The Continuous Dynamic Robbins-Monro Procedure

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The proving methods developed in the book by Nevel'son and Has'minskij [4] are utilized to prove the asymptotic normality of the multidimensional continuous dynamic Robbins-Monro procedure, under assumptions similar to those usually made in the theory of stochastic approximation.

1. Suppose the zero point of a regression function is a time-varying parameter, its evolution law being known to certain extent; the Robbins-Monro stochastic approximation procedure can be then adapted to track this moving point. We shall consider the continuous-time case (investigated already by Cypkin [1] from the viewpoint of adaptive systems theory). Exploiting the proving methods developed (for the non-dynamic case) in Nevel'son and Has'minskij [3], [4], we obtain results concerning mean-square convergence, the rate of a.s. convergence, and asymptotic normality of the procedure.

2. We shall make the following assumptions:

(i) $R^0(t, x)$, $\sigma_r^0(t, x)$, $1 \leqq r \leqq k$, are continuous mappings of $[t_0, +\infty) \times E_l$ into E_l; $t_0 > 0$.

(ii) For every bounded region $D \subset [t_0, +\infty) \times E_l$, there is a $K_D > 0$ such that

$$|R^0(t, x) - R^0(t, y)| + \sum_{r=1}^{k} |\sigma_r^0(t, x) - \sigma_r^0(t, y)| \leqq K_D|x - y|$$

everywhere in D.

(iii) $x = 0$ is the unique zero point of $R^0(t, x)$ for all $t \geqq t_0$.

(iv) There is a positive definite matrix C and a $\lambda > 0$ such that $(CR^0(t, x), x) \leqq -\lambda(Cx, x)$, for all $x \in E_l$, $t \geqq t_0$.

(v) $\sum_{r=1}^{k} |\sigma_r^0(t, x)| \leqq K(1 + |x|)$, for all $x \in E_l$, $t \geqq t_0$, and some $K > 0$.

(vi) $\xi_r(t)$, $1 \leqq r \leqq k$, are independent (standard) Wiener processes, consistent with a non-decreasing family $\{\mathscr{F}_t, t \geqq t_0\}$ of σ-fields of events.

(vii) $Q(t)$ and $q(t)$, $\theta(t)$ are matrix-valued and vector-valued functions, respectively; Q, q continuous, θ differentiable, satisfying

$$\mathrm{d}\theta(t)/\mathrm{d}t = Q(t)\,\theta(t) + q(t)\,, \quad t \geqq t_0\,;$$

Q is known, R^0, σ_r^0, q, θ are unknown in general.

(viii) $R(t, x) = R^0(t, x - \theta(t))$, $\sigma_r(t, x) = \sigma_r^0(t, x - \theta(t))$, $1 \leqq r \leqq k$.

(ix) $a(t)$ is a (given) positive function, $t \geqq t_0$.

(x) $X^x(t)$ is the regular solution of the stochastic differential equation

$$\mathrm{d}X(t) = Q(t)\,X(t)\,\mathrm{d}t + a(t)\left(R(t, X(t))\,\mathrm{d}t + \sum_{r=1}^{k} \sigma_r(t, X(t))\,\mathrm{d}\xi_r(t)\right), \quad t \geqq t_0\,,$$

with the initial condition $X(t_0) = x$, $x \in E_l$.

This is the dynamic Robbins-Monro procedure for tracking $\theta(t)$, corresponding to a situation, when at time t, the values of $R(t, x)$ are observable with experimental errors $\sum_{r=1}^{k} \sigma_r(t, x)\,\dot{\xi}_r(t)$; the term $Q(t)\,X(t)\,\mathrm{d}t$ is a correction for trend in $\theta(t)$.

Theorem 1. Under the assumptions (i)–(x) and

$$\int_{t_0}^{\infty} a(t)\,\mathrm{d}t = +\infty\,, \quad |Q(t)| = o(a(t))\,, \quad |q(t)| = o(a(t)) \quad \text{for} \quad t \to \infty\,,$$

we have

$$X^x(t) - \theta(t) \to 0 \quad \text{for} \quad t \to \infty\,,$$

in the mean square.

