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Hierarchical Solution Concept for Static and Multistage Decision Problems with Two Objectives

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A new concept of the so called hierarchical solution is introduced for decision problems with two objectives. This concept is then applied to static and multistage decision problems, and sufficient existence and necessary optimality conditions are always derived. For a special class of linear multistage decision problems with both objectives being quadratic it is possible to obtain the explicite analytic form of hierarchical solutions. As an illustration, also a simple example is included which is solved in detail.

1. INTRODUCTION

In this contribution an alternative approach to the solution of decision problems with two objectives (cost functionals) is suggested. The main idea of this approach is based on the definition of a certain hierarchical structure of the original bicriterion optimization problem. This hierarchical structure enables us to define the so called concept of hierarchical solutions in a straightforward way.

In the last few years we can find a number of papers and studies dealing with multiobjective optimization problems either static or dynamic, e.g. see the references [1] – [9]. The mentioned papers consider mostly the so called noninferior (nondominated, efficient) solution type, which fact seems not to be too much practical, especially because such solution concept is in principal nonunique and leads to the whole family of noninferior solutions. This feature then causes considerable troubles in connection with the derivation of practically important methods for its determination. We are thus forced to impose additional requirements on the studied multiobjective decision problem so as to transform it into the another problem with a scalar (single) objective, which can be solved, at least principally, by standart methods. Some details concerning such approach and also other possibilities can be found again in [1] – [9].

In our approach we avoid these difficulties, because the used solution concept leads generally to the finite number (hopefully one) of possible hierarchical solutions. The situation is in this sense similar to that of classical optimization problems with a scalar objective. It is also clear that for this advantage we must pay a something. Namely, the hierarchical solution obtained by our method need not be generally also noninferior, i.e. there can exist such admissible decision which will further improve both given objectives. Anyhow, it is clearly better to arrive to the concrete result obtained by a certain clear-cut procedure, than to waste a lot of effort and time by applying a general theory with only a little hope to come to some practical conclusions.

The concept of a hierarchical solution is introduced and studied in the next section. The all ideas are illustrated on the problem of simultaneous minimization of two given real functions in finite dimensional space – the so called static case. This case is further explored in Section 3. We derive there sufficient existence conditions for the hierarchical solution of general constrained static optimization problems, and for the unconstrained case we include also necessary optimality conditions.

Through Section 4 we formulate a multistage (discrete) decision problem with two objectives and we study sufficient excistence conditions for this problem. It was possible to guarantee the existence of hierarchical solutions under general and mild assumptions. The most important results concerning the necessary optimality conditions for the hierarchical solution of multistage decision problems with two objectives are given in Section 5. These conditions were obtained applying the results of the author [10] for discrete systems with a state-dependent admissible control region and they can be also denoted as a two-level discrete maximum principle.

The obtained results are applied in Section 6 to the case of linear multistage decision problems with both objectives assumed to be quadratic. Thus a set of discrete equations was derived, some of which are Riccati-like, i.e. we encounter a discrete boundary value problem. As an illustration and also for the possibility of a comparison with other solution types we included a simple example of this type which is solved in detail. The computed hierarchical solution is compared with the set of noninferior solutions and also with the so called equilibrium solution if we apply game-theoretic approach.

2. CONCEPT OF HIERARCHICAL SOLUTION

In this section we shall introduce a concept of the hierarchical solution which will give us an important possibility to solve optimization problems with two objectives with the aid of a two-level optimization. First, let us consider a general decision problem with two objectives.

For this purpose denote as Ω the set of admissible decisions and assume that two objectives J_1 and J_2, $J_i : \Omega \to E^1$, $i = 1, 2$ are given. Here and henceforth E^n denotes Euclidean space of the dimension indicated by the superscript. Our aim is to find such decision $\hat{w} \in \Omega$ for which both objectives will be "as small as possible". This requirement is sufficient to obtain a meaningful solution only in the scalar case, i.e. when only one objective exists, but in the studied problem the solution concept must be further specified.

It is clear that in our problem each decision results in certain values of both objectives J_1 and J_2, and that they can be interpreted as a point in E^2. As we know, it is not generally possible to compare various points (vectors) in E^2 as we can in E^1. This is because of the fact that only a partial order can be introduced among various outcomes of our problem. For completeness and for later comparison let us consider the following simple case.

Definition 1. Let $J_i : \Omega \to E^1$, $i = 1, 2$. The decision $\hat{w} \in \Omega$ is said to be noninferior if for every other decision $w \in \Omega$,

$$J_i(w) \leqq J_i(\hat{w}), \quad i = 1, 2 \Rightarrow J_i(w) = J_i(\hat{w}), \quad i = 1, 2 .$$

In other words the just stated definition says that for a noninferior decision $\hat{w}$ we cannot find any admissible decision w which would decrease at least one of the objectives and in the same time which would not increase the other objective. It is also not very hard to see that, in principal, we obtain the whole one-parameter family of noninferior solutions which is denoted as a noninferior set. Further details and extensions concerning this solution concept can be found in the references [1]–[9].

In practical applications the mentioned "nonuniqueness" property resulting from Definition 1 causes serious troubles, as it is practically impossible to compute the whole noninferior set. Also if this set is known, we have no general rules how to choose a suitable noninferior decision unless some additional conditions are postulated, which determine a certain point (decision) from the noninferior set, e.g. see [7].

Through this paper we follow another approach. We assume that in the studied problem the given objectives have various importance. In such situations it is formally possible to asign properly chosen weights to the given objectives and to solve the resulting problem with a scalar objective, obtained as a weighted combination of the original objectives, applying the standart methods. However, we suggest still another way to achieve to the meaningful solution provided that the original problem can be structured in the sense described in the sequel. Especially in the case of two objectives is our approach easy to apply and from the computational point of view only well-known methods of the calculus of extrema are needed.

In our further considerations let us assume that the admissible decision set lies in certain finite dimensional space. Such assumption will be satisfied in many decision problems. All vectors, if not otherwise specified, are supposed to be column vectors.

Problem 1. Let $J_i : \Omega \to E^1$, $i = 1, 2$ with $\Omega \subset E^n$. The problem is to find such $\hat{w} \in \Omega$ for which both objectives J_1 and J_2 are minimized in a vector sense, e.g. in the sense indicated in Definition 1.

From the previous discussion we know that in this setting the vector minimization problem is very difficult to solve. So we shall assume that one of the objectives, say J_1, is more important, i.e. the objectives are hierarchically ordered. Further we assume that some, at least one, of the components of a decision vector w, i.e. $n \geqq 2$, play a crutial role when minimizing this preferred objective J_1, while the remaining components, again at least one, have smaller significience for this minimization and which are then supposed to be used while minimizing the slacking objective J_2. In practical problems this assumption can be either directly verified if it holds, or can be given by the practical background of the decision problem, or can be applied dividing the components of the decision vector artificially into two groups mentioned above.

Therefore we can without any loss of generality assume that the first m, $m \geqq 1$, components of w, i.e. vector $u \in E^m$, are significient for the minimization of J_1, and that the remaining p, $p \geqq 1$, $m + p = n$, components, i.e. vector $v \in E^p$, are used for the minimization of J_2. To simplify further considerations let us also assume that $\Omega = \Omega_u \times \Omega_v$, where $\Omega_u \subset E^m$ and $\Omega_v \subset E^p$. Hence,

$$w = \begin{pmatrix} u \\ v \end{pmatrix} \in \Omega \Leftrightarrow u \in \Omega_u \,, \quad v \in \Omega_v \,.$$

For example, such assumption will be satisfied in problems with $a_j \leqq w_j \leqq b_j$, $j = 1, \ldots, n$. Now we have to solve the following problem.

