1}{(5+\beta)(4+\beta)} \\ \hline 0, & 0 & \frac{1}{5+\beta} \quad , & \frac{1}{(5+\beta)(4+\beta)} \\ 0, & 0 & 0 \quad , & \frac{1}{(4+\beta)} \end{array}\right],$$

$$\mathbf{G}^{-1}(0) = \frac{1}{60}\left[\begin{array}{cc|cc} 10 & 2 & 2 & 1 \\ 0 & 12 & 0 & 3 \\ \hline 0 & 0 & 12 & 3 \\ 0 & 0 & 0 & 15 \end{array}\right], \quad {}^{\sim}\mathbf{C}^{\sim}\mathbf{C}^{*} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$${}^{\sim}\mathbf{C}^{\sim}\mathbf{C}^{*}\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{Q} = \mathcal{V}^{-1}(\mathbf{G}^{-1}(0)\, {}^{\sim}\mathbf{C}^{\sim}\mathbf{C}^{*}\mathbf{v}) = \frac{1}{60}\begin{bmatrix} 1 & 3 \\ 3 & 15 \end{bmatrix};$$

$$\mathbf{N} = \frac{1}{60}\left[\begin{array}{cc|cc} 10 & 2 & 2 & 1 \\ 0 & 12 & 0 & 3 \\ \hline 0 & 0 & 12 & 3 \\ 0 & 0 & 0 & 15 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \frac{1}{60} \left[\begin{array}{cc|cc} 1 & 3 & 0 & 0 \\ 3 & 15 & 0 & 0 \\ \hline 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 15 \end{array}\right] =$$

$$= \frac{1}{60^2}\left[\begin{array}{cc|cc} 10 & 30 & 2 & 6 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 12 & 36 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$(\mathcal{V}(\mathbf{N}_{11}), \mathcal{V}(\mathbf{N}_{21})) = \frac{1}{60^2}\begin{bmatrix} 10 & 0 \\ 0 & 0 \\ 30 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\boldsymbol{L}_1 = \begin{bmatrix} \frac{1}{6+\beta}, & \frac{1}{(6+\beta)(5+\beta)}, & \frac{1}{(5+\beta)(6+\beta)}, & \frac{2}{(6+\beta)(5+\beta)(4+\beta)} \\ 0 \quad , & \frac{1}{5+\beta} \quad , & 0 \quad , & \frac{1}{(5+\beta)(4+\beta)} \end{bmatrix} .$$

$$. \frac{1}{60^2} \begin{bmatrix} 10 & 0 \\ 0 & 0 \\ 30 & 0 \\ 0 & 0 \end{bmatrix} = \boldsymbol{E}^{11} \frac{1}{60^2} \left(\frac{10}{6+\beta} + \frac{30}{(5+\beta)(6+\beta)} \right) =$$

$$= \boldsymbol{E}^{11} \frac{1}{60^2} \left(\frac{30}{5+\beta} - \frac{20}{6+\beta} \right);$$

$$\boldsymbol{R}^{11}(0) = 2\sigma^2 \frac{1}{2 \operatorname{Re} \beta} (\boldsymbol{L}_1 + \boldsymbol{L}_1^*) = \boldsymbol{E}^{11} \sigma^2 \left(\frac{1}{10} \frac{1}{36 + \omega_0^2} - \frac{7}{90} \frac{1}{49 + \omega_0^2} \right).$$

Let us compute the error: let $\sigma = 0, 1$; $\omega_0 = 1$,

$$m^2 = \max_{p \in Im} \|(p\boldsymbol{E} - \boldsymbol{A})^{-1} (p\boldsymbol{E} - \boldsymbol{A}^*)^{-1}\| = \frac{1}{\min\limits_i \min\limits_p \lambda_i(p)},$$

where $\lambda_i(p)$ are (positive) eigenvalues of $(p\boldsymbol{E} - \boldsymbol{A})(-p\boldsymbol{E} - \boldsymbol{A}^*)$; for our example $m = 0{\cdot}542769$; $\|\boldsymbol{C}\| = 1$;

$\beta \quad = 3\sigma \,.\, 0{\cdot}542769 = 0{\cdot}162831; \ \dfrac{\beta}{1-\beta} = 0{\cdot}194502;$

${}_1\sigma_y \ = \sqrt{(\|{}_1\boldsymbol{K}_y(0)\|)} = \sqrt{(\|\boldsymbol{Q}\|)} = 0{\cdot}510160;$