Further assume:

(xi) $a(t) = a/t^\alpha$, $a > 0$, $1/2 < \alpha < 1$.

(xii) $|Q(t)| = o(1/t^\alpha)$, $|q(t)| = O(1/t^{3\alpha/2})$, $t \to \infty$.

(xiii) $R^0(t, x) = Bx + \delta(t, x)$, $|\delta(t, x)| = o(|x|)$ for $x \to 0$, uniformly in $t \in [t_0, +\infty)$; B is a matrix such that all its eigenvalues have negative real parts.

(xiv) $\lim_{t \to \infty, x \to 0} \sigma_r^0(t, x) = s_r$ exists.

(xv) $\lim_{t \to \infty} t^{3\alpha/2} q(t) = q_\infty$ exists (with $q_\infty = 0$ if $|q(t)| = o(t^{-3\alpha/2})$).

Theorem 2. Under the assumptions (i)–(xii), we have for any γ, $0 \leqq \gamma < \alpha - 1/2$,

$$t^{\gamma}(X^{x}(t) - \theta(t)) \to 0 \quad \text{a.s.} \quad \text{for} \quad t \to \infty\,.$$

Theorem 3. Unter the assumptions (i)–(xv), the asymptotic distribution of $t^{\alpha/2}(X^{x}(t) - \theta(t))$ for $t \to \infty$ is normal with mean value $a^{-1}B^{-1}q_{\infty}$ and covariance matrix $a \int_0^{\infty} e^{Bv} S e^{B^T v}\, dv$, with $S = \sum_{r=1}^{k} s_r s_r^T$.

Remark. The conditions (vii), (xii), (xv) are satisfied especially if $\theta(t) = bt^Q + c$ (Q a known matrix of constants, b, c unknown vectors) and $\alpha = 2/3$. The differential equation (vii) then becomes $d\theta/dt = Qt^{-1}\,\theta(t) - Qct^{-1}$, i.e., $Q(t) = Qt^{-1}$, $q(t) = -Qct^{-1}$, $q_{\infty} = -Qc$. If $Q = I$, we have the linear trend in $\theta : \theta(t) = bt + c$.

3. Proof of Theorem 1. Subtract $d\theta(t)$ from both sides of the equation (x); using (vii) and denoting $Z(t) = X(t) - \theta(t)$, $z = x - \theta(t_0)$, we get

$$(1)\quad dZ(t) = Q(t)\,Z(t)\,dt - q(t)\,dt + a(t)\,(R^0(t, Z(t))\,dt + \sum_{r=1}^{k} \sigma_r^0(t, Z(t))\,d\xi_r(t)),\ t \geqq t_0$$

$$Z(t_0) = z\,.$$

Let L be the differential operator corresponding to (1):

$$(2)\quad L = \partial/\partial t + (Q(t)\,z - q(t) + a(t)\,R^0(t, z), \partial/\partial z) + (1/2)\,a^2(t) \sum_{r=1}^{k} (\sigma_r^0(t, z), \partial/\partial z)^2\,.$$

Putting $V(z) = (Cz, z)$, C that of (iv), we have

$$(3)\quad L\,V(z) = 2a(t)\,(CR^0(t, z), z) + 2(CQ(t)\,z, z) - 2(Cq(t), z) + a^2(t) \sum_{r=1}^{k} (C\sigma_r^0, \sigma_r^0)\,.$$

The first term on the right is less than $-2\lambda\, a(t)\, V(z)$, according to (iv); all the other terms are bounded by $b(t)\,(1 + V(z))$, with $b(t) = o(a(t))$, which follows from $|Q| = o(a(t))$, $|q| = o(a(t))$, from (v) and from the inequality $|z| \leqq 1 + |z|^2$. Hence,

$$L\,V(z) \leqq -\lambda\, a(t)\, V(z) + b(t)\,, \quad t \geqq t_1\,,$$

$$\int_{t_0}^{\infty} a(t)\,dt = +\infty\,, \quad b(t) = o(a(t))\,, \quad V(z) \geqq K(z, z)\,.$$

(Here, as well as in the sequel, K with or without subscript will denote positive constants, possibly of different values in different formulas.)