Problem 2. Let $J_i : \Omega_u \times \Omega_v \to E^1$, $i = 1,2$ with $\Omega_u \subset E^m$ and $\Omega_v \subset E^p$. The problem is to find $\tilde{u} \in \Omega_u$ and $\tilde{v} \in \Omega_v$ such that the objective J_1, resp. J_2, is minimized with respect to u, resp. with respect to v.

In this statement of the decision problem we have for a moment neglected the given hierarchical structure, i.e. that J_1 is the preferred objective. Then Problem 2 can be interpreted also as a problem with conflicting decisions – two decision-markers with different objectives, which is usually denoted as a two-player, nonzero-sum game, see e.g. [2]. The mostly used solution concept for such situations is the so called equilibrium solution.

Definition 2. Let $J_i : \Omega_u \times \Omega_v \to E^1$, $i = 1, 2$. The decision pair (u^*, v^*) satisfying

$$J_1(u^*, v^*) \leq \min_{u \in \Omega_u} J_1(u, v^*) \,,$$

$$J_2(u^*, v^*) \leq \min_{v \in \Omega_v} J_2(u^*, v) \,,$$

is denoted as an equilibrium solution.

The meaning of this definition is clear. Namely, if deviating from his equlibrium decision each decision-maker (player) can only loose provided that the opponent uses also corresponding equilibrium decision. Let us also remark that the existence of the equilibrium decisions (u^*, v^*) does not exclude the possibility of the existence of noninferior decisions $\hat{w} = (\hat{u}, \hat{v})$ which give the lower value of both objectives in comparison with the equilibrium decisions – the so called "prisoner's dilemma" situation in the language of the game theory. To these questions we shall return later in connection with the hierarchical solution concept.

Now let us study Problem 2 from the different point of view. In fact, we have to solve two parametric optimization (decision) problems. This means that for the given $u \in \Omega_u$ we are generally able to determine the set $V_0(u)$, (minimizing decision not necessarily unique) such that

$$V_0(u) = \{v \in \Omega_v | \; J_2(u, v) = \min_{\bar{v} \in \Omega_v} J_2(u, \bar{v})\}, \quad u \in \Omega_u . \tag{2.1}$$

The set-valued mapping $V_0(u)$ given by (2.1) can be evidently denoted as a reflective mapping for the decision set Ω_u. In a quite analogous way for the preferred objective being J_2 we can for any $v \in \Omega_v$ determine also the corresponding set of minimizing decisions $U_0(v)$, i.e.

$$U_0(v) = \{u \in \Omega_u | \; J_1(u, v) = \min_{\bar{u} \in \Omega_u} J_1(\bar{u}, v)\}, \quad v \in \Omega_v . \tag{2.2}$$

It is clear that we cannot proceed our reasoning further unless some additional requirements in Problem 2 are specified, which would relate the above mentioned parametric optimization problems. One way of doing it is to take advantage of the additionally assumed hierarchical order for the both objective in Problem 2 as discussed earlier in this section. Therefore we assume that J_1 is the preferred objective, while J_2 is the slacking one. It is required that J_2 is minimized with respect to $v \in \Omega_v$. Then we choose such decision $\tilde{u} \in \Omega_u$ which together with the corresponding decision $\tilde{v} \in V_0(\tilde{u})$ minimize the preferred objective, i.e. J_1 is minimized over the so called reflective set

$$\tilde{\Omega}_u = \{(u, v) | \; u \in \Omega_u , \quad v \in V_0(u)\} \tag{2.3}$$

where $V_0(u)$ is defined by (2.1), i.e. $\tilde{\Omega}_u$ is the graph of $V_0(u)$. Similarly also $\tilde{\Omega}_v$ can be defined if J_2 is the preferred objective:

$$\tilde{\Omega}_v = \{(u, v) | \; v \in \Omega_v , \quad u \in U_0(v)\} \tag{2.4}$$

with $U_0(v)$ determined according to (2.2).

This means that the preferred objective J_1 is minimized not only with respect to $u \in \Omega_u$, but also with respect to the corresponding set of minimizing decisions $V_0(u)$ for the slacking objective J_2. In this way we used the assumed hierarchical structure of Problem 2, and it resulted in certain discrimination of the less important objective J_2. Let us also remark that the mappings $U_0(v)$, $V_0(u)$ are generally denoted as multivalued mappings. This concept will be further specified in the next section when dealing with sufficient existence conditions for the hierarchical solution.

Definition 3. Let $J_i : \Omega_u \times \Omega_v \to E^1$, $i = 1, 2$ with $\Omega_u \subset E^m$ and $\Omega_v \subset E^p$. Further let J_1 be the preferred objective. The pair of decisions $(\tilde{u}, \tilde{v})$ minimizing the preferred objective over the set $\tilde{\Omega}_u$ given by (2.3) is denoted as a hierarchical solution of the decision problem with two objectives.

In this definition we implicitely assume that the minimization in (2.1) exists. Otherwise it would not be possible to define set $\tilde{\Omega}_u$ as given in (2.3).

To some extent the introduced concept of a hierarchical solution resembles the so called Stackelberg solution for the competitive situations as detailly discussed in [11]–[14]. However, it is necessary to note that in our case, besides of the evident conceptual reasons, we admit also a possibility of the nonunique "reflection", i.e. $V_0(u)$ can consist of more points. If such case arise in a game situation, the preferred player, denoted as a leader through [11]–[13], is not able to determine his Stackelberg strategy, because he do not know which $\tilde{v} \in V_0(\tilde{u})$ will be used by the other player-follower. Thus in this situations the Stackelberg solution concept cannot solve the problem completely.

Let us now assume $E = \tilde{\Omega}_u \cap \tilde{\Omega}_v$ to be nonempty. We immediately see that the set $E \subset \Omega_u \times \Omega_v$ contains all possible equilibrium solutions. Namely, if $(u^*, v^*) \in E$, then from (2.3) and (2.4) one obtains that

$$(2.5) \qquad \begin{aligned} (u^*, v^*) \in \tilde{\Omega}_u &\Rightarrow J_2(u^*, v^*) \leqq J_2(u^*, v)\,, \quad v \in \Omega_v\,, \\ (u^*, v^*) \in \tilde{\Omega}_v &\Rightarrow J_1(u^*, v^*) \leqq J_1(u, v^*)\,, \quad u \in \Omega_u\,. \end{aligned}$$

This is clearly nothing else than Definition 3.

In the light of the previous discussion we see that the hierarchical solution concept with J_1 as the preferred objective gives for this objective generally better outcome in comparison with equilibrium solutions. On the other hand, no such conclusion is possible for the slacking objective J_2, the value of which can be generally higher or lower than the equilibrium one depending on the particular decision problem.

