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Generalized Cooperative Games and Markets

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In the last time, we are witnesses of an unprecendented development of mathematical applications in economy including the theory of games and markets. It is sufficient to name the works of Aumann, Peleg, Rosenmüller, Hildenbrand, Vind and others. This paper is a contribution to that field. It suggests a generalization of one part of theory of games and markets in which the existence of side payments is not assumed, and in which we suppose the validity of so called Direct democracy law. The presented work is restricted to games and markets with finite number of participants only. The theory, instituted on continuum or on countably infinite number of participants, is mathematically nice, though, hardly applicable, as the condition of so called absolute competition does not seem to be practically real.

The paper is divided to three chapters. In the first chapter, some contributions to the game theory are given, the second chapter is devoted to market theory, and, eventually, in the last one, some relations between markets and their correspondent games are presented.

ABBREVIATIONS AND NOTATIONS

$\Omega = \{1, \ldots, n\}$	set of players participants
$\exp \Omega$	class of all subsets of Ω
$\mathscr{K} \subset \exp \Omega$	set of all available coalitions
$\lvert S \rvert$	number of all elements of set S
$y^S = (y^i)_{i \in S}$	
$x \geqq y$	$x^i \geqq y^i$ for all i
$v^S = (v^i)_{i \in S}$	
$x \not> y$	$x \leqq y$ and $x \neq y$
$x > y$	$x^i > y^i$ for all i
$S(y/x)$	set of players from S which prefer y to x
$h(S)$	decision coefficient of S
$\mathscr{F}(\mathscr{K})$	set of all mappings from $\mathscr{K}$ to $(0, 1\rangle$
$h \in \mathscr{F}(\mathscr{K})$	decision function

$S(x > y) = \{i \in S : x^i > y^i\}$	
R^n	n-dimensional real space; $R = R^1$
R^m_+	non-negative orthant of R^m
$\succ_{G(S)}$	domination via S in game G
$\succ_G$	domination in game G
$I(G) = \{i \in \Omega : \{i\} \in \mathscr{K}\}$	
$G = (\Omega, \mathscr{K}, v, H, h)$	cooperative game
$\mathrm{dom}_{G(S)} P = \{y : y \in R^n \; \exists x \in P, \; x \succ_{G(S)} y\}$	
$\mathrm{dom}_G P = \bigcup_{S \in \mathscr{K}} \mathrm{dom}_{G(S)} P$	
$\mathfrak{C}^G(P) = P - \mathrm{dom}_G P$	P-core
$\mathfrak{C}(G) = = \mathfrak{C}^G(H)$	core of game
$J(H) = \{x \in H : \neg(\exists y \in H; \; y > x)\}$	
$f_T(x) = \max_{i,j \in \Omega - T} (x^i/x^j)$ for such an x that $x^{\Omega - T} > 0$	
$\lVert y \rVert_S = \lVert y^S \rVert = \max_{i \in S} \lvert y^i \rvert$	
$[x, y]$	segment joining x with y
$\mathfrak{S}^G(P)$	P-solution
$\mathfrak{S}(G) = \mathfrak{S}^G(\bar{A})$	solution of game
$\bar{A}$	set of individually rational distributions in game
E	set of group rational distributions in game
$\bar{E} = H, \; A = \bar{A} \cap E$	
$\geqq$	preference*
$\sim$	equivalence
$\succ$	pure preferences
$(\geqq_i)_{i \in \Omega}$	system of preference
$(a^i)_{i \in \Omega}$	system of initial quantities of goods
$m = (\Omega, \mathscr{K}, R^m_+, (\geqq_i)_{i \in \Omega}, (a^i)_{i \in \Omega}, h)$	cooperative market
$m(K) = \{(x^i)_{i \in \Omega} : x^i \in R^m_+, \; i \in \Omega$ and $\sum_K x^i \leqq \sum_K a^i\}$	
$P = \{p = (p^1 \dots p^m)' \in R^m_+, \; \sum_{i=1}^{m} p^i = 1\}$	price space
$B^K_p = \{(x^i)_{i \in \Omega} : x^i \in R^+_m, \; i \in \Omega, \; \sum_K p' x^i \leqq \sum_K p' a^i\}$	budget-set
$K(x \succ y) = \{i \in K : x^i \succ_i y^i\}$	
$\succ_{m(K)}$	domination via K in market
$\succ_m$	domination in market
$\mathrm{dom}_{m(K)} P = \{x \in \times_{\Omega} R^m_+ : \exists y \in P : y \succ_{m(K)} x\}$	
$\mathrm{dom}_m P = \bigcup_{K \in \mathscr{K}} \mathrm{dom}_{m(K)} P$	

* Symbol $\geqq$ is used instead of more usual $\succsim$ which was not typographically acceptable.

$\mathfrak{C}^m(P) = P - \mathrm{dom}_m P$	P-core
$\mathfrak{C}(\boldsymbol{m}) = \mathfrak{C}^m(\boldsymbol{m}(\Omega))$	core of market
$\mathfrak{S}^m(P)$	P-solution
$\mathfrak{S}(\boldsymbol{m}) = \mathfrak{S}^m(\boldsymbol{m}(\Omega))$	solution of market
$G_{\boldsymbol{m}}$	game corresponding to market $\boldsymbol{m}$
$\bar{A}_{\boldsymbol{m}}$	set of individually rational distributions in market
$E_{\boldsymbol{m}}$	set of group rational distributions in market
$A_{\boldsymbol{m}} = \bar{A}_{\boldsymbol{m}} \cap E_{\boldsymbol{m}}$	

CHAPTER I: COOPERATIVE GAMES WITHOUT TRANSFERABLE UTILITY

In this chapter, we present a generalized model of game without transferable utility. On establishing the concept of domination we suppose that in each admissible coalition the direct democracy law (it means each decision in each coalition certified by voting among all players of the coalition) holds.

1. Fundamental definitions

Definition 1.1. The triple $(\Omega, \mathscr{K}, \boldsymbol{v})$ is called *characteristic function* if Ω is a finite set, $\mathscr{K} \subset \exp \Omega$, $\Omega \in \mathscr{K}$ and $\boldsymbol{v}$ is a mapping from $\mathscr{K}$ to $\exp R^{|\Omega|}$ which fulfills the following conditions

(1) $\boldsymbol{v}(S)$ is convex for all $S \in \mathscr{K}$,

(2) $\boldsymbol{v}(S)$ is closed for all $S \in \mathscr{K}$,

(3) $\boldsymbol{v}(\emptyset) = R^{|\Omega|}$,

(4) $x \in \boldsymbol{v}(S)$, $y \in R^{|\Omega|}$, $y^S \leqq x^S \Rightarrow y \in \boldsymbol{v}(S)$ for all $S \in \mathscr{K}$.

Definition 1.2. Let $S \in \mathscr{K}$ be an arbitrary coalition. Let x and y be two possible payment distributions among players from S. Let $S(y/x)$ denotes the set of the players from S who prefer y to x. Then the number $h(S)$, $h(S) \in (0, 1\rangle$ is called *decision coefficient*, if coalition S accepts y if and only if

$$\frac{|S(y/x)|}{|S|} \geqq h(S) .$$

Let $\mathscr{F}(\mathscr{K})$ be the set of all mappings from $\mathscr{K}$ to $(0, 1\rangle$. $h \in \mathscr{F}(\mathscr{K})$ is called *decision function* of $\mathscr{K}$ if $h(S)$ is decision coefficient for all coalitions $S \in \mathscr{K}$.

Definition 1.3. *Cooperative game* with characteristic function is a quintuple $G = (\Omega, \mathscr{K}, v, H, h)$, where $(\Omega, \mathscr{K}, v)$ is a characteristic function, H is a convex and compact subset of $v(\Omega)$, and $h \in \mathscr{F}(\mathscr{K})$ is a decision function.

Definition 1.4. Cooperative game G is called *guaranted* if it fulfills

(5) $$x \in v(S_i)\,, \quad i = 1, 2, \ldots, k\,; \quad S_i \cap S_j = \emptyset \quad \text{for} \quad i \neq j\,, \quad S_i \in \mathscr{K}\,,$$
$$i = 1, 2, \ldots, k \Rightarrow \exists y \in R^{|\Omega - S|} : (x^S, y) \in v(\Omega)\,, \quad \text{where} \quad S = \bigcup_{i=1}^{k} S_i\,.$$

Definition 1.5. Cooperative game G is called *ordinary* if

(6) $$x \in v(\Omega) \Leftrightarrow \exists y \in H : x \leqq y\,.$$

2. Domination, rationality

Definition 2.1. Let G be a cooperative game. Let $x, y \in H$, $S \in \mathscr{K}$. We say that x *dominates* y *via* S and write $x \succ_{G(S)} y$ if $x \in v(S)$, $x^S \nless y^S$ and

$$\frac{|S(x > y)|}{|S|} \geqq h(S)\,, \quad \text{where} \quad S(x > y) = \{i \in S : x^i > y^i\}\,.$$

Definition 2.2. Let $x, y \in H$. We say that x *dominates* y and write $x \succ_G y$ if there exists $S \in \mathscr{K}$ such that x dominates y via S.

Note. Let us denote $I(G) = \{i : \{i\} \in \mathscr{K}\}$. It follows from Definition 1.1 that if $i \in I(G)$ then there exists $v^i \in R$ such that $v(\{i\}) = \{x \in R^{|\Omega|} : x^i \leqq v^i\}$.

Definition 2.3. Let $P \subset H$, $S \in \mathscr{K}$, then we denote

$$\text{dom}_{G(S)} P = \{y : \exists x \in F,\ x \succ_{G(S)} y\}\,,$$
$$\text{dom}_G P = \{y : \exists x \in P,\ x \succ_G y\}\,.$$

Definition 2.4. Payments vector $x \in H$ is called *individually rational* if $x^i \geqq v^i$ for all $i \in I(G)$. x is called *group rational* if there exists no $y \in H$ such that $y \succ_{G(\Omega)} x$. We denote $\bar{E} = H$, E the set of all group rational vectors from H, $\bar{A}$ the set of all individually rational vectors from H, $A = \bar{A} \cap E$. If $I(G) = \emptyset$ we put $\bar{A} = H$.

3. Core

Definition 3.1. Let G be a cooperative game, $P \subset H$ be an arbitrary set. Then we call the set $\mathfrak{C}(GP) = P^G - \text{dom}_G\, P$ a *P-core*. The set $\mathfrak{C}^G(H)$ is called *game core* and it is denoted also by $\mathfrak{C}(G)$.

Theorem 3.1. Let $G_1 = (\Omega, \mathscr{K}_1, \mathfrak{v}_1, H, h_1)$, $G_2 = (\Omega, \mathscr{K}_2, \mathfrak{v}_2, H, h_2)$ be cooperative games that fulfil

$$\mathscr{K}_1 \subseteq \mathscr{K}_2\,, \tag{3.1.1}$$

$$S \in \mathscr{K}_1 \Rightarrow \mathfrak{v}_1(S) \subset \mathfrak{v}_2(S) \quad \text{and} \quad h_1(S) \geqq h_2(S)\,. \tag{3.1.2}$$

Then $\mathfrak{C}^{G_2}(P) \subset \mathfrak{C}^{G_1}(P)$ for arbitrary $P \subset H$.