According to Lemma 1.2 in Nevel'son, Has'minskij [3], the assertion of Theorem 1 follows.

Proof of Theorem 2. The first term on the right hand side of (3) is now less than $-2\lambda a t^{-\alpha} V(z)$, the second one is bounded by $\varepsilon(t)\, t^{-\alpha} V(z)$ with $\varepsilon(t) \searrow 0$, and the fourth one by $Kt^{-2\alpha}(1 + V(z))$; we have used (xi) and (xii). Using the inequality

$$(4) \qquad |z| \leqq \delta^{-1} t^{-\alpha/2} + \delta t^{\alpha/2} |z|^2 , \quad \delta > 0 ,$$

and (xii), we obtain a bound for the third term:

$$2|(Cq(t), z)| \leqq K_1 t^{-2\alpha} + \delta K_2 t^{-\alpha} V(z) ,$$

K_2 independent of δ; choosing δ sufficiently small, we get

$$(5) \qquad L\, V(z) \leqq -\lambda a t^{-\alpha} V(z) + K_3 t^{-2\alpha} , \quad t \geqq t_1 .$$

Now put $V_1(t, z) = t^{2\gamma} V(z) + t^{-\varepsilon}$ where

$$(6) \qquad 0 < \gamma < \alpha - 1/2 , \quad 0 < \varepsilon < 2(\alpha - \tfrac{1}{2} - \gamma) .$$

Obviously,

$$L\, V_1(t, z) = t^{2\gamma} L\, V(z) + 2\gamma t^{2\gamma - 1} V(z) - \varepsilon t^{-\varepsilon - 1} ;$$

inserting (5) for $L\, V(z)$, we have

$$L\, V_1(t, z) \leqq -\lambda a t^{2\gamma - \alpha} V(z) + K_3 t^{2\gamma - 2\alpha} + 2\gamma t^{2\gamma - 1} V(z) - \varepsilon t^{-\varepsilon - 1} .$$

The sum of terms containing $V(z)$ is negative for $t \geqq t_1$, since (xi) implies $2\gamma - \alpha > 2\gamma - 1$, and so is the sum of the remaining two terms, since (6) implies $-\varepsilon - 1 > 2\gamma - 2\alpha$. Hence, $L\, V_1(t, z) < 0$, $t \geqq t_2$.

According to Nevel'son Has'minskij [4], Corollary 3.8.1., $\{V_1(t, Z(t)), \mathscr{F}_t\}$ is a nonnegative supermartingale, which implies the a.s. existence of finite $\lim_{t \to \infty} V_1(t, Z(t))$, i.e., of finite $\lim_{t \to \infty} t^{2\gamma} V(Z(t))$. Hence, $t^{2\gamma} |Z(t)|^2 \to 0$ a.s., which entails the assertion of Theorem 2.

Proof of Theorem 3. Owing to the uniformity condition in (xiii), there are $\varepsilon > 0$ and $K > 0$ such that $|R^0(t, z)| \leqq K$ for all $|z| \leqq \varepsilon$ and $t \geqq t_0$. Let ε be chosen in such a way that also $|\sigma_r^0(t, z)| \leqq K_1$ for all $|z| \leqq \varepsilon$ and $t \geqq t_0$; this can be done owing to (i) and (xiv). With this ε, define (for $t \geqq t_0$)

$$(7) \qquad \hat{R}(t, z) = \begin{cases} R^0(t, z) , & |z| \leqq \varepsilon , \\ R^0(t, \varepsilon z/|z|)\, |z|/\varepsilon , & |z| < \varepsilon ; \end{cases}$$

$$\hat{\sigma}_r(t, z) = \begin{cases} \sigma_r^0(t, z) & |z| \leqq \varepsilon , \\ \sigma_r^0(t, \varepsilon z/|z|) , & |z| > \varepsilon , \end{cases} \qquad 1 \leqq r \leqq k ;$$