${}_2\sigma_y \ = \sqrt{(\|{}_1\boldsymbol{K}_y(0) + \boldsymbol{R}^{11}(0)\|)} = 0{\cdot}510160;$

$\boldsymbol{Q}_1 \ = \mathscr{V}^{-1}(\boldsymbol{G}^{-1}(0)\,\boldsymbol{E}v) = \dfrac{1}{60} \begin{bmatrix} 11 & 3 \\ 3 & 15 \end{bmatrix};$

${}_1\sigma_{y1} = \sqrt{(\|\boldsymbol{Q}_1\|)} = 0{\cdot}526079.$

$\boldsymbol{R}_1^{11}(0)$ will be computed as $\boldsymbol{R}^{11}(0)$ with $\boldsymbol{Q}_1$ instead of $\boldsymbol{Q}$. We get

$\sqrt{(\|\boldsymbol{R}_1^{11}(0)\|)} = 0{\cdot}009447;$

$\eta_1 = 0{\cdot}102324, \ \eta_2 = 0{\cdot}001838;$

$\|\varepsilon_1(t_1, t_2)\| \leqq 0{\cdot}114873;$

$\|\varepsilon_2(t_1, t_2)\| \leqq 0{\cdot}001879.$

Let us summarize:

1. Estimate by nominal system ($\tau > 0$; $\sigma = 0, 1$; $\omega_0 = 1$) $\boldsymbol{K}_y(\tau) = {}_1\boldsymbol{K}_y(\tau) + \varepsilon_1(t_1, t_2)$, where

$$ {}_1\boldsymbol{K}_y(\tau) = \frac{1}{60}\left(\begin{bmatrix} 3 & 15 \\ 3 & 15 \end{bmatrix} e^{-2\tau} - \begin{bmatrix} 2 & 12 \\ 0 & 0 \end{bmatrix} e^{-3\tau}\right), $$

$$ \|\varepsilon_1(t_1, t_2)\| \leqq 0{\cdot}114873 $$

$$ |{}_1\sigma_y - \sigma_y| \leqq 0{\cdot}102324 \quad (\text{here } {}_1\sigma_y = \sqrt{(\|{}_1\boldsymbol{K}_y(0)\|)}) . $$

For component $z(t, \alpha)$ we have

$$ K_z(\tau) = \frac{1}{60}(3e^{-2\tau} - 2e^{-3\tau}) + \varepsilon , \quad |\varepsilon| \leqq 0{\cdot}114873 . $$

2. Estimate by iteration step

$$ \boldsymbol{K}_y(\tau) = {}_2\boldsymbol{K}_y(\tau) + \varepsilon_2(t_1, t_2) ; \quad {}_2\boldsymbol{K}_y(\tau) = {}_1\boldsymbol{K}_y(\tau) + \boldsymbol{R}^{11}(\tau) \quad (\text{see } (15)) $$

$$ \|\varepsilon_2(t_1, t_2)\| \leqq 0{\cdot}001879 ; $$

specially

$$ \boldsymbol{K}_y(0) = \frac{1}{60}\begin{bmatrix} 1 & 3 \\ 3 & 15 \end{bmatrix} + \begin{bmatrix} 0{\cdot}000011 & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon_2(t, t) , $$

$$ \|\varepsilon_2(t, t)\| \leqq 0{\cdot}001879 , $$

$$ |{}_2\sigma_y - \sigma_y| \leqq 0{\cdot}001838 , $$

$$ K_z(0) = \frac{1}{60} + 0{\cdot}000011 + \varepsilon ; \quad |\varepsilon| < 0{\cdot}001879 . $$

APPENDIX

Lemma A. Let $(X, \mathscr{X}, \mu)$ be a measure space and $\boldsymbol{T}$ be a measurable function which maps X into the set of complex valued quadrat $n \times n$ matrices. Let

$$ \|\boldsymbol{T}\|^2 = \left\|\int_X \boldsymbol{T}(x)\,(\boldsymbol{T}(x))^*\,d\mu(x)\right\| ; \quad \|\boldsymbol{T}\|_a^2 = \left\|\int_X (\boldsymbol{T}(x))^*\,\boldsymbol{T}(x)\,d\mu(x)\right\| ; $$

$\|\cdot\|$ resp. $\|\cdot\|_a$ is a norm.