Proof. The statement of this Theorem obviously follows from the previous definitions.

Theorem 3.2. Let $G = (\Omega, \mathscr{K}, \mathfrak{v}, H, h)$ be an arbitrary cooperative game and let $\bar{A} \neq \emptyset$. Then $\mathfrak{C}^G(\bar{E}) \subset \mathfrak{C}^G(\bar{A})$.

Proof. If $I(G) = \emptyset$ then $\bar{A} = \bar{E}$ and $\mathfrak{C}^G(\bar{E}) = \mathfrak{C}^G(\bar{A})$. Let, hence, $I(G) \neq \emptyset$ and let $x \in \mathfrak{C}^G(\bar{E})$. If $x \notin \bar{A}$ then there exists $i \in I(G)$ such that $x^i < v^i$. Let z be an arbitrary vector from $\bar{A}$, obviously $z^i \geqq v^i$. There exists $y \in H$ such that $y \geqq z$ for G is ordinary. Let y_1 lie on the segment $[x, y]$ so near to x that $v^i \geqq y_1^i > x^i$. $y_1 \in \bar{E}$ for convexity of $\bar{E}$. Then $y_1 \succ_{G(\{i\})} x$; that contradicts to $x \in \mathfrak{C}^G(\bar{E})$. Hence $x \in \mathfrak{C}^G(\bar{A})$.

Theorem 3.3. Let G be an ordinary and guaranted cooperative game. Then $\bar{A} \neq \emptyset$ and $\mathfrak{C}^G(\bar{E}) = \mathfrak{C}^G(\bar{A})$.

Proof. If $I(G) = \emptyset$ then $\mathfrak{C}^G(\bar{A}) = \mathfrak{C}^G(\bar{E})$ for $\bar{A} = \bar{E}$. Suppose that $I(G) \neq \emptyset$. By (5) there exists $x \in R^{|\Omega - I(G)|}$ such that $(v^{I(G)}, x) \in \mathfrak{v}(\Omega)$. By (6) there exists $y \in H$ such that $y \geqq (v^{I(G)}, x)$. Hence, $y \in \bar{A} \Rightarrow \bar{A} \neq \emptyset$. By the foregoing Theorem is $\mathfrak{C}^G(\bar{E}) \subset \mathfrak{C}^G(\bar{A})$. It remains to prove $\mathfrak{C}^G(\bar{A}) \subset \mathfrak{C}^G(\bar{E})$. Let $z \in \mathfrak{C}^G(\bar{A})$. Then obviously $z \in \bar{E}$. If $z \notin \mathfrak{C}^G(\bar{E})$ then there exists $z_1 \in \bar{E}$ and $S \in \mathscr{K}$ such that $z_1 \succ_{G(S)} z$. We set

$$z_2 = \left(v^{I(G)-S}, z^{S \cup (\Omega - I(G))}\right),$$

then evidently $z_2 \in \mathfrak{v}(S)$ and $z_2 \in \mathfrak{v}(\{i\})$ for $i \in I(G) - S$. Hence, by (5), there exists $x_1 \in R^{|\Omega - (I(G) \cup S)|}$ so that

$$z_3 = \left(z^{S \cup I(G)}, x_1\right) \in \mathfrak{v}(\Omega)\,.$$

By (6) there exists $w \in H$ such that $w \geqq z_3$. Obviously,

$$w^{I(G)-S} \geqq z_3^{I(G)-S} = v^{I(G)-S}\,,$$

and

$$w^{S \cap I(G)} \geqq z_3^{S \cap I(G)} = z_2^{S \cap I(G)} = z_1^{S \cap I(G)} \geqq z^{S \cap I(G)} \geqq v^{S \cap I(G)}\,,$$

thus $w \in \bar{A}$. Let w_1 lie on the segment $[z, w]$ so near to z that

$$z_1^{S(z_1 > z)} \geqq w_1^{S(z_1 > z)} > z^{S(z_1 > z)}\,,$$

(evidently $z_1^{S-S(z_1>z)} = w_1^{S-S(z_1>z)} = z^{S-S(z_1>z)}$). $w_1 \in \bar{A}$ as $\bar{A}$ is convex. Further, by (4), $w_1 \in v(S)$, hence $w_1 \succ_{G(S)} z$. It contradicts to $z \in \mathfrak{C}^G(\bar{A})$. Hence, $z \in \mathfrak{C}^G(\bar{E})$.

Theorem 3.4. Let $G = (\Omega, \mathscr{K}, v, H, h)$ be an ordinary cooperative game, let $h(S) = 1$ for all $S \in \mathscr{K}$ and let H be a polyhedron. Then

$$\mathfrak{C}^G(E) = \mathfrak{C}^G(\bar{E}) \quad \text{and} \quad \mathfrak{C}^G(A) = \mathfrak{C}^G(\bar{A}) .$$

Proof. See the proof given in [3], Theorem 8, page 546.

Theorem 3.5. Let G be an ordinary and guaranted cooperative game, let $h(S) = 1$ for all $S \in \mathscr{K}$ and let H be a polyhedron. Then

$$\mathfrak{C}^G(E) = \mathfrak{C}^G(\bar{E}) = \mathfrak{C}^G(A) = \mathfrak{C}^G(\bar{A}) .$$

Proof. The assertion follows immediately from Theorems 3.3 and 3.4.

Corollary 3.1. Let G be an ordinary and guaranted cooperative game, let H be a polyhedron and let $h(S) > (|S| - 1)/|S|$ for all $S \in \mathscr{K}$, $S \neq \emptyset$. Then $\mathfrak{C}^G(E) = \mathfrak{C}^G(\bar{E}) = \mathfrak{C}^G(A) = \mathfrak{C}^G(\bar{A})$.

Let $H \in R^n$, we denote

$$J(H) = \{x \in H : \neg(\exists y \in H : y > x)\} .$$

Further, for arbitrary $T \subset \Omega$ we define

$$f_T(x) = \max_{i,j \in \Omega - T} (x^i/x^j) \quad \text{for such } x \text{ that} \quad x^{\Omega - T} > 0 .$$

Lemma 3.1. Let $x, y \in R^n$, $x^{\Omega-T} > 0$, $y^{\Omega-T} > 0$. Then

$$f_T(x + y) \leqq \max \{f_T(x), f_T(y)\} .$$

Proof. Let $c = \max \{f_T(x), f_T(y)\}$. Then $x^i/x^j \leqq c$, $y^i/y^j \leqq c$ for all $i, j \in \Omega - T$. Hence

$$(x^i + y^i)/(x^j + y^j) \leqq (cx^j + cy^j)/(x^j + y^j) = c$$

for all $i, j \in \Omega - T$, so that $f_T(x + y) \leqq c$.

Lemma 3.2. Let H be a closed polyhedron in R^n and let it fulfills the following condition

$$(7) \qquad x \in H\,, \quad y \in H\,, \quad y^T \leqq x^T\,, \quad T \subset \Omega\,, \quad |T| \leqq \max_{\mathscr{K} - \{\Omega\}} |S|(1 - h(S)) \Rightarrow$$
$$\Rightarrow (x^{\Omega - T}, y^T) \in H\,.$$

Then there exists for arbitrary $T \subset \Omega$ a number $K > 0$ such that for each $x \in H - J(H)$ there exists $x' \in H$ for which $x'^{\Omega-T} > x^{\Omega-T}$, $x'^T = x^T$ and $f_T(x' - x) \leqq K$.

Proof. Let $H = \{x \in R^n : L_1(x) \geqq b_1, \ldots, L_m(x) \geqq b_m\}$, where L_i, $i \in M = \{1, \ldots, m\}$, is a linear functional in R^n, and let $Q \subset M$. We define

$$H_Q = \{x \in H : L_i(x) > b_i \text{ for } i \in Q \text{ and } L_i(x) = b_i \text{ for } i \in M - Q\} .$$

Nonempty sets H_Q are mutually disjoint and

$$H = \bigcup_{Q \subset M} H_Q .$$

For such a Q for which $H_Q - J(H) \neq \emptyset$, we choose $x_Q \in H_Q - J(H)$. Then there exists $y'_Q \in H$ such that $y'_Q > x_Q$. If we set

$$y_Q = (y'^{\Omega-T}_Q, x^T_Q)$$

then, by (7), $y_Q \in H$. Now, let $x \in H_Q - J(H)$, $\delta > 0$. We define

$$y_\delta = x + \delta(y_Q - x_Q) .$$

For sufficiently small $\delta > 0$ we have

$$L_i(y_\delta) > b_i \quad \text{for} \quad i \in Q , \tag{3.2.1}$$

and for arbitrary $\delta > 0$

$$L_i(y_\delta) \geqq b_i \quad \text{for} \quad i \in M - Q . \tag{3.2.2}$$

We fix such a $\delta > 0$ for which (3.2.1) and (3.2.2) hold. It means that $y_\delta \in H$ and obviously $y^{\Omega-T}_\delta > x^{\Omega-T}$, $y^T_\delta = x^T$. Let $x' = y_\delta$. Then $x' - x = \delta(y_Q - x_Q)$, hence

$$f_T(x' - x) = f_T(y_Q - x_Q) .$$

We put

$$K = \max_{Q \subset M} f_T(y_Q - x_Q) .$$

Then $f_T(x' - x) \leqq K$ for all $x \in H - J(H)$ because

$$H - J(H) = \bigcup_{Q \subset M} (H_Q - J(H)) .$$

Lemma 3.3. Let $H \subset R^n$ be a compact polyhedron fulfilling condition (7). Then there exists a number $K > 0$ such that $\forall T \subset \Omega$, $\forall x \in H - J(H)$, $\exists x'' \in J(H)$ such that $x''^{\Omega-T} > x^{\Omega-T}$, $x''^T = x^T$ and $f_T(x'' - x) \leqq K$.

Proof. For $x \in H - J(H)$, we denote

$$F_x = \{x' \in H : x' \text{ fulfills Lemma 4.2}\}$$

and define

$$g(y) = \max_{\Omega - T} |y^i - x^i| = \|y - x\|_{\Omega - T} \quad \text{for} \quad y \in F_x .$$

F_x is bounded, hence there exists

(3.3.1) $$L = \sup_{F_x} g(y) .$$

Let $\{y_k\}_{k=1}^{\infty}$, $y_k \in F_x$ for all k such that $g(y_k)$ converges to L and $y_k \to x''$.

$$f_T(y_k - x) \leqq K \Rightarrow f_T(x'' - x) \leqq K \Rightarrow x''^{\Omega - T} > x^{\Omega - T} .$$

If $x'' \notin J(H)$ then there exists $y' \in H$ such that $y'^{\Omega - T} > x''^{\Omega - T}$, $y'^T = x''^T$ and $f_T(y' - x'') \leqq K$ according to Lemma 3.1.

$$f_T(y' - x) = f_T(y' - x'' + x'' - x) \leqq \max \{f_T(y' - x''), f_T(x'' - x)\} \leqq K .$$

Thus $y' \in F_x$. But

$$g(y') = \|y' - x\|_{\Omega - T} = \|(y' - x'') + (x'' - x)\|_{\Omega - T} > \|x'' - x\|_{\Omega - T}$$

and it is a contradiction with (3.3.1). Hence $x'' \in J(H)$.