$$\hat{\delta}(t, z) = \hat{R}(t, z) - Bz .$$

Together with (1), consider the auxiliary equation

$$\text{(8)} \qquad d\hat{Z}(t) = Q(t)\,\hat{Z}(t)\,dt - q(t)\,dt + at^{-\alpha}(\hat{R}(t, \hat{Z}(t))\,dt + \sum_{r=1}^{k} \hat{\sigma}_r(t, \hat{Z}(t))\,d\xi_r(t)),$$
$$t \geqq s\,(\geqq t_0),$$

with the initial condition $\hat{Z}(s) = \zeta$, ζ being a $\mathscr{F}_s$-measurable random variable, $\mathsf{E}|\zeta|^2 < +\infty$. The corresponding differential operator is

$$L = \partial/\partial t + (Q(t)\,z - q(t) + at^{-\alpha}\,\hat{R}(t, z), \partial/\partial z) + (1/2)\,a^2 t^{-2\alpha} \sum_{r=1}^{k} (\hat{\sigma}_r(t, z), \partial/\partial z)^2 .$$

Put $V(z) = (Cz, z)$; we have as in (5)

$$LV(z) \leqq -\lambda a\,V(z) + Kt^{-2\alpha}, \quad t \geqq t_1 ;$$

hence (see Nevel'son, Has'minskij [4], formula 3.5.5, which is valid here, owing to the definition of $\hat{R}$, $\hat{\sigma}_r$)

$$\frac{d}{dt}\mathsf{E}\,V(\hat{Z}^{\zeta}(t)) = \mathsf{E}L\,V(\hat{Z}^{\zeta}(t)) \leqq -\lambda a t^{-\alpha}\mathsf{E}\,V(\hat{Z}^{\zeta}(t)) + Kt^{-2\alpha}, \quad t \geqq s .$$

From this differential inequality, we get (see Lemmas 1, 2. 3 in Dupač [2])

$$\text{(9)} \qquad \mathsf{E}|\hat{Z}^{\zeta}(t)|^2 \leqq K_1 t^{-\alpha} .$$

Denoting $\hat{Y}(t) = t^{\alpha/2}\,\hat{Z}^{\zeta}(t)$, we obtain from (8) the equation

$$d\hat{Y} = (\tfrac{1}{2}\alpha I t^{-1} + aBt^{-\alpha})\hat{Y}\,dt + Q(t)\,\hat{Z}t^{\alpha/2}\,dt - q(t)\,t^{\alpha/2}\,dt +$$
$$+ at^{-\alpha/2}\,\hat{\delta}(t, \hat{Z})\,dt + at^{-\alpha/2}\sum_{r=1}^{k}\hat{\sigma}_r(t, \hat{Z})\,d\xi_r(t), \quad t \geqq s,$$
$$\hat{Y}(s) = s^{\alpha/2}\zeta .$$

Its solution is

$$\hat{Y}(t) = t^{\alpha/2}\exp\{a(1-\alpha)^{-1}B(t^{1-\alpha} - s^{1-\alpha})\}\,\zeta +$$
$$\text{(10)} \qquad + \int_s^t (t/u)^{\alpha/2}\exp\{a(1-\alpha)^{-1}B(t^{1-\alpha} - u^{1-\alpha})\} \,.$$
$$.\,[(Q(u)\,\hat{Z}(u)\,u^{\alpha/2} - q(u)\,u^{\alpha/2} + a\,\hat{\delta}(u, \hat{Z})\,u^{-\alpha/2})\,du + au^{-\alpha/2}\sum_{r=1}^{k}\hat{\sigma}_r(u, \hat{Z})\,d\xi_r(u)] .$$