For convenience, the hierarchical solution concept is illustrated in Fig. 1, where $\Omega_u = \Omega_v = E^1$. In the depicted case it is assumed that J_1 and J_2 are strictly convex and twice continuously differentiable, e.g. positive definite quadratic forms in u and v. In this case the reflective mapping $V_0(u)$, resp. $U_0(v)$, is a common single-valued function on Ω_u, resp. Ω_v. Set $\tilde{\Omega}_u$, resp. $\tilde{\Omega}_v$, is easily obtained as the collection of the all tangency points between the constant J_2, resp. J_1, contour lines and the lines of constant u, resp. v. If J_1 is the preferred objective we see that its minimum over $\tilde{\Omega}_u$ is achieved at the point $H_1 = (\tilde{u}, \tilde{v})$ which is therefore the hierarchical solution of this problem. Analogously, point H_2 will be the hierarchical solution provided that the objective J_2 will be preferred.

For comparison we have included in Fig. 1 also the corresponding equilibrium solution E, and the curve M_1M_2 denoting the set of all noninferior solutions. Point E is given as the intersection of $\tilde{\Omega}_u$ and $\tilde{\Omega}_v$, as indicated by (2.5). Curve M_1M_2 is the collection of the all "touching" points between constant J_1 contour lines with the constant J_2 contour lines. We further see that in the depicted case both possible hierarchical solutions strictly dominate the equilibrium solution. On the other hand, there exist noninferior solutions, which are given by the curve $D_1 D_2$, and which give lower values of both objectives than any of the hierarchical solutions H_1 or H_2. Such situation can be viewed as a typical one in the studied decision problems. Let us also remark that if this simple case is interpreted as a game, the hierarchical and Stackelberg solutions will coincide, e.g. see [12].

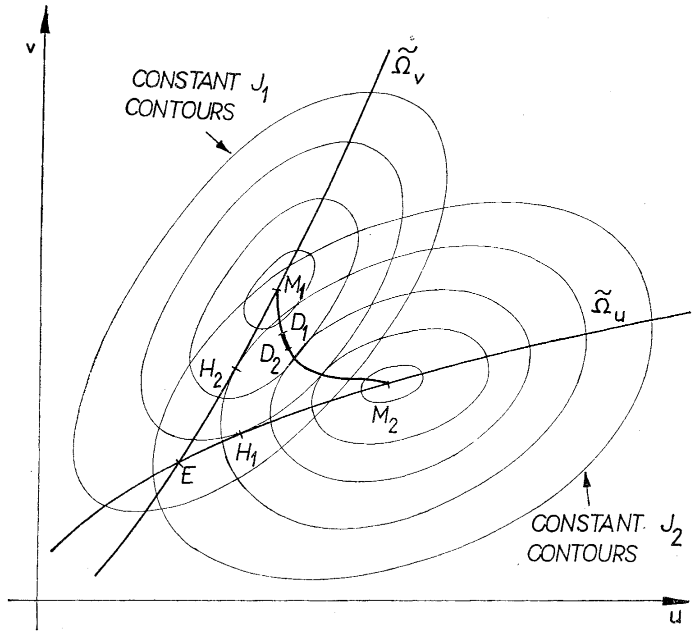

Fig. 1. Schematic plot for the hierarchical solution concept.

3. EXISTENCE AND OPTIMALITY CONDITIONS FOR STATIC CASE

Through this section we shall study static decision problems with two objectives from the point of view of a hierarchical solution as given in Definition 3. First we shall show that under certain general assumptions we are able to guarantee the existence of such hierarchical decision pair $(\tilde{u}, \tilde{v})$. For this purpose some fundamental concepts concerning the theory of multivalued mappings and mathematical programming will be necessary, which are briefly summerized.

We shall work with a finite dimensional decision problem denoted as Problem 3 through the previous section. However, this restriction can be considerably released, especially in the connection with sufficient existence conditions. The further presented concepts and results can be found in the monograph of Berge [15] and partially also in the paper of the author [16]. The all topological spaces used further are assumed to be Hausdorff and denoted as X, Y, Z etc., which is evidently fulfilled in our finite dimensional case. As $\mathscr{A}(X)$ we denote the collection of all nonempty subsets of X and as $\mathscr{C}(X)$ the collection of all nonempty and compact subsets of X. Now let us define the most important concepts.

Definition 4. The mapping F which associates with every point $x \in X$ a nonempty subset of Y, i.e. $F : X \to \mathscr{A}(Y)$, is called a multivalued mapping. If $A \subset X$, then we define $F(A) = \bigcup_{x \in A} F(x)$.

Definition 5. We say that the multivalued mapping $F : X \to \mathscr{A}(Y)$ is upper, resp. lower, semicontinuous at the point $x_0 \in X$ if for all open sets $\Gamma \subset Y$ with $F(x_0) \subset \Gamma$, resp. $F(x_0) \cap \Gamma \neq \emptyset$, there exists a neighbourhood O_{x_0} of x_0 such that for all $x \in O_{x_0}$ we have $F(x) \subset \Gamma$, resp. $F(x) \cap \Gamma \neq \emptyset$. We say that F is continuous at x_0 if F is both, upper and lower semicontinuous at x_0.

We say that F is upper semicontinuous, resp. lower semicontinuous, resp. continuous, if this is true at every point $x \in X$.

Further let us consider the following parametrized mathematical programming problem. Let $f : X \times Y \to E^1$ be a continuous real function and $G : X \to \mathscr{C}(Y)$ a multivalued mapping. Denote

$$m(x) = \min_{y \in G(x)} f(x, y) \tag{3.1}$$

and

$$F(x) = \{y \in Y \mid y \in G(x), \ f(x, y) = m(x)\} . \tag{3.2}$$

It is clear that due to the continuity of f and compactness of $G(x)$ the both expressions (3.1) and (3.2) are meaningful. From [15] we know that the following results hold.

Proposition 1. Let $F : X \to \mathscr{C}(Y)$ be upper semicontinuous and $\Gamma \subset X$ compact. Then the set $F(\Gamma)$ is also compact.

Proposition 2. Let $F : X \to \mathscr{C}(Y)$ be upper semicontinuous. Then its graph, i.e. the set

$$\tilde{F}\{(x, y) \in X \times Y \mid y \in F(x)\} ,$$

is closed.

Proposition 3. Let $f : X \times Y \to E^1$ be continuous real function and $G : X \to \mathscr{C}(Y)$ continuous multivalued mapping. Then the minimum value function $m(x)$ in (3.1) is continuous on X and its solution set function $F(x)$ in (3.2) is upper semicontinuous multivalued mapping from X to $\mathscr{C}(Y)$.

It is easy to see that if in Definition 5 the mapping F is a usual single-valued function, then all three continuity concepts introduced for multivalued mappings are equivalent and coincide with a classical continuity concept. Hence, if in Proposition 3 we additionally assume that for each $x \in X$ there exists just one $y \in G(x)$ which minimizes f, i.e. F is a single-valued mapping, we can conclude that F will be a continuous function in this case.

Now let us return to our decision problem. Here G is a constant multivalued mapping, i.e. $G(x) \equiv A \subset Y$ for all $x \in X$, which is clearly continuous. Hence, from Propositions 2 and 3 we can conclude that the next result is valid for Problem 2.

Corollary 1. Let J_2, resp. J_1, be a continuous function on $\Omega_u \times \Omega_v$ and let Ω_v, resp. Ω_u, be nonempty and compact set, and Ω_u, resp. Ω_v, nonempty and closed set. Then the multivalued mapping $V_0 : \Omega_u \to \mathscr{C}(\Omega_v)$, resp. $U_0 : \Omega_v \to \mathscr{C}(\Omega_u)$, is upper semicontinuous and the set $\tilde{\Omega}_u$, resp. $\tilde{\Omega}_v$, is therefore nonempty and closed.