Lemma 3.4. Let $H \subset R$ be a compact polyhedron fulfilling condition (7). Then there exists a number $K > 0$ such that

$$\forall T \subset \Omega , \quad \forall x \in H - J(H) , \quad \exists x'' \in J(H) : x''^T =$$
$$= x^T \,\&\, x''^{\Omega - T} > x^{\Omega - T} \,\&\, \forall (i \in \Omega - T) : x''^i - x^i \geqq (\|x'' - x\|_{\Omega - T})/K .$$

Proof. The assertion follows immediately from Lemma 3.3.

Theorem 3.6. Let $G = (\Omega, \mathscr{K}, v, H, h)$ be a cooperative game. Let H be a polyhedron fulfilling condition (7) and let $y \in J(H)$. If there exist $z \in H$ and $S \in \mathscr{K}$ such that $z \succ_{G(S)} y$ then there exists $w \in J(H)$ such that $w \succ_{G(S)} y$.

Proof. Suppose, without loss of generality, that $y = 0$. Let $V = R^n_+ = \{x \in R^n : : x \geqq 0\}$ and V° be the interior of V. $0 = y \in J(H) \Rightarrow H \cap V^\circ = \emptyset$. As H is a compact polyhedron, there exists a hyperplane $g(x) = \sum_{\Omega} c^i x^i = 0$ which separates H from V. Without loss of generality we suppose that $g(x) \leqq 0$ for $x \in H$ and $g(x) \geqq 0$ for $x \in V$. Thus $c^i \geqq 0$ for all i. Note that

(3.6.1) $$x \in H , \quad g(x) \geqq 0 \Rightarrow x \in J(H) .$$

If $z \in J(H)$ then there is nothing to prove. Let $z \notin J(H)$, $z \succ_{G(S)} 0 \Rightarrow S \neq \Omega$, otherwise it leads to contradiction with $z \notin J(H)$. Set

$$f(x) = \sum_{\Omega - S} c^i x^i ,$$

 then the implication

$$(3.6.2) \qquad x \succ_{G(S)} 0\,, \quad f(x) \geqq 0 \Rightarrow x \in J(H)$$

follows from (3.6.1) and

$$\sum_{\Omega} c^i x^i = \sum_{S} c^i x^i + f(x)\,.$$

$z \notin J(H) \Rightarrow f(z) < 0$. Put

$$(3.6.3) \qquad k = \min_{S(z>0)} z^i/2 > 0$$

and

$$M = \{x \in H : x^{S(z>0)} \not< 0,\ x^{S-S(z>0)} = 0 \text{ and } \|x^{S(z>0)}\| = k\}\,.$$

As H is convex, there exists a point of the segment $[z, 0]$ which lies in M, thus $M \neq \emptyset$. For M is compact and f is continuous, there exists $x_1 \in M$ such that

$$(3.6.4) \qquad f(x_1) = \max_{M} f(x)\,.$$

If $x_1 \notin J(H)$ then there exists $x_2 \in H$, $x_2 > x_1$. We have

$$(3.6.5) \qquad \|x_2^{S(z>0)}\| > \|x_1^{S(z>0)}\| = k\,,$$

$$(3.6.6) \qquad f(x_2) \geqq f(x_1)\,.$$

Put

$$x_3 = (k/\|x_2^{S(z>0)}\|)\, x_2\,.$$

Obviously, $\|x_3^{S(z>0)}\| = k$ and $x_3^{S(z>0)} > 0$. $x_3 \in H$ for convexity of H. Set $w_3 = (x_3^{(\Omega-S)\cup S(z>0)}, o^{S-S(z>0)})$. Then $w_3 \in H$ for (7), hence $w_3 \in M$. It follows from (3.6.4) that

$$(3.6.7) \qquad f(x_1) > f(w_3)\,.$$

Obviously, $z^{S(z>0)} > x_3^{S(z>0)} = w_3^{S(z>0)} > 0$, so $w_3 \in \mathfrak{v}(S)$ according to (4), and $w_3 \succ_{G(S)} 0$. If $w_3 \in J(H)$, proof is finished ($w = w_3$), otherwise (3.6.2) implies $f(w_3) < 0$. But it follows from $f(w_3) = (k/\|x_2^{S(z>0)}\|) f(x_2)$ and $k/\|x_2^{S(z>0)}\| < 1$ that $f(x_2) < f(w_3)$ what contradicts to (3.6.6) and (3.6.7). We conclude $x_1 \in J(H)$. Obviously $0 \not> x_1^{S(z>0)} < z^{S(z>0)}$ and $x_1^{S-S(z>0)} = 0$, so $x_1 \in \mathfrak{v}(S)$ according to (4). If $x_1^{S(z>0)} > 0$ then $x_1 \succ_{G(S)} 0$ and proof is finished ($w = x_1$). Therefore, it remains only to deal with the case in which one of the coordinates of $x_1^{S(z>0)}$ vanishes. Then we construct x_4 such that $x_4 = \alpha_1 0 + \alpha_2 x_1 + \alpha_3 z$, where $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\alpha_i > 0$, $i = 1, 2, 3$, and x_4 is so near to x_1 that

$$(3.6.8) \qquad \|x_4 - x_1\|_{\Omega-T} \leqq \delta\,, \quad \text{where} \quad \delta = k/(K+1)\,,$$

where K is the constant from Lemma 3.4 for $T = S - S(z > 0)$. Convexity of H implies that $x_4 \in H$. Further, $x_4^{S-S(z>0)} = 0$, $x^{S(z>0)} < z^{S(z>0)}$. So, $x_4 \in \mathfrak{v}(S)$ and $x_4 \succ_{G(S)} 0$. If $x_4 \in J(H)$, the proof is finished ($w = x_4$). If $x_4 \in H - J(H)$, we choose x_4'' for $T = \Omega - S(z > 0)$ according to Lemma 3.4. It may have two cases.

(i) $\|x_4'' - x_4\|_{\Omega - T} > K\delta$, then $x_4''^i - x_4^i > \delta$, $i \in \Omega - T$, according to Lemma 3.4. Hence,

$$x_4''^i - x_1^i = x_4''^i - x_4^i + x_4^i - x_1^i > \delta - \|x_4 - x_1\|_{\Omega - T} \geqq 0\,,$$

for $i \in \Omega - T$, according to (3.6.8). If $x_4''^{S(z>0)} \leqq z^{S(z>0)}$ then $x_4'' \in \mathfrak{v}(S)$ and $x_4'' \succ_{G(S)} 0$ and proof is finished ($w = x_4''$). Otherwise we set $x_5 = \alpha x_1 + (1 - \alpha) x_4''$ which is so near to x_1 that $0 < x_5^{S(z>0)} < z^{S(z>0)}$. Obviously, $x_5 \succ_{G(S)} 0$. $x_5 \geqq x_1$ so $x_5 \in J(H)$ and proof is finished ($w = x_5$).

(ii) $\|x_4'' - x_4\|_{\Omega - T} \leqq K\delta$. Then from

$$\|x_4'' - x_1\|_{\Omega - T} \leqq \|x_4'' - x_4\|_{\Omega - T} + \|x_4 - x_1\|_{\Omega - T} \leqq K\delta + \delta = k$$

follows that

$$\|x_4''\|_{S(z>0)} \leqq \|x_4'' - x_1\|_{S(z>0)} + \|x_1\|_{S(z>0)} \leqq 2k = \min_{S(z>0)} z^i\,.$$

Thus $x_4''^{S(z>0)} \leqq z^{S(z>0)}$. So $x_4'' \in \mathfrak{v}(u)$ and $x_4'' \succ_{G(S)} 0$. By setting $w = x_4''$ the proof is complete.

Theorem 3.7. Let G be a cooperative game and let H be a polyhedron fulfilling condition (7). Then

$$\mathfrak{C}^G(J(H)) = \mathfrak{C}^G(H) \cap J(H)\,.$$

Proof. Let $y \in \mathfrak{C}^G(J(H))$. Obviously $y \in J(H)$. If $y \notin \mathfrak{C}^G(H)$ then there exists $z \in H$ such that $z \succ_G y$. According to the foregoing Theorem, there exists $w \in J(H)$, $w \succ_G y$ and it contradicts to $y \in \mathfrak{C}^G(J(H))$. On the other hand, if $y \in \mathfrak{C}^G(H) \cap J(H)$ then obviously $y \in \mathfrak{C}^G(J(H))$.

Similarly we derive the following statement.

Corollary 3.2. Let G be a cooperative game, let H be a polyhedron fulfilling (7) and let $\bar{A} \neq \emptyset$. Then

$$\mathfrak{C}^G(J(\bar{A})) = \mathfrak{C}^G(\bar{A}) \cap J(\bar{A})\,.$$

Theorem 3.8. Let all assumptions of Theorem 3.7 hold. Then

$$\mathfrak{C}^G(H) = \mathfrak{C}^G(J(H))\,.$$

 If, in addition, $\bar{A} \neq \emptyset$, then

$$\mathfrak{C}^G(\bar{A}) = \mathfrak{C}^G(J(\bar{A}))\,.$$

Proof. We show that $\mathfrak{C}^G(H) \subset J(H)$. $y \in \mathfrak{C}^G(H) \Rightarrow \neg(\exists z : z \succ_{G(\Omega)} y) \Rightarrow \Rightarrow \neg(\exists z \in H : z > y) \Rightarrow y \in J(H)$. Now, using Theorem 3.7, we obtain the assertion. Similarly we proceed for the second equality.

The following theorem follows from Theorems 3.3 and 3.8.

Theorem 3.9. Let $G = (\Omega, \mathscr{K}, v, H, h)$ be a cooperative guaranted and ordinary game, and let H be a polyhedron fulfilling (7). Then

$$\mathfrak{C}^G(\bar{E}) = \mathfrak{C}^G(J(\bar{E})) = \mathfrak{C}^G(\bar{A}) = \mathfrak{C}^G(J(\bar{A})\,.$$

Remark 3.1. Without condition (7), Theorem 3.6 can fail. We show the following counter-example. Let $G = (\Omega, \mathscr{K}, v, H, h)$, where $\Omega = \{1, 2, 3, 4\}$, $\mathscr{K} = \exp \Omega$, $v(\Omega) = \{x \in R^4 : x^i \leqq 100,\ i = 1, 2, 3, 4\}$, $v(S^4) = \{x \in R^4 : (x^1, x^2, x^3) \leqq (1, 1, 0)\}$ $S^4 = \{1,2,3\}$, $v(S)$ is arbitrary for other $S \in \exp \Omega$.

$H = \{x \in v(\Omega) : x^i \geqq -100,\ i = 1, 2, 3, 4,\ x^4 \leqq 0,\ x^1 + x^2 + x^4 - x^3 \leqq 0\}$, $h \in \in \mathscr{F}(\mathscr{K})$ such that $h(S^4) \in (1/3, 2/3)$. Obviously, $0 \in J(H)$. Set $z = (1, 1, 0, -2)$ then $z \in H$, $z \succ_{G(S^4)} 0$, but $z \notin J(H)$. If there exists $w \in J(H)$, $w \succ_{G(S^4)} 0$, then $w \in \in v(S^4)$. It means $w^3 = 0$ and $0 < w^1, w^2 \leqq 1$. But $w \in H$, hence $w^1 + w^2 + w^4 - - w^3 \leqq 0$, $x^4 \leqq 0 \Rightarrow w^4 \leqq -(w^1 + w^2) < 0$. It follows that we can construct $y \in H$ such that $y > w$ and it is in contradiction to $w \in J(H)$.