Disclosing the brackets, the integral in (10) splits into four ones; the first of them tends to zero in the mean and hence also in probability:

$$\text{(11)} \qquad \mathsf{E}\left|\int_s^t (t/u)^{\alpha/2}\exp\{a(1-\alpha)^{-1}B(t^{1-\alpha} - u^{1-\alpha})\}\;Q\hat{Z}u^{\alpha/2}\,du\right| \leqq$$

$$\leqq \int_s^t (t/u)^{\alpha/2} |\exp\{.\}| |Q| |\hat{Z}| u^{\alpha/2} \, du \leqq$$

$$\leqq K \int_s^t \exp\{-\lambda_1(t^{1-\alpha} - u^{1-\alpha})\} \, \varepsilon(u) \, u^{-\alpha} \, du \,, \quad \lambda_1 > 0 \,, \quad \varepsilon(u) \searrow 0 \,,$$

where we have utilized the properties of the matrix B and Q ((xiii) and (xii)) and the inequality (9); after the substitution $t^{1-\alpha} - u^{1-\alpha} = v$, the last line of (11) is transformed into

$$K(1-\alpha)^{-1} \int_0^{t^{1-\alpha}-s^{1-\alpha}} e^{-\lambda_1 v} \, \varepsilon(t(1 - v/t^{1-\alpha})^{1/(1-\alpha)}) \, dv \,,$$

which tends to 0 for $t \to \infty$.

The second integral can be written (in view of (xv)) as

$$- \int_s^t (t/u)^{\alpha/2} \exp\{.\} \, (q_\infty + \varepsilon_1(u)) \, u^{-\alpha} \, du \,, \quad \varepsilon_1(u) \searrow 0 \,;$$

the same substitution changes it into

$$- (1-\alpha)^{-1} \int_0^{t^{1-\alpha}-s^{1-\alpha}} (1 - v/t^{1-\alpha})^{-\alpha/(2-2\alpha)} \exp\{a(1-\alpha)^{-1} Bv\} \, .$$
$$. \, (q_\infty + \varepsilon_1(t(1 - v/t^{1-\alpha})^{1/(1-\alpha)})) \, dv \,,$$

which tends for $t \to \infty$ to

$$- q_\infty (1-\alpha)^{-1} \int_0^\infty \exp\{a(1-\alpha)^{-1} Bv\} \, dv = -q_\infty a^{-1} \int_0^\infty e^{Bw} \, dw =$$
$$= q_\infty a^{-1} B^{-1} \,.$$

The third integral, $a \int_s^t (t/u)^{\alpha/2} \exp\{.\} \, \hat{\delta} u^{-\alpha/2} \, du$, can be again shown to tend to 0 in probability (cf. Lemma 6 in Dupač [2]), as well as the integral

$$a \int_s^t (t/u)^{\alpha/2} \exp\{.\} \, u^{-\alpha/2} \sum_{r=1}^k (\hat{\sigma}_r(u, \hat{Z}) - s_r) \, d\xi_r(u)$$

(cf. the same paper, formulas (13), (14)).

As the first term in (10), $t^{\alpha/2} \exp\{.\} \, \zeta$, tends obviously to 0 owing to the properties of B, we get thus that the distribution of $\hat{Y}(t) - q_\infty a^{-1} B^{-1}$ is asymptotically equivalent to the distribution of

$$a \int_s^t (t/u)^{\alpha/2} \exp\{a(1-\alpha)^{-1} B(t^{1-\alpha} - u^{1-\alpha})\} \, u^{-\alpha/2} \sum_{r=1}^k s_r \, d\xi_r(u) \,,$$

which is, however, a Gaussian process with zero mean and a covariance matrix, which can be calculated in a straightforward way, using the same substitution as above, and shown to tend to $a \int_0^\infty e^{Bv} S\, e^{B^T v}\, dv$, for $t \to \infty$. The rest of the proof consists in proving the asymptotic equivalence of distributions of

$$t^{\alpha/2}(X^x(t) - \theta(t)) = t^{\alpha/2}\, Z^z(t) \quad \text{and of} \quad \hat{Y}(t) = t^{\alpha/2} Z^\zeta(t)$$

for properly related z and ζ; it is exactly the same as the end of the proof of the Theorem in Dupač [2].

It should be pointed out, that the proofs in the present paper as well as in the paper Dupač [2] more or less follow the pattern of proofs in Nevel'son, Has'minskij [4], Chapt. 6.

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