For the definition of all necessary items in Corollary 1 see (2.1)–(2.4). With these preliminary results we can prove sufficient existence conditions for the hierarchical solution of Problem 2.

Theorem 1. Suppose that the objectives $J_i : E^m \times E^p \to E^1$, $i = 1,2$ are continuous and that the sets $\Omega_u \subset E^m$ and $\Omega_v \subset E^p$ are nonempty and compact. Finally let J_1 be the preferred objective. Then the corresponding hierarchical solution exists.

Proof. From Corollary 1 we known that the multivalued mapping $V_0(u)$ is upper semicontinuous. Hence the set $\Omega_u \times V_0(\Omega_u)$ is nonempty and compact by Proposition 1. Then $\tilde{\Omega}_u$ is clearly a nonempty and closed subset of the compact set $\Omega_u \times \times V_0(\Omega_u)$, and $\tilde{\Omega}_u$ is nonempty and compact. Therefore there exists a point $(\tilde{u}, \tilde{v}) \in \tilde{\Omega}_u$ for which the minimum of J_1 is attained. □

Remark 1. It is not very hard to see that Theorem 1 holds also in more general cases in which the admissible decision sets are elements of Hausdorff topological spaces. In this way the existence of hierarchical solutions can be guaranteed also for infinite dimensional decision problems with two objectives.

If the hierarchical decisions happen to lie in the interior of $\Omega_u \times \Omega_v$, or when $\Omega_u \times \Omega_v$ is the whole Euclidean E^n space, the necessary conditions for the existence of a hierarchical solution are obtained in a simple way. Recall that various gradients are treated as row-vectors.

Theorem 2. Let $J_i : E^m \times E^p \to E^1$, $i = 1, 2$ be twice continuously differentiable and let $\Omega_u \subset E^m$ and $\Omega_v \subset E^p$ be open. Further let J_1 be the preferred objective. If $(\tilde{u}, \tilde{v})$ is a hieararchical solution, then there exist a number $\mu \leqq 0$ and a row-vector multiplier $\lambda \subset E^p$ not both zero and such that the following conditions are satisfied:

$$\frac{\partial}{\partial v} J_2(\tilde{u}, \tilde{v}) = 0 , \tag{3.3}$$

$$\mu \frac{\partial}{\partial u} J_1(\tilde{u}, \tilde{v}) + \lambda \frac{\partial^2}{\partial v\, \partial u} J_2(\tilde{u}, \tilde{v}) = 0 , \tag{3.4}$$

$$\mu \frac{\partial}{\partial v} J_1(\tilde{u}, \tilde{v}) + \lambda \frac{\partial^2}{\partial v\, \partial v} J_2(\tilde{u}, \tilde{v}) = 0 , \tag{3.5}$$

where $\partial^2 J_2/\partial v\, \partial u$, resp. $\partial^2 J_2/\partial v\, \partial v$, denotes the $p \times m$, resp. $p \times p$, matrix with ij-th element given by $\partial^2 J_2/\partial v_i\, \partial u_j$, resp. $\partial^2 J_2/\partial v_i\, \partial v_j$.

Proof. The condition (3.3) is the necessary one for the decision pairs $(\tilde{u}, \tilde{v})$ to be in $\tilde{\Omega}_u$. Thus from the point of view of the necessary optimality conditions for a hierarchical solution we have to minimize J_2 subject to (3.3). But this is nothing else than a mathematical programming problem with equality constraints, for which the stated conditions (3.4) and (3.5) must hold – see [17]. □

The theorem just stated is interesting only provided that $\mu \neq 0$. This can be achieved if the so called constraint qualification is satisfied, e.g. see [17]. In our case we obtain the following result.

Corollary 2. Suppose that the $(m + p)$-dimensional vectors

$$\left[\frac{\partial}{\partial u}\left(\frac{\partial}{\partial v_i} J_2(u, v)\right), \ \frac{\partial}{\partial v}\left(\frac{\partial}{\partial v_i} J_2(u, v)\right)\right], \quad i = 1, \ldots, p$$

are linearly independent for $u \in E^m$, $v \in E^p$. Then the constraint qualification is satisfied in Theorem 2, i.e. we can put $\mu = -1$ in (3.4) and (3.5).

4. MULTISTAGE HIERARCHICAL DECISION PROBLEMS

Further we shall study the so called multistage decision problems with two objectives. Such problems can be also denoted as discrete optimal control problems. In previous author's papers [10] and [16] the multistage optimal control (decision) problems with only one objective were studied in detail. Here an attempt is made to apply the hierarchical solution concept also to multistage systems with two objectives (cost functionals).

To begin, let us assume that the behaviour of the system in question is described by difference equation

(4.1) $$x_{k+1} = f_k(x_k, u_k, v_k), \quad k = 0, 1, \ldots, K - 1, \quad x_0 \text{ given},$$

where a positive integer K denotes the prescribed number of stages, $x_k \in E^n$ denotes state of the system at the stage k, $u_k \in E^m$, $v_k \in E^p$ are controls (decisions) at the stage k and $f_k : E^n \times E^m \times E^p \to E^1$. As usual, x_k, u_k and v_k are assumed to be column-vectors.

The cost functionals (objectives) J_1 and J_2 are given by the relations

(4.2) $$J_i = g^i(x_k) + \sum_{k=0}^{K-1} h_k^i(x_k, u_k, v_k), \quad i = 1, 2,$$

where $g^i : E^n \to E^1$ and $h_k^i : E^n \times E^m \times E^p \to E^1$. Here and henceforth it is assumed that J_1 is the preferred functional.

Admissible controls must satisfy the constraints:

(4.3) $$u_k \in U_k, \quad k = 0, 1, \ldots, K - 1,$$

(4.4) $$v_k \in V_k, \quad k = 0, 1, \ldots, K - 1,$$

where $U_k \subset E^m$ and $V_k \subset E^p$, In this case as admissible decisions we denote the control sequences $u = \{u_0, u_1, \ldots, u_{K-1}\}$ and $v = \{v_0, v_1, \ldots, v_{K-1}\}$ satisfying (4.3) and (4.4), respectively. As an admissible process let us denote a triplet (x, u, v), where $x = \{x_0, x_1, \ldots, x_K\}$ is the state trajectory of the system, satisfying (4.1), (4.3) and (4.4).

First let us study the existence of hierarchical solutions for the given multistage system (4.1)–(4.4). Assume that all functions appearing in (4.1) and (4.2) are continuous and the admissible control regions (4.3) and (4.4) nonempty and compact.

To be able to apply Theorem 1 directly let us transcribe the original problem (4.1)–(4.4) in the following way. It is evident that if (4.1) is successively used we obtain that

(4.5) $$x_{k+1} = \tilde{f}_k(x_0, u_1, u_2, \ldots, u_k, v_0, v_1, \ldots, v_k), \quad k = 0, 1, \ldots, K - 1.$$

As the initial state x_0 is given, it follows from (4.5) that for the cost functionals (4.2) one can write:

(4.6) $$J_i = J_i(u, v),$$

where u and v are admissible decisions (control sequences), i.e.