Remark 3.2. Without the assumption that H is a polyhedron, Theorem 3.6 can fail, too. Conter-example (see [3]).

$\Omega = \{1, 2, 3\}$, $\mathscr{K} = \exp \Omega$, $h \equiv 1$, H is the convex hull of sets C and D, where

$$C = \{x : x^1 \geqq 0,\ x^2 \geqq 0,\ x^3 = 0,\ (x^1)^2 + (x^2)^2 \leqq 1\}\,,$$
$$D = \{x : x^1 \geqq 0,\ x^2 \geqq 0,\ x^3 = 1,\ (x^1)^2 + (x^2)^2 \leqq 4\}\,.$$

Then $J(H) = D \cup \{x : x^1 = 0,\ x^2 = 1,\ 0 \leqq x^3 \leqq 1\}$. We define

$$v(\Omega) = \{x : \exists y \in H,\ y \geqq x\}\,, \quad v^\Omega = 0\,,$$
$$v(\{i, j\}) = \{x : x^i \leqq 1/2,\ x^j \leqq 1/2\}\,, \quad i, j \in \Omega\,.$$

Then obviously $(1/2, 1/2, 1/2) \in H$, $(0, 1, 0) \in J(H)$ and

$$(1/2, 1/2, 1/2) \succ_{G(\{1,3\})} (0, 1, 0)\,,$$

but there exists no $y \in J(H)$, $y \succ_G (0, 1, 0)$.

Theorem 3.10. Let $G = (\Omega, \mathscr{K}, v, H, h)$ be a cooperative game. Then $\mathfrak{C}^G(J(H)) \subset \subset \mathfrak{C}^G(E)$. If, in addition, $\bar{A} \neq \emptyset$ then $\mathfrak{C}^G(J(\bar{A})) \subset \mathfrak{C}^G(A)$.

Proof. Let $x \in \mathfrak{C}^G(J(H))$ then $x \in E$. If $x \notin \mathfrak{C}^G(E)$ then $x \notin \mathfrak{C}^G(J(H))$ for $E \subset J(H)$ what is a contradiction. Analogically for the second inclusion.

Theorem 3.11. Let G be a cooperative game, let H fulfil condition (7) and let the following inequality

$$|\Omega|\; h(\Omega) > \max_{K-\{\Omega\}} |S| \tag{3.11.1}$$

holds. Then $\mathfrak{C}^G(J(H)) = \mathfrak{C}^G(E)$; in addition, if $\bar{A} \neq \emptyset$ then $\mathfrak{C}^G(J(\bar{A})) = \mathfrak{C}^G(A)$.

Proof. Let $x \in \mathfrak{C}^G(E)$. We prove that $x \in \mathfrak{C}^G(J(H))$. Suppose that $x \notin \mathfrak{C}^G(J(H))$. Then there exists $y \in J(H) - E$, $S \in \mathcal{K}$, $y \succ_{G(S)} x$. Set

$$M_x = \{z \in J(H) : z^S = y^S\}\,.$$

As M_x is nonempty, closed and bounded, there exists $y_1 \in M_x$ such that

$$\sum_{\Omega - S} (y_1^i - y^i) = \max_{M_x} \sum_{\Omega - S} (z^i - y^i)\,. \tag{3.11.2}$$

If $y_1 \notin E$ then there exists $y_2 \in H$, $y_2 \succ_{G(\Omega)} y_1$. It follows from (3.11.1) that there exists $k \in \Omega - S$ such that $y_2^k > y_1^k$. Set $y_3 = (y^S, y_2^{\Omega - S})$. Then $y_3 \in H$, according to (7), and obviously $y_3 \in J(H)$. So $y_3 \in M_x$. But,

$$\sum_{\Omega - S} (y_3^i - y^i) > \sum_{\Omega - S} (y_1^i - y^i)$$

what contradicts to (3.11.2). Hence $y_1 \in E$ and $y_1 \succ_{G(S)} x$, what contradicts to $x \in \mathfrak{C}^G(E)$. Consequently, $x^G \in \mathfrak{C}(J(H))$. We have proved that $\mathfrak{C}^G_{\mathfrak{s}}(E) \subset \mathfrak{C}^G(J(H))$. Using Theorem 3.10, we obtain $\mathfrak{C}^G(E) = \mathfrak{C}^G(J(H))$. For the second equality we can proceed analogously.

Lemma 3.5. Let G be a cooperative game. Then

$$E = J(H) - \mathrm{dom}_{G(\Omega)}\, J(H)\,, \quad A = J(\bar{A}) - \mathrm{dom}_{G(\Omega)}\, J(\bar{A})\,.$$

Proof. Lemma follows from the respective definitions.

Lemma 3.6. Let G be a cooperative game. Then

$$\mathrm{dom}_{G(\Omega)}\, E = \mathrm{dom}_{\,G(\Omega)}\, J(H)$$

and

$$\mathrm{dom}_{G(\Omega)}\, A = \mathrm{dom}_{G(\Omega)}\, J(\bar{A})\,.$$

Proof. It follows from Lemma 3.5 immediately that

$$\mathrm{dom}_{G(\Omega)}\, E \subset \mathrm{dom}_{G(\Omega)}\, J(H)\,.$$

 It remains to show that

$$\mathrm{dom}_{G(\Omega)} J(H) \subset \mathrm{dom}_{G(\Omega)} E .$$

Let $x \in \mathrm{dom}_{G(\Omega)} J(H)$ then there exists $x_1 \in J(H)$, $x_1 \succ_{G(\Omega)} x$.

$$M_{x_1} = \{y \in J(H) : y \geqq x_1\} ,$$
$$g(y) = \sum_{\Omega}(y^i - x^i) .$$

M_{x_1} is compact and nonempty, hence there exists $z \in M_{x_1}$ such that

$$g(z) = \max_{M_x} g(y) .$$

It is easy to verify that $z \in E$, so $x \in \mathrm{dom}_{G(\Omega)} E$. Analogically for the second equality.

Theorem 3.12. Let G be a two-person cooperative game. Then $\mathfrak{C}^G(A) = \mathfrak{C}^G(J(\bar{A}))$.

Proof. Note that in two-person game only the domination via Ω can be realized. Hence, by lemma 3.6 and Lemma 3.5,

$$\begin{aligned}\mathfrak{C}^G(A) &= A - \mathrm{dom}_G A = A - \mathrm{dom}_{G(\Omega)} A = A - \mathrm{dom}_{G(\Omega)} J(\bar{A}) = \\ &= [J(\bar{A}) - \mathrm{dom}_{G(\Omega)} J(\bar{A})] - \mathrm{dom}_{G(\Omega)} J(\bar{A}) = \\ &= J(\bar{A}) - \mathrm{dom}_{G(\Omega)} J(\bar{A}) = \mathfrak{C}^G(J(\bar{A})) .\end{aligned}$$

Remark 3.3. We show an example in which $\mathfrak{C}^G(A) \neq \mathfrak{C}^G(J(\bar{A}))$. $\Omega = \{1, 2, 3\}$, $\mathscr{K} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \Omega\}$, $v^1 = v^2 = v^3 = 0$, $v(\{1, 2\}) = \{x \in R^3 : x^2 + x^3 \leqq \leqq 2,\ x^2 + 2x^1 \leqq 2,\ x^3 + 2x^1 \leqq 2\}$, $H = \{x \in R^3_+ : x^2 + x^3 \leqq 2,\ x^2 + x^1 \leqq 2,\ x^3 + x^1 \leqq 2\}$, $\boldsymbol{v}(\Omega) = \{x \in R^3 : \exists y \in H : y \geqq x\}$, $h(\Omega) = 1/3$, $h(\{1, 2\}) = h(\{i\}) = = 1$. Game $G = (\Omega, \mathscr{K}, \boldsymbol{v}, H, h)$ is guaranted and ordinary. H is a compact polyhedron fulfilling (7). Let $x = (0, 0, 2)$, obviously $x \in A = E$. Let $0 < \varepsilon < 0{,}1$ and $y = (\varepsilon/3, \varepsilon, 2 - \varepsilon)$. We can easily verify that $y \in J(\bar{A}) = J(H)$ and $y \in v(\{1, 2\})$, thus $y \succ_{G(\{1,2\})} x$. We conclude $x \notin \mathfrak{C}^G(J(\bar{A}))$. We show that $x \in \mathfrak{C}^G(A)$ by contradiction. Let there exists $z \in A$ such that $z \succ_G x$. We see that z dominates x only via $\{1, 2\}$. Hence $0 < z^1, z^2 < 2$, $z^2 + 2z^1 \leqq 2$, $z^3 + 2z^1 \leqq 2$. Set $z_1 = (2z^1, z^2, z^3)$ then obviously $z_1 \in H = \bar{A}$, $z_1 \succ_{G(\Omega)} z$ what contradicts to $z \in A$. So $x \in \mathfrak{C}^G(A)$.

4. Solution

Definition 4.1. Let $G = (\Omega, \mathscr{K}, \boldsymbol{v}, H, h)$ be a cooperative game and let $P \subset H$. Then a set $V \subset P$ is called *P-solution* iff

$$V = P - \mathrm{dom}_G V .$$

We denote it by $\mathfrak{S}^G(P)$.

Remark 4.1. If $\mathfrak{S}^G(P)$ exists then $\mathfrak{C}^G(P) \subset \mathfrak{S}^G(P)$.

Remark 4.2. Suppose that

$$x \in P \cap \mathrm{dom}_G P \Rightarrow \exists y \in P - \mathrm{dom}_G P : y \succ_G x . \tag{4.2.1}$$

Then $\mathfrak{C}^G(P) = \mathfrak{S}^G(P)$.

Proof. $\mathfrak{C}^G(P) = P - \mathrm{dom}_G P$. It follows from (4.2.1) that

$$P \cap \mathrm{dom}_G P \subset P \cap \mathrm{dom}_G \mathfrak{C}^G(P) \Rightarrow P \cap \mathrm{dom}_G P = P \cap \mathrm{dom}_G \mathfrak{C}^G(P)$$

for $\mathrm{dom}_G \mathfrak{C}^G(P) \subset \mathrm{dom}_G P$. Hence, $P - \mathrm{dom}_G \mathfrak{C}^G(P) = P - P \cap \mathrm{dom}_G \mathfrak{C}^G(P) = P - P \cap \mathrm{dom}_G P = P - \mathrm{dom}_G P = \mathfrak{C}^G(P) \Rightarrow \mathfrak{C}^G(P)$ is P-solution.

Theorem 4.1. Let G be a cooperative game. Then for every $x \in H - E$ there exists $y \in E$ such that $y \succ_{G(\Omega)} x$. In addition, if $\bar{A} \neq \emptyset$ then for every $x \in \bar{A} - A$ there exists $y \in A$ such that $y \succ_{G(\Omega)} x$.