Evidently it is sufficient to prove this statement for the case of $\boldsymbol{T}$ being a simple function: $\boldsymbol{T} = \sum_{i=1}^{k} \boldsymbol{A}_i\,\chi_{E_i}(x)$ where $\chi_{E_i}(x)$ is the characteristic function of the set E_i. The matrix

$$ \begin{bmatrix} \boldsymbol{A}_1 \sqrt{(\mu(E_1))} \\ \cdot \\ \cdot \\ \boldsymbol{A}_k \sqrt{(\mu(E_k))} \end{bmatrix} $$

maps n-dimensional space into the nk-dimensional space and its norm is

$$\sqrt{\left(\left\|\sum_i^k \mathbf{A}_i^* \mathbf{A}_i \mu(E_i)\right\|\right)} = \sqrt{\left(\left\|\int_X (\mathbf{T}(x))^* \mathbf{T}(x) \, d\mu(x)\right\|\right)} = \|\mathbf{T}\|_a .$$

Similarly, when we consider the matrix

$$\begin{bmatrix} \mathbf{A}_1^* \sqrt{(\mu(E_1))} \\ \cdot \\ \cdot \\ \mathbf{A}_k^* \sqrt{(\mu(E_k))} \end{bmatrix}$$

we get the norm $\|\mathbf{T}\|$.

Lemma B. Let

$$(\mathbf{T}, \mathbf{U}) = \int_X \mathbf{T}(x) \, (\mathbf{U}(x))^* \, d\mu(x) \, ;$$

then

$$\|(\mathbf{T}, \mathbf{U})\| \leqq \|\mathbf{T}\| \; \|\mathbf{U}\| .$$

Evidently, let $\mathbf{T}(x) = \sum_i^k \mathbf{A}_i \, \chi_{E_i}(x)$, $\mathbf{U}(x) = \sum_i^k \mathbf{B}_i \, \chi_{E_i}(x)$;

$$(\mathbf{T}, \mathbf{U}) = [\mathbf{A}_1 \sqrt{(\mu(E_1))}, \ldots, \mathbf{A}_k \sqrt{(\mu(E_k))}] \begin{bmatrix} \mathbf{B}_1^* \sqrt{(\mu(E_1))} \\ \cdot \\ \cdot \\ \mathbf{B}_k^* \sqrt{(\mu(E_k))} \end{bmatrix} .$$

Lemma C.

$$\left\|\int_X \mathbf{T}(x) \, \mathbf{U}(x) \, (\mathbf{U}(x))^* \, (\mathbf{T}(x))^* \, d\mu(x)\right\| \leqq \|\mathbf{T}\|^2 \sup_x \|\mathbf{U}(x)\|^2 .$$

Proof. Let us consider the simple functions given in the lemma B. Let us denote $\mathbf{A}_i^* \mathbf{y} = \mathbf{u}_i$. It is known that for a selfadjoined matrix $\mathbf{S}$ the relation

$$\|\mathbf{S}\| = \max_{\|\mathbf{v}\| = 1} |\mathbf{v}^* \mathbf{S} \mathbf{v}|$$

holds; thus we get

$$\left\|\int_X \mathbf{T}(x) \, \mathbf{U}(x) \, (\mathbf{U}(x))^* \, (\mathbf{T}(x))^* \, d\mu(x)\right\| = \left\|\sum_i^k \mathbf{A}_i \mathbf{B}_i \mathbf{B}_i^* \mathbf{A}_i^* \, \mu(E_i)\right\| =$$

$$= \max_{\|\mathbf{y}\|=1} \left|\sum_i^k \mathbf{u}_i \mathbf{B}_i \mathbf{B}_i^* \mathbf{u}_i \, \mu(E_i)\right| \leqq \max_{\|\mathbf{y}\|=1} \left(\sum_i^k \mathbf{u}_i^* \mathbf{u}_i \, \mu(E_i) \, \|\mathbf{B}_i \mathbf{B}_i^*\|\right) \leqq$$

$$\leqq \max_{\|\mathbf{y}\|=1} \left(\sum_i^k \mathbf{u}_i^* \mathbf{u}_i \, \mu(E_i)\right) \max_i \|\mathbf{B}_i \mathbf{B}_i^*\| =$$

$$= \max_{\|\mathbf{y}\|=1} \mathbf{y}^* \left(\sum_i^k \mathbf{A}_i \mathbf{A}_i^* \, \mu(E_i)\right) \mathbf{y} \max_i \|\mathbf{B}_i \mathbf{B}_i^*\| = \|\mathbf{T}\|^2 \sup_x \|\mathbf{U}(x)\|^2 .$$

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