(4.7) $$u \in U = \prod_{k=0}^{K-1} U_k, \quad v \in V = \prod_{k=0}^{K-1} V_k.$$

It is clear that U and V are nonempty and compact subsets of E^{mK} and E^{pK}, respectively, and that both cost functionals J_i, $i = 1, 2$ are continuous. From Theorem 1 we immediately obtain that the hierarchical solution for the studied multistage problem exists. We summarize this result in the following theorem.

Theorem 3. Suppose that a multistage decision problem with two objectives is described by (4.1)–(4.4). Let us further assume that J_1 is the preferred objective and that

(a) the functions f_k, h_k^i, g^i, $k = 0, 1, \ldots, K - 1$, $i = 1, 2$, are continuous in the corresponding domains of definition,

(b) the sets U_k, V_k, $k = 0, 1, \ldots, K - 1$ are nonempty and compact.

Then the hierarchical solution exists.

Remark 2. It is easy to see that Theorem 3 remains valid also if the initial state x_0 is not given, but it is only required $x_0 \in A \subset E^n$, where A is nonempty and compact initial set. On the other hand, it is not possible to prove such existence theorem for more general cases, e.g. with state constraints or with state-dependent admissible control regions, applying this simple reasoning or the scheme described in [16]. The main difficulty lies in the fact that the finite intersection of lower semicontinuous mappings is not necessarily lower semicontinuous and we cannot apply Theorem 1 directly.

5. NECESSARY OPTIMALITY CONDITIONS FOR MULTISTAGE CASE

In this section we shall derive necessary optimality conditions for the hierarchical solution of multistage decision problems. For the sake of simplicity we assume the unconstrained case, i.e. $U_k = E^m$ and $V_k = E^p$, $k = 0, 1, \ldots, K - 1$ in (4.3) and (4.4), respectively. In principle, it is then possible to use Theorem 2 in connection with the transcription (4.5)–(4.7), but the very high dimensionality and complexity of the resulting problem will prevent us to solve such problem in a general case.

Our position is the same as when studying problems with one cost functional in [10] and [17], where the resulting mathematical programming problem is decomposed and a discrete maximum principle obtained. Therefore we shall try to use the results of [10] as much as possible, in order to derive the necessary optimality conditions for hierarchical solutions in a more straightforward way. For this purpose certain technical assumptions will be needed.

Assumption 1. All function appearing in (4.1) and (4.2) are twice continuously differentiable in their domains of definition.

Further we shall proceed quite analogously as in the static case – see Theorem 2. Analogical to (3.3) are now necessary optimality conditions for minimization of J_2 with respect to control sequence v, while control sequence u is being fixed. To apply results from [10] or [17], certain convexity properties must hold.

Definition 6. Let e be any vector in E^n. A set $\Omega \subset E^n$ is said to be e-directionally convex if for every vector w^* in the convex hull of Ω there exists a vector $w \in \Omega$ such that $w = w^* + \alpha e$, $\alpha \geqq 0$.

As long as usually only the cost functionals without the terminal part are treated, we perform the following substitution for both cost functionals. Namely, we introduce for $i = 1, 2$ the functions

$$\hat{h}^i_k(x, u, v) \quad = h^i_k(x, u, v)\,, \quad k = 0, 1, \dots, K - 2\,,$$

$$\hat{h}^i_{K-1}(x, u, v) = h^i_{K-1}(x, u, v) + g^i(f_{K-1}(x, u, v))\,. \tag{5.1}$$

Now we can define functions $\mathscr{F}_k : E^n \times E^m \times E^p \to E^{n+1}$ by the relation

$$\mathscr{F}_k(x, u, v) = \begin{pmatrix} \hat{h}^2_k(x, u, v) \\ f_k(x, u, v) \end{pmatrix} k = 0, 1, \dots, K - 1\,. \tag{5.2}$$

Finally, let us consider in E^{n+1} sets

$$W_k(x, u) = \{w \in E^{n+1} \mid w = \mathscr{F}_k(x, u, v),\ v \in E^p\}\,, \quad k = 0, 1, \dots, K - 1\,, \tag{5.3}$$

and a vector $e = (-1, 0, \dots, 0)$.

Assumption 2. For every $x \in E^n$ and $u \in E^m$ the sets $W_k(x, u)$, $k = 0, 1, \dots, K - 1$ given by (5.3) are e-directionally convex.

The necessary conditions for (u, v) to belong to the corresponding reflective set (recall that J_1 is the preferred objective) are then given as necessary optimality conditions for the discrete optimal control problem with J_2 as a cost functional, v as a control and u as a parameter. From [10] it follows that there exist row-vectors $\lambda_k \in E^n$, $k = 0, 1, \dots, K$ satisfying

$$\lambda_k = \frac{\partial}{\partial x} H^2_{k+1}(x_k, u_k, v_k)\,, \quad k = 0, 1, \dots, K - 1\,, \tag{5.4}$$

$$\lambda_K = -\frac{\partial}{\partial x} g^2(x_K)\,,$$

where the Hamiltonian is written as

$$H^2_{k+1}(x, u, v) = -h^2_k(x, u, v) + \lambda_{k+1} f_k(x, u, v)\,, \quad k = 0, 1, \dots, K - 1\,, \tag{5.5}$$

and

$$\frac{\partial}{\partial v} H_{k+1}^2(x_k, u_k, v_k) = 0\,, \quad k = 0, 1, \ldots, K-1\,. \tag{5.6}$$

Here $x_0, x_1, \ldots, x_K$ denote the state trajectory corresponding to (u, v). In this way the relations (4.1), (5.4) and (5.6) are the multistage analogy of condition (3.3) and are, in fact, a discrete maximum principle for the above stated discrete optimal control problem. Therefore the hierarchical solution $(\tilde{u}, \tilde{v})$ must necessarily satisfy these relations and in the same time the cost functional J_1 should be minimized. This is a nonstandart form of the discrete optimal control problem and hence some preliminary steps will be necessary before we can apply the results from [10].

Assumption 3. The $n \times n$ matrices

$$\frac{\partial}{\partial x} f_k(x, u, v)\,, \quad k = 0, 1, \ldots, K-1$$

are regular for every $x \in E^n$, $u \in E^m$ and $v \in E^p$.

From (5.4) and (5.6) we then obtain (arguments are dropped out for simplicity)

$$\lambda_{k+1} = \left(\lambda_k + \frac{\partial h_k^2}{\partial x}\right)\left(\frac{\partial f_k}{\partial x}\right)^{-1} = F_k^T(x_k, \lambda_k, u_k, v_k)\,, \tag{5.7}$$

$$\lambda_K = -\frac{\partial g^2}{\partial x}\,,$$

$$\frac{\partial H_{k+1}^2}{\partial v} = -\frac{\partial h_k^2}{\partial v} + \left(\lambda_k + \frac{\partial h_k^2}{\partial x}\right)\left(\frac{\partial f_k}{\partial x}\right)^{-1}\frac{\partial f_k}{\partial v} = G_k^T(x_k, \lambda_k, u_k, v_k) = 0\,, \tag{5.8}$$

where $k = 0, 1, \ldots, K-1$. Here F_k and G_k denote n-dimensional and p-dimensional functions, respectively, and T denotes transposition. Constraints (4.1) and (5.7) can be written in a more compact form

$$y_{k+1} = \begin{pmatrix} x_{k+1} \\ \lambda_{k+1}^T \end{pmatrix} = \begin{pmatrix} f_k(y_k, u_k, v_k) \\ F_k(y_k, u_k, v_k) \end{pmatrix}, \quad k = 0, 1, \ldots, K-1\,. \tag{5.9}$$

Necessary optimality conditions for the hierarchical solution are then obtained as necessary optimality conditions for the discrete optimal control problem with the cost functional J_1 subject to (5.8) and (5.9), where u and v are the controls, and with boundary conditions

$$x_0 \text{ given,} \quad \lambda_K = -\frac{\partial}{\partial x} g^2(x_K)\,. \tag{5.10}$$

The resulting problem is a discrete optimal control problem with a state-dependent admissible control region (5.8). Such problems were studied in the paper [10] from which we know that certain regularity assumption for the constraints (5.8) is necessary.