Proof. Let $x \in H - E$. If $x \in J(H)$ then, according to Lemma 3.5 and Lemma 3.6, there exists $y \in E$, $y \succ_{G(\Omega)} x$. If $x \notin J(H)$, we set

$$M_x = \{z \in H : z > x\}, \quad f(z) = \min_{i \in \Omega} (z^i - x^i) \quad \text{for} \quad z \in H .$$

H is compact, consequently there exists $y_1 \in H$ such that

$$f(y_1) = \max_H f(z) .$$

$M_x \neq \emptyset \Rightarrow f(y_1) > 0 \Rightarrow y_1 \in M_x$, i.e. $y_1 > x$. Obviously $y_1 \in J(H)$ and, applying the foregoing reasoning, we obtain the assertion. Analogically for the second assertion.

Theorem 4.2. Let G be a cooperative game and let $\bar{A} \neq \emptyset$. Let $A \subset P \subset \bar{A}$ be an arbitrary set. Then a set K is A-solution if and only if it is P-solution.

Proof. Let K be A-solution. We show that

$$P - A \subset \mathrm{dom}_G K . \tag{4.2.2}$$

According to Theorem 4.1

$$x \in P - A \Rightarrow \exists y \in A , \quad y \succ_{G(\Omega)} x .$$

If $y \in K$ then $x \in \mathrm{dom}_G K$. If $y \in \mathrm{dom}_G K$ then $\exists z \in K$, $z \succ_G y \Rightarrow z \succ_G x \Rightarrow x \in \mathrm{dom}_G K$. It follows from (4.2.2) that

$$P - \mathrm{dom}_G K = A - \mathrm{dom}_G K = K .$$

So, K is really P-solution. Let, now, K be P-solution. Then, applying the same reasoning, we obtain (4.2.2), and, consequently, $K \subset A$; hence K is A-solution.

Definition 4.2. $\mathfrak{S}^G(\bar{A})$, provided that it exists, is called a *solution of game* G and is denoted by $\mathfrak{S}(G)$.

Theorem 4.3. Every two-person cooperative game has its unique solution and it is the set A.

Proof. Theorem follows immediately from foregoing definitions and theorems.

Definition 4.3. Cooperative game $G = (\Omega, \mathscr{K}, v, H, h)$ is called a constant-sum game if H is a subset of the set

$$\{x : \sum_{\Omega} x^i = e\}$$

where e is a constant number.

Theorem 4.4. Let G be a three-person cooperative constant-sum game. Let $h(S) > 1/2$ for all two-person coalitions from $\mathscr{K}$. Then $\mathfrak{S}^G(\bar{A})$ exists.

Proof. If $\mathscr{K} \neq \exp \Omega$ then for all $S \in \exp \Omega - \mathscr{K}$ we define $v(S)$ such that $v(S) \cap H = \emptyset$. It is easy to verify that in such game the relation of domination may be realized only via two-person coalition. Now, we can use the result given in [4] for the game $G_1 = (\Omega, \exp \Omega, v, H, 1)$, where $1(S) = 1$ for all $S \in \exp \Omega$, and the solutions of G_1 are identical with the ones of G.

Remark 4.3. Without the condition $h(S) > 1/2$ for all two-person coalitions from $\mathscr{K}$ Theorem 4.4 could fail. We give the following counter-example. $\mathscr{K} = \exp \Omega$, $\Omega = \{1, 2, 3\}$, $v^i = 0$; $H = \{x \geqq 0 : x^1 + x^2 + x^3 = 1\} = A$, $v(\Omega) = \{x : \exists y \in H, y \geqq x\} = v(\{i, j\})$, $i, j = 1, 2, 3$; $h \equiv 1/2$. Then $x, y \in A \Rightarrow x \succ_G y$ or $y \succ_G x$, hence there exists no solution of G.

CHAPTER II: COOPERATIVE MARKET WITHOUT TRANSFERABLE UTILITY

In this chapter we shall deal with markets with possibility of cooperation among participants but without any transferable utility. We shall define the concepts of core, solution, optimum and equilibrium in general sense. We shall derive some relations among these concepts and Theorem about the existence of equilibrium.

5. Preference, Utility Function

Definition 5.1. Linear relation $\succeq$ on $R^m_+ \times R^m_+$ is called a *preference* if it fulfills the following conditions

1. Reflexivity $x \in R^m_+ \Rightarrow x \gtrsim x$.
2. Transitivity $x, y, z \in R^m_+$, $x \gtrsim y$ and $y \gtrsim z \Rightarrow x \gtrsim z$.
3. Completness $x, y \in R^m_+ \Rightarrow x \gtrsim y$ or $y \gtrsim x$.

Further we denote

$x \sim y$ if $x \gtrsim y$ and $y \gtrsim x$,
$x \succ y$ if $x \gtrsim y$ and $\neg(y \gtrsim x)$.

Definition 5.2. Let $\gtrsim$ be a preference on $R^m_+ \times R^m_+$. A function $u : R^m_+ \to R$ is called *utility function* corresponding to $\gtrsim$, if $x \gtrsim y \Leftrightarrow u(x) \geqq u(y)$ for all $x, y \in R^m_+$.

Definition 5.3. Let $\gtrsim$ be a preference on $R^m_+ \times R^m_+$. For every $\bar{x} \in R^m_+$ we set

$$G_{\bar{x}}(\gtrsim) = \{x \in R^m_+ : \bar{x} \gtrsim x\},$$
$$F_{\bar{x}}(\gtrsim) = \{x \in R^m_+ : x \gtrsim \bar{x}\}.$$

Then $\gtrsim$ is called *continuous* if $G_{\bar{x}}(\gtrsim)$ and $F_{\bar{x}}(\gtrsim)$ are closed for all $\bar{x} \in R^m_+$.

Theorem 5.1. A preference $\gtrsim$ is continuous if and only if there exists a continuous utility function corresponding to $\gtrsim$.

Proof. See [1], page 4.2, Theorem 2.1.

Definition 5.4. A preference $\gtrsim$ is called *convex* if

$$x \gtrsim y, \quad 0 \leqq \alpha \leqq 1 \Rightarrow \alpha x + (1 - \alpha) y \gtrsim y \quad \text{for all} \quad x, y \in R^m_+,$$

strictly convex if

$$x \gtrsim y, \quad x \neq y, \quad 0 < \alpha < 1 \Rightarrow \alpha x + (1 - \alpha) y \succ y,$$

monotonous if

$$x \geqq y \Rightarrow x \gtrsim y,$$

strictly monotonous if

$$x \neq y, \quad x \geqq y \Rightarrow x \succ y,$$

and *positively monotonous* if

$$x > y \Rightarrow x \succ y.$$

6. Definition of Market

We suppose that there are altogether n participants (players) and m sorts of goods (commodities). Every participant has certain initial quantity of goods and values it according to his preference. Participants barter their goods in order to make the consequent distribution as advantageous as possible for all of them. There are only some permissible coalitions in which we suppose so called direct democracy law (see Chapter I).

Definition 6.1. A *cooperative market* is the sixtuple

$$\boldsymbol{m} = (\Omega, \mathscr{K}, R^m_+, (\geqq_i)_{i\in\Omega}, (a^i)_{i\in\Omega}, h),$$

where $\Omega = \{1, 2, \ldots, n\}$ is the set of participants, $\mathscr{K} \subset \exp \Omega$, $\Omega \in \mathscr{K}$, is the set of available coalitions, $\geqq_i$, $i \in \Omega$, is the preference of participant i, $a^i \in R^m_+$, $a^i \nless 0$, is the initial quantity of goods of player i, $h \in \mathscr{F}(\mathscr{K})$ is a decision function (see Chapter I, Definition 1.2).

Further we denote

$$\boldsymbol{m}(K) = \{(x^i)_{i\in\Omega} : x^i \in R^m_+ \text{ for all } i \in \Omega, \sum_K x^i \leqq \sum_K a^i\}$$

for $K \in \mathscr{K}$,

$$\boldsymbol{m}(\emptyset) = \mathop{\times}_{\Omega} R^m_+ = \{(x^i)_{i\in\Omega} : x^i \in R^m_+, i \in \Omega\}.$$

Definition 6.2. The set

$$\mathscr{P} = \{p = (p_1, p_2, \ldots, p_m)' \in R^m_+, \sum_{i=1}^m p_i = 1\}$$

is called the *space of price vectors*.

For $p \in \mathscr{P}$, $K \in \mathscr{K}$ we denote $\boldsymbol{B}^\Omega_p = \boldsymbol{m}(\Omega)$ and

$$\boldsymbol{B}^K_p = \{(x^i)_{i\in\Omega} : x^i \in R^m_+, i \in \Omega, \sum_K px^i \leqq \sum_K pa^i\},$$

and we call it the *budget-set* of coalition K according to price vector p.

Definition 6.3. A pair (x, p), where $x \in \boldsymbol{m}(\Omega)$, $p \in \mathscr{P}$, is called *market state*.

7. Core, Solution, Optimum and Equilibrium

Definition 7.1. Let $\boldsymbol{m}$ be a cooperative market, let $x, y \in \mathop{\times}_{\Omega} R^m_+$. Then we say that x *dominates* y *via* $K \in \mathscr{K}$ and write $x \succ_{\boldsymbol{m}(K)} y$, if $x \in \boldsymbol{m}(K)$, $x^K \geqq y^K$; it means $x^i \geqq_i y^i$ for all $i \in K$, and

$$\frac{|K(x \succ y)|}{|K} \geqq h(K),$$

where $K(x \succ y) = \{i \in K : x^i \succ_i y^i\}$. Further, we say that x *dominates* y and write $x \succ_{\boldsymbol{m}} y$ if there exists $K \in \mathscr{K}$ such that $x \succ_{\boldsymbol{m}(K)} y$.

Definition 7.2. Let $P \in \boldsymbol{m}(\Omega)$. We define

$$\operatorname{dom}_{\boldsymbol{m}(K)} P = \{x \in \underset{\Omega}{\times} R^m_+ : \exists y \in 0 : y \succ_{\boldsymbol{m}(K)} x\},$$

$$\operatorname{dom}_{\boldsymbol{m}} P = \bigcup_K \operatorname{dom}_{\boldsymbol{m}(K)} P .$$

The set $\mathfrak{C}^m(P) = P - \operatorname{dom}_{\boldsymbol{m}} P$ is called *P-core*; the $\boldsymbol{m}(\Omega)$-core is called the *core of market* and we denote it by $\mathfrak{C}(\boldsymbol{m})$. A set $Q \subset P$ is called *P-solution* and denoted by $\mathfrak{S}^m(P)$ if $Q = P - \operatorname{dom}_{\boldsymbol{m}} Q$. $\boldsymbol{m}(\Omega)$-solution is called a *solution of market.*

Definition 7.3. Let $(\bar{x}, \bar{p})$ be a state of market $\boldsymbol{m}$. Then $(\bar{x}, \bar{p})$ is called *optimum* of $\boldsymbol{m}$ if there exists no $K \in \mathscr{K}$ with $y \in \boldsymbol{B}^K_{\bar{p}}$, $y^K \geqq x^K$ and $|K(y \succ x)|/|K| \geqq h(K)$. In addition, if $\bar{x} \in B^K_{\bar{p}}$ for all $K \in \mathscr{K}$ then $(\bar{x}, \bar{p})$ is called an *equilibrium* of $\boldsymbol{m}$. An optimum resp. equilibrium $(\bar{x}, \bar{p})$ is *strong* if

$$\sum_\Omega x^i = \sum_\Omega a^i .$$

Theorem 7.1. Let $(\bar{x}, \bar{p})$ be an optimum of $\boldsymbol{m}$, then $\bar{x} \in \mathfrak{C}(\boldsymbol{m})$.

Proof. If $\bar{x} \notin \mathfrak{C}(\boldsymbol{m})$ then $\exists K \in \mathscr{K}\ \exists y \in \boldsymbol{m}(K)$, $y^K \geqq x^K$ and $|K(y \succ x)|/|K| \geqq h(K)$. $\boldsymbol{m}(K) \subset \boldsymbol{B}^K_p$ for all $p \in \mathscr{P} \Rightarrow y \in \boldsymbol{B}^K_{\bar{p}}$. It contradicts the property of optimum $(\bar{x}, \bar{p})$. So $\bar{x} \in \mathfrak{C}(\boldsymbol{m})$.

Theorem 7.2. Let $\boldsymbol{m}$ be a market and let $(\bar{x}, \bar{p})$ be an optimum of $\boldsymbol{m}$. If there exists at least one player from Ω with monotonous preference then there exists $\hat{x} \in \boldsymbol{m}(\Omega)$ such that $(\hat{x}, \bar{p})$ is a strong optimum of $\boldsymbol{m}$.

Proof. Let $k \in \Omega$, $\geqq_k$ is monotonous. We set

$$\hat{x}^i = \bar{x}^i , \quad i \neq k , \quad \hat{x}^k = \bar{x}^k + \sum_\Omega a^i - \sum_\Omega \bar{x}^i .$$

Then we can easily verify that $(\hat{x}, \bar{p})$ is a strong optimum of $\boldsymbol{m}$.

Definition 7.4. Let $\mathscr{S} \subset \exp \Omega$. Then we set

$$[\mathscr{S}] = \{K : \exists K_1, \ldots, K_r \in \mathscr{S},\ K = \bigcup_{i=1}^r K_i\} .$$

$\mathscr{S}$ is a *coalition structure* if all coalitions from $\mathscr{S}$ are disjoint and $\bigcup_{K \in \mathscr{S}} K = \Omega$.

Theorem 7.3. Let $\boldsymbol{m}$ be a market. Let $\mathscr{S} \subset \mathscr{K} \subset [\mathscr{S}]$ where $\mathscr{S}$ is a coalition structure in which each coalition has at least one player with monotonous preference. Let $(\bar{x}, \bar{p})$ be an equilibrium of $\boldsymbol{m}$. Then there exists $\hat{x}$ such that $(\hat{x}, \bar{p})$ is a strong equilibrium of $\boldsymbol{m}$.

 Proof. Set

$$z = \sum_{\Omega} a^i - \sum_{\Omega} \bar{x}^i .$$

Suppose that $z \nless 0$ as otherwise there is nothing to prove. Let

$$\mathscr{S} = \{K_j,\ j = 1, 2, \ldots, r\} .$$

If $\bar{p}'z > 0$, we set

$$\varepsilon_j = \sup \{\varepsilon : \bar{p}'(\sum_{K_j} \bar{x}^i + \varepsilon z) \leqq \bar{p}' \sum_{K_j} a^i\} , \quad j = 1, \ldots, r .$$

If $\bar{p}'z = 0$, we set

$$\varepsilon_j = \frac{1}{r} , \quad j = 1, \ldots, r .$$

Let $k_j \in K_j$ be the mentioned player whose preference is monotonous; we set

$$\hat{x}^i = \bar{x}^i \quad \text{for} \quad i \neq k_j , \quad j = 1, \ldots, r ,$$
$$\hat{x}^{k_j} = \bar{x}^{k_j} + \varepsilon_j z , \qquad j = 1, \ldots, r .$$

Then

$$\sum_{\Omega} \hat{x}^i = \sum_{\Omega} \bar{x}^i + z \sum_{j=1}^{r} \varepsilon_j = \sum_{\Omega} a^i \Rightarrow \hat{x} \in \boldsymbol{m}(\Omega) ,$$
$$\sum_{K} \bar{p}'\hat{x}^i = \sum_{K} \bar{p}'a^i \quad \text{for} \quad K \in \mathscr{S} \Rightarrow \hat{x} \in \boldsymbol{B}_{\bar{p}}^{K} \quad \text{for} \quad K \in \mathscr{K}$$

and $\hat{x} \succeq \bar{x}$ for the monotonity of $\succeq_{k_j}$, $j = 1, \ldots, r$. So $(\hat{x}, \bar{p})$ is a strong equilibrium of $\boldsymbol{m}$.

Remark 7.1. Suppose, in addition, that for every $j = 1, 2, \ldots, r$ there exists $K_j^* \subset K_j$ such that $|K_j^*|/|K_j| \geqq h(K_j)$ and $\succeq_i$, $i \in K_j$, are strictly monotonous. Then every equilibrium is already a strong equilibrium.

Theorem 7.4. Let $\boldsymbol{m}$ be a market. Let $\mathscr{K}$ fulfil condition

(7.4) $\forall S \in \mathscr{K}\ \exists S_1, \ldots, S_r \in \mathscr{K}$, $S, S_1, \ldots, S_r$ are mutually disjoint, $S \cup S_1 \cup S_2 \cup \ldots \cup S_r = \Omega$ and $\exists S^* \subset S$, $|S^*|/|S| \geqq h(S)$ such that $\succeq_i$, $i \in S^*$, are positively monotonous.

Then every optimum of $\boldsymbol{m}$ is also an equilibrium of $\boldsymbol{m}$.

Proof. If $(\bar{x}, \bar{p})$ is an optimum and it is not equilibrium then there exists $S \in \mathscr{K}$ such that $\bar{x} \notin \boldsymbol{B}_{\bar{p}}^{S}$, it means

$$\sum_{S} \bar{p}'\bar{x}^i > \sum_{S} \bar{p}'a^i . \tag{7.4.1}$$

Let $S_1, S_2, \ldots, S_r \in \mathscr{K}$ fulfil (7.4). Since

$$\bar{x} \in \mathbf{m}(\Omega) \Rightarrow \sum_{\Omega} \bar{x}^i \leqq \sum_{\Omega} a^i \Rightarrow \bar{p}' \sum_{\Omega} \bar{x}^i \leqq \bar{p}' \sum_{\Omega} a^i$$

then also

$$(7.4.2) \quad \sum_{S} \bar{p}'\bar{x}^i + \sum_{S_1} \bar{p}'\bar{x}^i + \ldots + \sum_{S_r} \bar{p}'\bar{x}^i \leqq \sum_{S} \bar{p}'a^i + \sum_{S_1} \bar{p}'a^i + \ldots + \sum_{S_r} \bar{p}'a^i .$$

It follows from (7.4.1) and (7.4.2) that there exists $S_j = K$ among $S_1, \ldots, S_r$ such that

$$(7.4.3) \qquad \sum_{K} \bar{p}'\bar{x}^i < \sum_{K} \bar{p}'a^i .$$

Let $K^* = \{i \in K : \succsim_i \text{ positively monotonous}\}$ then, according to (7.4), $|K^*|/|K| \geqq h(K)$. We choose $z > 0$ such that

$$\bar{p}'z = \sum_{K} \bar{p}'a^i - \sum_{K} \bar{p}'\bar{x}^i > 0 .$$

Set

$$\hat{x}^i = \bar{x}^i \quad \text{for} \quad i \notin K^* ,$$

$$\hat{x}^i = \bar{x}^i + \frac{1}{|K^*|} z \quad \text{for} \quad i \in K^* .$$

Then obviously $\hat{x} \in \mathbf{B}_{\bar{p}}^K$, $\hat{x}^i \succ_i \bar{x}^i$ for $i \in K^*$, and it contradicts to the property of optimum $(\bar{x}, \bar{p})$. Hence $(\bar{x}, \bar{p})$ is an equilibrium.

Theorem 7.5. Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be markets,

$$\mathbf{m}_1 = (\Omega, \mathscr{H}, R_+^m, (\succsim_i)_{i\in\Omega}, (a^i)_{i\in\Omega}, h_1) ,$$

$$\mathbf{m}_2 = (\Omega, \mathscr{K}, R_+^m, (\succsim_i)_{i\in\Omega}, (a^i)_{i\in\Omega}, h_2) .$$

Let $\mathscr{H} - \{\Omega\}$ include only mutually disjoint coalitions, $\mathscr{K} \subset [\mathscr{H}]$, $h_2(S) \geqq h_1(S)$ for $S \in \mathscr{K} \cap \mathscr{H}$ and $h_2(S) \geqq \max\{h_1(S_1), \ldots, h_1(S_r)\}$ for $S \in K$, $S = S_1 \cup \ldots \cup S_r$, $S_i \in \mathscr{H}$, $i = 1, \ldots, r$. Further, we suppose that for all $S \in \mathscr{H}$ is $|S^*|/|S| \geqq h_1(S)$ where

$$S^* = \{i \in S : \succsim_i \text{ is positively monotonous}\} .$$

Then every optimum, resp. equilibrium, of $\mathbf{m}_1$ is the one of $\mathbf{m}_2$.

Proof. Let $(\bar{x}, \bar{p})$ be an optimum of $\mathbf{m}_1$. If $(\bar{x}, \bar{p})$ is no optimum of $\mathbf{m}_2$ then there exists $K \in \mathscr{K}$ and $y \in \mathbf{B}_{\bar{p}}^K$ so that $y^K \geqq x^K$ and $|K(y \succ \bar{x})|/|K| \geqq h_2(K)$, where $K(y \succ \bar{x}) = \{i \in K : y^i \succ_i x^i\}$. Let $K_1, \ldots, K_r \in \mathscr{H}$ such that $K = K_1 \cup \ldots \cup K_r$, $K_i \neq \Omega$. Then:

(i) Either $y \in B_{\bar{p}}^{K_i}$, $i = 1, \ldots, r$. Then there exists K_j among $K_1, \ldots, K_r$ such that $|K_j(y \succ \bar{x})|/|K_j| \geqq h_1(K_j)$ for $h_2(K) \geqq \max\{h_1(K_1), \ldots, h_2(K_r)\}$ and it contradicts to the assumption that $(\bar{x}, \bar{p})$ is optimum of $\boldsymbol{m}_1$.

(ii) Or there exists $K_i \in \{K_1, \ldots, K_r\}$ such that $y \notin B_{\bar{p}}^{K_i}$; $y \in B_{\bar{p}}^{K} \Rightarrow \exists K_j \in \in \{K_1, \ldots, K_r\}$ such that

$$\sum_{K_j} \bar{p}' y^i < \sum_{K_j} \bar{p}' a^i .$$

Let us choose $z > 0$, $z \in R_+^m$ such that

$$\bar{p}' z = \sum_{K_j} \bar{p}' a^i - \sum_{K_j} \bar{p}' y^i$$

and set

$$\bar{y}^i = y^i \quad \text{for} \quad i \notin K_j^* = \{i \in K_j : \succsim_i \text{ positively monotonous}\} ,$$

$$\bar{y}^i = y^i + \frac{1}{|K_j^*|} z \quad \text{for} \quad i \in K_j^* .$$

Obviously $\bar{y} \in B_{\bar{p}}^{K_j}$, $\bar{y}^K \succsim \bar{x}^K$ and $(K_j(\bar{y} \succ \bar{x})|/|K_j| \geqq h_1(K_j)$ and it is a contradiction with properties of the optimum $(\bar{x}, \bar{p})$. Hence, $(\bar{x}, \bar{p})$ is an optimum of $\boldsymbol{m}_2$. In the same way we may prove the statement for equilibrium.