Assumption 4. For every $k = 0, 1, \dots, K-1$ the $(m+p)$-dimensional row-vectors (gradients of the components of G_k)

$$\left(\frac{\partial G_k^j}{\partial u}, \frac{\partial G_k^j}{\partial v}\right), \quad j = 1, \dots, p$$

are linearly independent for every $x \in E^n$, $u \in E^m$ and $v \in E^p$.

Using (5.1) we define functions $\mathscr{G}_k : E^n \times E^n \times E^m \times E^p \to E^{2n+1}$

$$\mathscr{G}_k(x, \lambda, u, v) = \begin{pmatrix} \hat{h}_k^1(x, u, v) \\ f_k(x, u, v) \\ F_k(x, \lambda, u, v) \end{pmatrix}, \quad k = 0, 1, \dots, K-1,$$

where $\hat{h}_k^1$ is defined by (5.1). Consider in E^{2n+1} sets

$$Z_k(x, \lambda) = \{z \in E^{2n+1} \mid z = \mathscr{G}_k(x, \lambda, u, v),\ u \in E^m,\ v \in E^p\}, \quad k = 0, 1, \dots, K-1 \tag{5.11}$$

and a vector $\hat{e} = (-1, 0, \dots, 0)$.

Assumption 5. For every $x \in E^n$ and $\lambda \in E^n$ the sets $Z_k(x, \lambda)$, $k = 0, 1, \dots, K-1$ given by (5.11) are $\hat{e}$-directionally convex.

Under Assumptions 1–5 the results of [10] can be applied, but from the practical reasons let us make one additional assumption. The terminal condition for λ_K in (5.10) must be clearly treated as a state constraint. However, in this case we are not allowed to put the multiplier -1 in the definition of the corresponding Hamiltonian as it was done in (5.5). On the other hand, the case with this multiplier equal to zero is pathological, and of no interest from both, practical and computational point of view. Hence we impose the following "normality" assumption, which is not restrictive from the practical view-point.

Assumption 6. For the discrete optimal control problem with the cost functional J_1 and subject to constraints (5.8)–(5.10) the corresponding Hamiltonian can be written as

$$\mathscr{H}_{k+1}(x, \lambda, v) = -h_k^1 + \mu_{k+1} f_k(x, u, v) + \nu_{k+1} F_k(x, \lambda, u, v), \quad k = 0, 1, \dots, K-1, \tag{5.12}$$

where μ and ν are n-dimensional row-vector multipliers.

Now the basic result of [10, Theorem 5] can be rigorously applied to our problem. After some straightforward manipulations, omitted here for the sake of brevity, we obtain the desired necessary optimality conditions for the hierarchical solution of multistage decision problems with two objectives. These conditions are summarized in the following theorem, where we use the notation

$$H^1_{k+1}(x, u, v) = -h^1_k(x, u, v) + \mu_{k+1} f_k(x, u, v), \quad k = 0, 1, \ldots, K-1 .$$

Theorem 4. Consider a multistage decision problem with two objectives (4.1) and (4.2), where J_1 is the preferred one. Further suppose that the Assumptions 1–6 are fulfilled and that the pair $(\tilde{u}, \tilde{v})$ is a hierarchical solution of this problem. The corresponding state trajectory let us denote as $x_0, x_1, \ldots, x_K$.

Then there exist row-vector multipliers λ_k, μ_k, ν_k belonging to E^n, $k = 0, 1, \ldots, K$ and $\zeta_k \in E^p$, $k = 0, 1, \ldots, K-1$ such that the following conditions are satisfied.

a) $$\lambda_k = \frac{\partial H^2_{k+1}}{\partial x}, \quad \lambda_K = -\frac{\partial g^2}{\partial x};$$

b) $$\mu_k = \frac{\partial H^1_{k+1}}{\partial x} + \nu_{k+1}\frac{\partial F_k}{\partial x} + \zeta_k \frac{\partial G_k}{\partial x}, \quad \mu_K = -\frac{\partial g^1}{\partial x} + \nu_K \frac{\partial^2 g^2}{\partial x^2};$$

c) $$\nu_{k+1} = \nu_k \left(\frac{\partial f_k}{\partial x}\right)^T - \zeta_k \left(\frac{\partial f_k}{\partial v}\right)^T, \quad \nu_0 = 0;$$

d) $$\frac{\partial H^2_{k+1}}{\partial v} = 0;$$

e) $$\frac{\partial H^1_{k+1}}{\partial u} + \nu_{k+1}\frac{\partial F_k}{\partial u} + \zeta_k \frac{\partial G_k}{\partial u} = 0;$$

f) $$\frac{\partial H^1_{k+1}}{\partial v} + \nu_{k+1}\frac{\partial F_k}{\partial v} + \zeta_k \frac{\partial G_k}{\partial v} = 0,$$

where always $k = 0, 1, \ldots, K-1$ and all expressions are evaluated at the corresponding values of x_k, $\tilde{u}_k$, $\tilde{v}_k$ and λ_k.

Remark 3. Looking through the Theorem 4 it is not very difficult to see that Assumption 1 can be somewhat released, e.g. functions g^1, h^1_k, $k = 0, 1, \ldots, K-1$ can be only continuously differentiable, or functions $\partial f_k/\partial u$, $\partial h^2_k/\partial u$, $k = 0, 1, \ldots, K-1$ can be only continuous with respect to u. However, we cannot neglect any of the Assumptions 2–5, because then the described approach would not be valid.

6. LINEAR MULTISTAGE SYSTEM WITH QUADRATIC COST FUNCTIONALS

Fairly deep results can be obtained if we assume that the system (4.1) is linear and that the cost functionals (4.2) are quadratic. In this case we are able to derive an analytical scheme for the computation of hierarchical solutions in general. Relations (4.1) and (4.2) are then replaced by

$$x_{k+1} = Ax_k + B_1u_k + B_2v_k\,, \quad k = 0, 1, \ldots, K - 1\,, \tag{6.1}$$

$$J_i = \tfrac{1}{2}x_K^TM_ix_K + \tfrac{1}{2}\sum_{k=0}^{K-1} x_k^TQ_ix_k + u_k^TR_{i1}u_k + v_k^TR_{i2}v_k\,, \quad i = 1, 2\,. \tag{6.2}$$

Again we assume that the initial state x_0 is given. The dimensions of x, u and v are the same as in preceding sections. In this way also all dimensions of various matrices in (6.1) and (6.2) are determined.