Remark 7.2. Let us denote

$$\mathfrak{N}(\boldsymbol{m}) = \{x : \exists p \in \mathscr{P}, (x, p) \text{ is an equilibrium of } \boldsymbol{m}\} ,$$

$$\mathfrak{O}(\boldsymbol{m}) = \{x : \exists p \in \mathscr{P}, (x, p) \text{ is an optimum of } \boldsymbol{m}\} .$$

We know that $\mathfrak{N}(\boldsymbol{m}) \subset \mathfrak{O}(\boldsymbol{m}) \subset \mathfrak{C}(\boldsymbol{m})$ and $\mathfrak{N}(\boldsymbol{m}) = \mathfrak{O}(\boldsymbol{m})$ under relatively weak assumptions. But, if the set of players is finite then $\mathfrak{O}(\boldsymbol{m}) \neq \mathfrak{C}(\boldsymbol{m})$ even if $\boldsymbol{m}$ fulfills much stronger assumptions. We testify it by the following example.

$$\Omega = \{1, 2\} , \quad \mathscr{K} = \exp \Omega , \quad a^1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad a^2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

u_1, u_2 are utility functions corresponding to $\succsim_1, \succsim_2$,

$$u_1(x) = u_2(x) = x_1 . x_2 .$$

We can easily verify that $\succsim_i$, $i = 1, 2$, are positively monotonous and convex. Let $h \in \mathscr{F}(\mathscr{K})$ be an arbitrary decision function. Let

$$\bar{x}^1 = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}, \quad \bar{x}^2 = \begin{pmatrix} 3 - \sqrt{2} \\ 3 - \sqrt{2} \end{pmatrix}, \quad \bar{x} = (x^i)_{i=1,2} .$$

We show that $\bar{x} \in \mathfrak{C}(\boldsymbol{m})$; let, on the contrary, $\bar{x} \notin \mathfrak{C}(\boldsymbol{m})$. Then there exists y such that $y \succ_{\boldsymbol{m}} \bar{x}$.

$$u_1(\bar{x}^1) = \sqrt{2} \,.\, \sqrt{2} = 2 = u_1(a^1)\,,$$
$$u_2(\bar{x}^2) = (3 - \sqrt{})^2 > 2 = u_2(a^2)\,.$$

Consequently, y dominates $\bar{x}$ only via $\{1, 2\}$. Without loss of generality we suppose that

$$y^1 \succ_1 \bar{x}^1 \Rightarrow y_1^1 \,.\, y_2^1 > 2 \Rightarrow y_1^1 + y_2^1 > 2\sqrt{2}$$

and

$$y^2 \succeq_2 \bar{x}^2 \Rightarrow y_1^2 y_2^2 \geqq (3 - \sqrt{2})^2 \Rightarrow y_1^2 + y_2^2 \geqq 2(3 - \sqrt{2})\,.$$

Hence, $y_1^1 + y_2^1 + y_1^2 + y_2^2 > 2\sqrt{2} + 2(3 - \sqrt{2}) = 6$. It is not possible, as $y \in \boldsymbol{m}(\Omega) \Rightarrow y_1^1 + y_1^2 \leqq 3$ and $y_2^2 + y_2^1 \leqq 3$. It means that $\bar{x} \in \mathfrak{C}(\boldsymbol{m})$. Now, we show that $\bar{x} \notin \mathfrak{D}(\boldsymbol{m})$, it means that there exists no $p \in \mathscr{P}$ such that $(\bar{x}, p)$ is an optimum of $\boldsymbol{m}$. We divide the proof into three cases:

(i) $p = (p_1, p_2)' \in \mathscr{P}$, $p_1 = 0$ or $p_2 = 0$.
We choose $y_1 = 1$, $y_2 > 2$ resp. $y_1 > 2$, $y_2 = 1$ for $p_2 = 0$ resp. $p_1 = 0$. We may easily verify that

$$(*) \qquad \begin{cases} p_1 y_1 + p_2 y_2 = p_1 a_1^1 + p_2 a_2^1 = p_1 + 2p_2\,, \\ y_1 y_2 > 2 = u_1(\bar{x}^1)\,, \end{cases}$$

what means that $(\bar{x}, p)$ is not optimum of $\boldsymbol{m}$.

(ii) $p_i \neq 0$, $i = 1, 2$, $p_2 \neq \frac{1}{3}$. We set $y_1 = (1 + p_2)/2p_1$, $y_2 = (1 + p_2)/2p_2$. Obviously

$$p_1 y_1 + p_2 y_2 = 1 + p_2 = p_1 a_1^1 + p_2 a_2^1\,,$$
$$y_1 y_2 = (1 + p_2)^2/4p_1 p_2 = (p_1 + 2p_2)^2/4p_1 p_2 > 2\,.$$

So the pair $(y_1, y_2)'$ fulfills $(*)$, hence $(\bar{x}, p)$ is not optimum of $\boldsymbol{m}$.

(iii) $p_1 = \frac{2}{3}$, $p_2 = \frac{1}{3}$:
Let us choose

$$y_2 = \tfrac{5}{2}\,, \quad y_1 = \tfrac{5}{4}\,.$$

Then

$$p_1 y_1 + p_2 y_2 = \tfrac{2}{3} \,.\, \tfrac{3}{4} + \tfrac{1}{3} \,.\, \tfrac{5}{2} = 1 + \tfrac{2}{3} = a_1^2 p_1 + a_2^2 p_2\,,$$
$$y_1 y_2 = \tfrac{5}{2} \,.\, \tfrac{1}{4} = \tfrac{25}{8} > 11 - 6\sqrt{2} = u_2(\bar{x}^2)$$

and it means $(\bar{x}, p)$ is not optimum of $\boldsymbol{m}$.

Definition 8.1. Let $\boldsymbol{m}$ be a market. Market state $(\bar{x}, \bar{p})$ is called a *competitive equilibrium* if

1) $$\bar{x} \in \boldsymbol{B}_{\bar{p}}^{i} \quad \text{for all} \quad i = 1, \ldots, n ,$$

2) $$y \in \underset{\Omega}{\times} R_{+}^{m} , \quad y^{i} \succ_{i} \bar{x}^{i} \Rightarrow y \notin \boldsymbol{B}_{\bar{p}}^{i} .$$

A competitive equilibrium $(\bar{x}, \bar{p})$ is called *strong* if

3) $$\sum_{\Omega} \bar{x}^{i} = \sum_{\Omega} a^{i} .$$

Theorem 8.1. Let $\boldsymbol{m}$ be a market, let $\succ_{i}$, $i \in \Omega$, be monotonous and let $(\bar{x}, \bar{p})$ be a competitive equilibrium of $\boldsymbol{m}$. Then there exists $\hat{x}$ such that $(\hat{x}, p)$ is a strong competitive equilibrium of $\boldsymbol{m}$.

Proof. The proof is analogous to the proof of Theorem 7.3.

Theorem 8.2. (Existence of competitive equilibrium.) Let $\succeq_{i}$, $i \in \Omega$, be continuous and convex. Then there exists a competitive equilibrium of $\boldsymbol{m}$. Moreover, if $\succeq_{i}$, $i \in \Omega$, are monotonous, then there exists a strong competitive equilibrium of $\boldsymbol{m}$.

Proof. See [1], page 54, Theorem 4.8.

Theorem 8.3. Let $\boldsymbol{m}$ be a market. Let $\succeq_{i}$ be positively monotonous for all $i \in \Omega$. Then every (strong) competitive equilibrium of $\boldsymbol{m}$ is a (strong) equilibrium of $\boldsymbol{m}$.

Proof. Let $(\bar{x}, \bar{p})$ be a (strong) competitive equilibrium. Suppose that $(\bar{x}, \bar{p})$ is not (strong) equilibrium. Then there exists $K \in \mathscr{K}$, $y \in \boldsymbol{B}_{\bar{p}}^{K}$ such that

$$y^{K} \geqq \bar{x}^{K} \quad \text{and} \quad |K(y \succ x)|/|K| \geqq h(K) .$$

According to Definition of competitive equilibrium we have

(8.3.1) $$\bar{p}' y^{i} > \bar{p}' a^{i} , \quad i \in K(y \succ \bar{x}) .$$

Then there exists $j \in K$ such that $\bar{p}' y^{j} < \bar{p}' a^{j}$ for $y \in \boldsymbol{B}_{\bar{p}}^{K}$ and (8.3.1). Let us choose $z > 0$ such that $\bar{p}' z = \bar{p}' a^{j} - \bar{p}' y^{j} > 0$ and set $\bar{y}^{j} = y^{j} + z$. Obviously, $\bar{p}' \bar{y}^{j} = \bar{p}' a^{j}$ and $\bar{y}^{j} \succ_{j} \bar{x}^{j}$ what contradicts to the assumption that $(\bar{x}, \bar{p})$ is competitive equilibrium.

Previous Theorems 8.2 and 8.3 imply the following one.

Theorem 8.4. (Existence of equilibrium) Let $\boldsymbol{m}$ be a market. Let $\succeq_{i}$, $i \in \Omega$, be continuous, convex and positively monotonous. Then there exists an equilibrium of $\boldsymbol{m}$. In addition, if $\succeq_i$, $i \in \Omega$, are monotonous then there exists a strong equilibrium of $\boldsymbol{m}$.

CHAPTER III: APPLICATION OF GAME THEORY IN MARKET THEORY

Let $\boldsymbol{m} = (\Omega, \mathscr{K}, R^m_+, (\succeq_i)_{i\in\Omega}, (a^i)_{i\in\Omega}, h)$ be a cooperative market defined in Chapter II. In this chapter we shall suppose that $\succeq_i$ are continuous for all $i \in \Omega$. Let u^i, $i \in \Omega$, be continuous utility functions corresponding to $\succeq_i$, $i \in \Omega$. Without loss of generality we assume that

$$u^i(0) = 0 \quad \text{for all} \quad i \in \Omega .$$

Let $u(.)$ be the mapping from $\underset{\Omega}{\times} R^m_+$ to R^n,

$$u : (x^i)_{i\in\Omega} \to [u^1(x^1), u^2(x^2), \ldots, u^n(x^n)] .$$

We denote

$$u(M) = \{x \in R^n : \exists \tilde{x} \in M,\ u(\tilde{x}) = x\} \quad \text{for} \quad M \subset \underset{\Omega}{\times} R^m_+ ,$$

$$u^{-1}(L) = \{\tilde{x} \in \underset{\Omega}{\times} R^m_+ : u(\tilde{x}) \in L\} \quad \text{for} \quad L \subset R^n ,$$

and $u(\boldsymbol{m}(\emptyset)) = R^n$.

Further, u^i is called nonnegative if $u^i(x) \geqq 0$ for $x \in R^m_+$.

9. Cooperative Game Corresponding to the Market

Lemma 9.1. Let u^i be concave and nonnegative for all $i \in \Omega$. Then $u(\boldsymbol{m}(K))$ are closed and convex for all $K \in \mathscr{K}$.