The multistage decision problem described by (6.1) and (6.2) is assumed to be autonomous, i.e. its parameters does not vary with k. The only reason for this assumption is to avoid notational complexity without any substantial gain. Not loosing any generality we may also assume that the matrices M_i, Q_i, R_{ij}, $i, j = 1, 2$ are symmetric. Further we assume that the matrix A is regular, the matrices M_i, Q_i, R_{ij}, $i, j = 1, 2$, $i \neq j$ positive semidefinite and the matrices R_{ii}, $i = 1, 2$ positive definite. Under these hypotheses the Assumptions $1-5$ will be satisfied as can be readily verified. The "normality" requirement in Assumption 6 would need a longer discussion of the corresponding quadratic programming problem. However, this is not the purpose of the presented paper and we, therefore, suppose that Assumption 6 is fulfilled, i.e. the studied problem is meaningful.

The necessary optimality conditions of Theorem 4 have now a simple form and result in the solution of the following discrete linear boundary value problem:

(6.3)

$$\left.\begin{aligned} x_{k+1} &= Ax_k + B_1u_k + B_2v_k\,, \\ \lambda_k &= -x_k^TQ_2 + \lambda_{k+1}A\,, \\ \mu_k &= -x_k^TQ_1 + \mu_{k+1}A + \nu_{k+1}(A^{-1})^T + \zeta_kB_2^T(A^{-1})^T\,Q_2\,, \\ \nu_{k+1} &= \nu_kA^T - \zeta_kB_2^T\,, \end{aligned}\right\}\ k = 0, 1, \ldots, K - 1\,,$$

where

$$\left.\begin{aligned} u_k &= R_{11}^{-1}B_1^T\mu_{k+1}^T\,, \\ v_k &= R_{22}^{-1}B_2^T\lambda_{k+1}^T\,, \\ \zeta_k &= (\mu_{k+1}B_2 - \lambda_{k+1}B_2R_{22}^{-1}R_{12})R_{22}^{-1}\,, \end{aligned}\right\}\ k = 0, 1, \ldots, K - 1 \tag{6.4}$$

 and with boundary conditions

$$(6.5)\qquad \begin{aligned} & x_0 \text{ given} \\ & \lambda_K = -x_K^T M_2 \\ & \mu_K = -x_K^T M_1 + \nu_K M_2 \\ & \nu_0 = 0 \,. \end{aligned}$$

For the solution of the stated boundary value problem let us assume that the varying unknown multipliers λ, μ and ν depend linearly on the state variable x, i.e.

$$(6.6)\qquad \lambda_k = x_k^T P_k\,, \quad \mu_k = x_k^T N_k\,, \quad \nu_k = x_k^T S_k\,, \quad k = 0, 1, \ldots, K_1$$

where the $n \times n$ matrices P_k, N_k, S_k are to be determined. Such assumption is quite often used when solving this type of discrete optimization problems. If we use (6.6) through (6.3)–(6.5) we obtain that the hierarchical solution $(\tilde{u}, \tilde{v})$ of our decision problem with the preferred cost functional J_1 satisfies the following relations:

$$(6.7)\qquad \left.\begin{aligned} \tilde{u}_k &= R_{11}^{-1} B_1^T N_{k+1}^T W_k \Gamma_k x_0\,, \\ \tilde{v}_k &= R_{22}^{-1} B_2^T P_{k+1}^T W_k \Gamma_k x_0\,, \end{aligned}\right\} k = 0, 1, \ldots, K-1\,,$$

where

$$(6.8)\qquad \left.\begin{aligned} P_k &= -Q_2 + W_k^T P_{k+1} A\,, \\ N_k &= -Q_1 + W_k^T N_{k+1} A + S_k Q_2\,, \\ S_{k+1} &= (W_k^T)^{-1} S_k A^T - P_{k+1} B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T + \\ &\quad + N_{k+1} B_2 R_{22}^{-1} B_2^T\,, \end{aligned}\right\} k = 0, 1, \ldots, K-1$$

with boundary conditions

$$(6.9)\qquad \begin{aligned} P_K &= -M_2 \\ N_K &= -M_1 + S_K M_2\,, \\ S_0 &= 0\,. \end{aligned}$$

For convenience we have denoted

$$(6.10)\qquad W_k = [1 - B_1 R_{11}^{-1} B_1^T N_{k+1}^T - B_2 R_{22}^{-1} B_2^T P_{k+1}^T]^{-1} A\,, \quad k = 0, 1, \ldots, K = 1\,,$$

where 1 denotes $(n \times n)$-dimensional unit matrix. Finally, the $(n \times n)$-dimensional matrices Γ_k, $k = 0, 1, \ldots, K$ are obtained as a fundamental solution of matrix difference equation

$$(6.11)\qquad \Gamma_{k+1} = [A + (B_1 R_{11}^{-1} B_1^T N_{k+1}^T + B_2 R_{22}^{-1} B_2^T P_{k+1}^T)\, W_k]\, \Gamma_k\,, \quad k = 0, 1, \ldots, K-1$$

with $\Gamma_0 = 1$, i.e.

(6.12) $$x_k = \Gamma_k x_0\,, \quad k = 0, 1, \ldots, K\,.$$

For the presented construction it is necessary that the inversion indicated in (6.10) exists, which fact is assumed. Otherwise it is not possible to apply the relations (6.7) to (6.9). The first two equations in (6.8)–(6.9) coincide with the discrete Riccati equations for the equilibrium solution except of the terms which contain S_k, see [18]. These terms resulted from the constrained minimization of J_1 over the corresponding reflective set. Both these equations are solved backwards as usual for the adjoint variables (multipliers). However, the equation for S_k is not Riccati-like and is solved forward in the discrete time k. This last conclusion is in accordance with the original boundary value problem (6.3)–(6.5), because for the multiplier ν the initial condition $\nu_0 = 0$ is specified.

Remark 4. Inserting formally from (6.12) into (6.7) we obtain the optimal hierarchical solution in a "closed-loop" form. However, having in mind certain conclusions stated in [13], our result can be interpreted only as a synthesis of "open-loop" hierarchical solutions. In fact, let us point out that through this paper only the open-loop hierarchical solution of multistage decision problems was considered and studied.

Remark 5. It would be clearly interesting to study also such cases of (6.2) which would contain the mixed terms of the type $u_k^T T_i v_k$, $i = 1, 2$. Principially the same approach is possible for such problems, but the resulting conditions are of rather great complexity, and thus not too much practical. Let us also note that as long as J_1 is the preferred cost functional, the term $u_k^T R_{21} u_k$ does not influence the hierarchical solution, i.e. R_{21} does not appear through (6.7)–(6.11).

Remark 6. Finally let us explore a possibility of the application of Theorem 3 to study the existence conditions for a hierarchical solution in this class of multistage decision problems. As now the admissible control regions are unbounded, the hypothesis of Theorem 3 is not satisfied. Hence this question must be investigated separately for each particular problem. We only note that the reflective mapping is linear in this case and, further, both cost functionals (6.2) are quadratic if we use the static interpretation (4.5)–(4.7). The existence problem for hierarchical solutions then results in problem of finding the necessary conditions for J_1 to be either positive definite quadratic form on $E^{(m+p)K}$ or at least on the reflective set (constraining subspace).

7. EXAMPLE

As an illustration we solve in this section a simple example of the multistage decision problem from the point of view of the hierarchical solution. For comparison also the equilibrium and noninferior solution concepts are applied to the same problem. All variables will be scalars and J_1 is again the preferred cost functional.

$$x_{k+1} = x_k + u_k + v_k\,, \quad k = 0, 1, \ldots, K-1\,, \quad x_0 \text{ given}\,, \tag{7.1}$$

$$J_1 = \tfrac{1}{2}x_K^2 + \tfrac{1}{2}\sum_{k=0}^{K-1} u_k^2\,, \quad J_2 = \tfrac{1}{2}x_K^2 + \tfrac{1}{2}\sum_{k=0}^{K-1} v_k^2\,. \tag{7.2}$$

In this really academic example it is advisable to prefer general conditions of Theorem 4 to the scheme given in the previous section. We omit the obvious manipulations and state only the final results. Let us only note the fact, that in this case Assumption 6 is a priori satisfied. Really, if we assume that the problem in question is not normal, we obtain that the multipliers μ, ν and ζ are identically zero, which contradicts with the general conclusions of [10]. In this way we obtain

$$\left.\begin{aligned} \tilde{u}_k &= -\frac{1}{K^2 + 3K + 1}\,x_0\,, \\ \tilde{v}_k &= -\frac{K+1}{K^2 + 3K + 1}\,x_0\,, \end{aligned}\right\} \; k = 0, 1, \ldots, K-1\,. \tag{7.3}$$

The corresponding values of the cost functionals are:

$$\tilde{J}_1 = \frac{1}{2}\,\frac{1}{K^2 + 3K + 1}\,x_0^2\,, \quad \tilde{J}_2 = \frac{1}{2}\,\frac{(K+1)^3}{(K^2 + 3K + 1)^2}\,x_0^2\,. \tag{7.4}$$

As far as the question of existence of the hierarchical solution is concerned it can be easily checked that the last conclusion in Remark 6 applies to this simple case. The cost functional is a quadratic function of the control sequence $u_0, u_1, \ldots, u_{K-1}$ over the reflective set with the corresponding matrix being positive definite. Hence, there exists a unique hierarchical solution in this case which is described by the above relations (7.3).

If now (7.1)–(7.2) are considered as a two-player, nonzero-sum multistage game, we can compute the equilibrium solution according to Definition 2. The necessary theory can the reader find in [18]. For the equilibrium pair (u^*, v^*) we thus obtain

$$u_k^* = v_k^* = -\frac{1}{2K+1}\,x_0\,, \quad k = 0, 1, \ldots, K-1\,, \tag{7.5}$$

and for equilibrium costs

$$J_1^* = J_2^* = \frac{1}{2}\,\frac{K+1}{(2K+1)^2}\,x_0^2\,. \tag{7.6}$$

It is a simple exercise to show that

$$\tilde{J}_1 < J_1^*, \quad \tilde{J}_2 > J_2^* \tag{7.7}$$

for general number of stages K. This result confirms the general conclusions of Section 2. It is also worth to note that the hierarchical controls corresponding to the preferred, resp. slacking, cost functional are strictly lower, resp. higher, than the equilibrium ones.

For the sake of completeness let us also consider the noninferior solutions of the studied problem as given in Definition 1. Again, the necessary optimality conditions for this solution type can be found in [18]. The set of noninferior controls has the following form:

$$\left.\begin{aligned} \hat{u}_k(\alpha) &= -\frac{1-\alpha}{\alpha(1-\alpha)+K}\, x_0 \\ \hat{v}_k(\alpha) &= -\frac{\alpha}{\alpha(1-\alpha)+K}\, x_0 \end{aligned}\right\} \quad 0 \leqq \alpha \leqq 1\,, \quad k = 0, 1, \ldots, K-1\,. \tag{7.8}$$

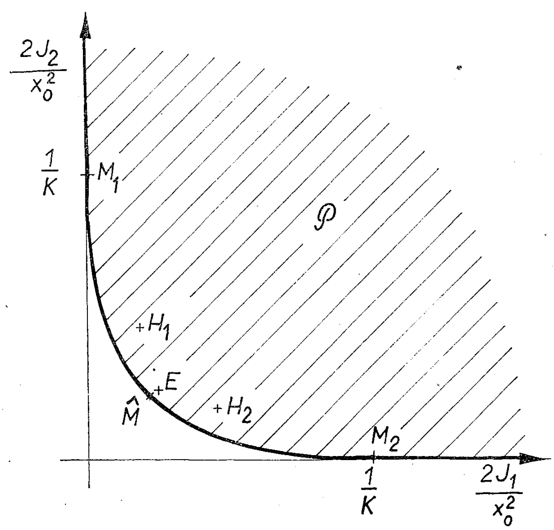

Fig. 2. Comparison of various outcomes for the illustrative example.

The corresponding values of the cost functionals:

$$(7.9)\qquad \left.\begin{aligned} \hat{J}_1(\alpha) &= \frac{1}{2}\frac{(1-\alpha)^2(\alpha^2+K)}{[\alpha(1-\alpha)+K]^2}\,x_0^2 \\ \hat{J}_2(\alpha) &= \frac{1}{2}\frac{\alpha^2[(1-\alpha)^2+K]}{[\alpha(1-\alpha)+K]^2}\,x_0^2 \end{aligned}\right\}\ 0 \leqq \alpha \leqq 1\,.$$

All discussed solution types are schematically depicted in Fig. 2. The shaded region $\mathscr{P}$ represents the set of all possible outcomes in E^2. The hierarchical outcome (7.4) is denoted as the point H_1 and, by the symmetry of the problem in question, H_2 corresponds to the case when J_2 is preferred. As point E we denoted the equilibrium outcome (7.6) and curve M_1M_2 stands for the set noninferior outcomes. This curve is parametrized by parameter α, points M_1 and M_2 correspond to $\alpha = 1$ and $\alpha = 0$, respectively. For example, choosing $\alpha = 0{\cdot}5$ we obtain the point $\hat{M}$ on the curve M_1M_2. From (7.6) and (7.9) we finally obtain that, for example, $\hat{J}_1(0)/J_1^*$ and $J_1^*/\hat{J}_1(0{\cdot}5)$ decrease monotonously with K, and

$$\lim_{K\to\infty}\frac{\hat{J}_1(0)}{J_1^*} = 4\,, \quad \lim_{K\to\infty}\frac{J_1^*}{\hat{J}_1(0{\cdot}5)} = 1\,.$$

8. CONCLUSIONS

In this paper a hierarchical solution concept for decision problems with two objectives was introduced. This concept made it possible to solve bicriterion decision problems applying the classical results concerning decision problems with only one objective. Also a comparison was made with the so called noninferior and equilibrium solution types.

For static decision problems we obtained sufficient existence and necessary optimality conditions. Further also the so called multistage decision problems were studied in detail. For certain classes of these problems we were able to derive sufficient existence and necessary optimality conditions. The latter ones can be also denoted as a two-level discrete maximum principle by an analogy with optimal control theory.

This result was then applied to linear multistage decision problems with both objectives being quadratic. It was possible to derive an analytic form of the hierarchical solution in this case in terms of matrix difference equations, some of which are of Riccati-type. To illustrate the presented theory also a simple example was solved and certain comparisons performed.

It is felt that the hierarchical solution concept can be successfully applied to various decision problems with two objectives arising for example in technology, or when studying economical or sociological problems. The main importance of this new concept lies in the fact that only standart optimization methods are used in comparison with the noninferior and equilibrium solution types. This feature will be especially useful when we are interested in the construction of various numerical approaches and algorithms for the numerical determination of hierarchical solutions.

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