Proof. Closeness: Let $x_\nu \in u(\boldsymbol{m}(K))$, $\nu = 1, 2, \ldots, x_\nu$, converge to $x \in R^n$. Let $\tilde{x}_\nu \in \boldsymbol{m}(K)$, $u(\tilde{x}_\nu) = x_\nu$. $\boldsymbol{m}(K)$ is compact and, hence, we can choose a subsequence from $\{\tilde{x}_\nu\}_{\nu=1}^\infty$ which converges to a certain $\tilde{x} \in \boldsymbol{m}(K)$. Obviously $u(\tilde{x}) = x$, so $x \in u(\boldsymbol{m}(K))$.

Convexity: Let $x, y \in u(\boldsymbol{m}(K))$, $0 \leqq \alpha, \beta \leqq 1$, $\alpha + \beta = 1$. Let $\tilde{x}, \tilde{y} \in \boldsymbol{m}(K)$, $u(\tilde{x}) = x$, $u(\tilde{y}) = y$. $\alpha\tilde{x} + \beta\tilde{y} \in \boldsymbol{m}(K)$ and $u(\alpha\tilde{x} + \beta\tilde{y}) \geqq \alpha\, u(\tilde{x}) + \beta\, u(\tilde{y}) = \alpha x + \beta y$ for concavity of u^i, $i \in \Omega$. It follows from continuity of u^i, $i \in \Omega$ that there exists such $\tilde{z}$ that $0 \leqq \tilde{z}^i \leqq (\alpha\tilde{x} + \beta\tilde{y})^i$ and $u^i(\tilde{z}^i) = (\alpha x + \beta y)^i$, $i \in \Omega$. Obviously $\tilde{z} \in \boldsymbol{m}(K)$, $u(\tilde{z}) = \alpha x + \beta y$. Hence, $\alpha x + \beta y \in u(\boldsymbol{m}(K))$.

Definition 9.1. Let $K \in \mathscr{K}$. Then we denote

$$\boldsymbol{v}(K) = \{x \in R^n : \exists y \in u(\boldsymbol{m}(K)) \text{ such that } x^K \leqq y^K\} .$$

The next statements follow from Lemma 9.1.

Theorem 9.1. Let u^i, $i \in \Omega$, be concave and non-negative. Then the triple $(\Omega, \mathscr{K}, \boldsymbol{v})$, where $\boldsymbol{v}$ is defined above, is a characteristic function and $v^i = u^i(a^i)$ for all

$$i \in I(\mathbf{m}) = \{i \in \Omega : \{i\} \in \mathscr{K}\}.$$

Lemma 9.2. Let u^i be concave and non-negative for all $i \in \Omega$. Then $H = u(\mathbf{m}(\Omega))$ is convex and compact.

Definition 9.2. Cooperative game $\boldsymbol{G}_{\mathbf{m}} = (\Omega, \mathscr{K}, \boldsymbol{v}, H, h)$, where $\boldsymbol{v}$ and H are defined by Definition 9.1 and Lemma 9.2, is called the *game corresponding to the market* **m**.

It is easy to verify the following theorem.

Theorem 9.2. Let u^i, $i \in \Omega$, be concave and non-negative. Then $\boldsymbol{G}_{\mathbf{m}}$ is guaranted and ordinary. If u^i, $i \in \Omega$, are monotonous, in addition, then H fulfills condition (7) from Chapter I.

10. Connection between Games and Markets

Definition 10.1. Let $\mathbf{m}$ be a cooperative market. Then we denote

$$\bar{A}_{\mathbf{m}} = \{x \in \mathbf{m}(\Omega) : x^i \gtrsim_i a^i \text{ for all } i \in I(\mathbf{m})\},$$
$$E_{\mathbf{m}} = \{x \in \mathbf{m}(\Omega) : \neg(\exists y : y \gtrsim x, \ |\Omega(y \succ x)|/|\Omega| \geqq h(\Omega)\},$$
$$A_{\mathbf{m}} = \bar{A}_{\mathbf{m}} \cap E_{\mathbf{m}}, \quad \bar{E}_{\mathbf{m}} = \mathbf{m}(\Omega).$$

$\bar{A}_{\mathbf{m}}$, resp. $E_{\mathbf{m}}$, are called the sets of individually, resp. group, rational distributions of goods.

Next theorem follows from Definition 10.1 immediately.

Theorem 10.1. Let u^i be concave and non-negative for all $i \in \Omega$. Let $\bar{A}$, E be the sets of individually and group rational payments distributions of the game $\boldsymbol{G}_{\mathbf{m}}$, $\bar{E} = H$, $A = E \cap \bar{A}$. Then

$$u(\bar{E}_{\mathbf{m}}) = \bar{E}, \quad u(E_{\mathbf{m}}) = E, \quad u(\bar{A}_{\mathbf{m}}) = \bar{A}, \quad u(A_{\mathbf{m}}) = A.$$
$$u^{-1}(\bar{E}) \cap \bar{E}_{\mathbf{m}} = \bar{E}_{\mathbf{m}}, \quad u^{-1}(E) \cap \bar{E}_{\mathbf{m}} = E_{\mathbf{m}},$$
$$u^{-1}(\bar{A}) \cap \bar{E}_{\mathbf{m}} = \bar{A}_{\mathbf{m}}, \quad u^{-1}(A) \cap \bar{E}_{\mathbf{m}} = A_{\mathbf{m}}.$$

Theorem 10.2. Let u^i, $i \in \Omega$, be concave and non-negative, let $P \subset \mathbf{m}(\Omega)$. Then

$$u(\mathfrak{C}^{\mathbf{m}}(P)) = \mathfrak{C}^{G\mathbf{m}}(u(P)).$$

Proof. We prove, first, that

$$u(\mathfrak{C}^{\mathbf{m}}(P)) \subset \mathfrak{C}^{G\mathbf{m}}(u(P)).$$

Let $x \subset u(\mathfrak{C}^{\mathbf{m}}(P))$ and let $x \notin \mathfrak{C}^{G\mathbf{m}}(u(P))$. Then there exists $y \in u(P)$ such that $y \succ_{G\mathbf{m}} x$. Let $\tilde{x} \in \mathfrak{C}^{\mathbf{m}}(P)$, $\tilde{y} \in P$ be such that $u(\tilde{x}) = x$ and $u(\tilde{y}) = y$. Obviously $\tilde{y} \succ_{\mathbf{m}} \tilde{x}$, and it contradicts to $\tilde{x} \in \mathfrak{C}^m(P)$. Hence, $x \in \mathfrak{C}^{G_m}(u(P))$. Now, we prove that

$$\mathfrak{C}^{G_m}(u(P)) \subset u(\mathfrak{C}^m(P)).$$

Let $x \in \mathfrak{C}^{G_m}(u(P))$, and $\tilde{x} \in P$, $u(\tilde{x}) = x$. Then $\tilde{x} \in \mathfrak{C}^m(P)$, otherwise it leads to contradiction with $x \in \mathfrak{C}^{G_m}(u(P))$. Hence $x \in u(\mathfrak{C}^m(P))$.

Theorem 10.3. Let u^i, $i \in \Omega$, be concave and non-negative, let $P \subset \boldsymbol{m}(\Omega)$. Then

$$\mathfrak{C}^m(P) = P \cap u^{-1}(\mathfrak{C}^{G_m}(u(P))).$$

Proof. The inclusion

$$\mathfrak{C}^m(P) \subset P \cap u^{-1}(\mathfrak{C}^{G_m}(u(P)))$$

follows from Theorem 10.2. It remains, therefore, to prove only the inclusion

$$P \cap u^{-1}(\mathfrak{C}^{G_m}(u(P))) \subset \mathfrak{C}^m(P).$$

Let $\tilde{x} \in P \cap u^{-1}(\mathfrak{C}^{G_m}(u(P)))$ and let $\tilde{x} \notin \mathfrak{C}_m(P)$. Then there exists $\tilde{y} \in P$, such that $\tilde{y} \succ_m \tilde{x}$. Let $u(\tilde{x}) = x$, $u(\tilde{y}) = y$. Then $y \succ_{Gm} x$, but it is a contradiction with $x \in \mathfrak{C}^{G_m}(u(P))$. Hence, $x \in \mathfrak{C}^m(P)$.

From Theorems 10.1 and 10.3 the next statement follows.

Corollary 10.1. Let u^i be concave and non-negative for all $i \in \Omega$. Then we have

$$\mathfrak{C}^m(\bar{E}_m) = \bar{E}_m \cap u^{-1}(\mathfrak{C}^{G_m}(\bar{E})), \quad \mathfrak{C}^m(E_m) = \bar{E}_m \cap u^{-1}(\mathfrak{C}^{G_m}(E)),$$
$$\mathfrak{C}^m(\bar{A}_m) = \bar{E}_m \cap u^{-1}(\mathfrak{C}^{G_m}(\bar{A})), \quad \mathfrak{C}^m(A_m) = \bar{E}_m \cap u^{-1}(\mathfrak{C}^{G_m}(A)).$$

Theorem 10.4. Let u^i be concave and non-negative for all $i \in \Omega$. Then $\bar{A}_m \neq \emptyset$ and $\mathfrak{C}^m(\bar{E}_m) = \mathfrak{C}^m(\bar{A}_m)$.

Proof. The assertion follows from the foregoing corollary and from Theorem 3.3.

Definition 10.2. Let $M \subset \times_{\Omega} R^m_+$. We denote

$$J(M) = \{x \in M : \neg(\exists y \in M \text{ such that } y \succ x)\}.$$

Theorem 10.5. Let u^i be concave and monotonous. Let H be a polyhedron. Then

$$\mathfrak{C}^m(\bar{E}_m) = \mathfrak{C}^m(J(E_m)) = \mathfrak{C}^m(\bar{A}^m) = \mathfrak{C}^m(J(\bar{A}_m)).$$

Proof. The assertion follows from Corollary 10.1 and from Theorem 3.9.

Theorem 10.6. Let u^i be concave and monotonous for all $i \in \Omega$. Let m fulfil the condition

$$|\Omega|\, h(\Omega) > \max_{\mathscr{K} - \{\Omega\}} |S|.$$

Then

$$\mathfrak{C}^m(J(\bar{E}_m)) = \mathfrak{C}^m(E_m) \quad \text{and} \quad \mathfrak{C}^m(J(\bar{A}_m)) = \mathfrak{C}_m(A_m) \,.$$

Proof. The statement follows from Corollary 10.1 and from Theorem 3.11.

Remark 10.1. Without the assumption of concavity of utility functions, $\mathfrak{v}(K)$ may be nonconvex. Let us consider the following example.

$$\Omega = \{1, 2\}\,, \quad \mathscr{K} = \exp \Omega,\, u^1(x) = u^2(x) = (x_1)^2 \,.$$

$$a^1 = a^2 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Let

$$x = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}; \quad y = \left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \quad \text{then}$$

$$u(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u(y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\tfrac{1}{2}\, u(x) + \tfrac{1}{2}\, u(y) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

But, it is easy to verify that there exists no $z \in \boldsymbol{m}(\Omega)$ such that

$$u(z) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Hence, $\mathfrak{v}(\Omega)$ is not convex.

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