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# The Spectrum of the Discrete Cesàro operator

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Some properties of the discrete Cesàro operator in $l_\infty$, especially the properties of the spectrum, are investigated.

## 1. INTRODUCTION

Let $\{x_1, x_2, \ldots\}$, $\{y_1, y_2, \ldots\}$ be complex sequences. The discrete Cesàro operator $C_0\{x_n\} = \{y_n\}$ is defined by

$$y_n = \frac{x_1 + \ldots + x_n}{n}, \quad n = 1, 2, \ldots \tag{1}$$

It is well known that $C_0$ is stable in the sense of bounded input-bounded output definition of stability.

In what follows the space of all bounded sequences with the usual metric will be denoted $l_\infty$.

Attempting to realize $C_0$ by feedback, the properties of the inverse (denoted $A_\lambda$ in what follows) to the more general operator $C_\lambda$, defined by

$$y_n = \frac{x_1 + \ldots + x_n}{n} - \lambda x_n \tag{2}$$

($\lambda$ complex), are substantial [6]. It is well known that the stability region of $C_\lambda$ (realized by feedback) in the complex plane $\lambda$ ($\lambda$ being the generalized gain) is the same what is named in the theory of operators the resolvent set of $C_0$. Its complement is called the spectrum of $C_0$.

In [1], the spectrum of the operator $C_1$ acting in the space $L_\infty(0,1)$ of functions bounded and Lebesgue integrable in the interval $(0,1)$ and defined by

$$y(t) = \frac{1}{t}\int_0^t x(t)\,\mathrm{d}t \tag{3}$$

has been found. In [2], there has been shown with the aid of $C_1$ that the spectrum of $C_0$ in $l_\infty$ is the same as that of $C_1$ in $L_\infty(0,1)$ and is given by the closed disk with the centre $1/2$ and the radius $1/2$.

Given a $\lambda$ in the spectrum and a sequence $\{y_n\} \in l_\infty$, one may ask what is the behaviour of $\{x_n\}$ computed from (2). The point spectrum of $C_0$ is defined as those $\lambda$ for which the inverse $A_\lambda$ is not existing, and the residual spectrum of $C_0$ is defined as those $\lambda$ for which the mapping of $l_\infty$ by $C_\lambda$ into $l_\infty$ is not dense in $l_\infty$ ([3], p. 182).

Among the points of the point spectrum, $\lambda$ fulfilling

$$C_\lambda\{x_n\} = \emptyset \tag{4}$$

called eigenvalues, are of special interest, with $\{x_n\}$ called eigenvectors of $C_0$ in $l_\infty$.

## 2. THE LINEAR EQUATIONS CONNECTED WITH $C_0$

The equation in (4) is identical with the infinite system of linear equations

$$\begin{aligned} (1 - \lambda)\, x_1 &= 0\,, \\ \tfrac{1}{2}x_1 + (\tfrac{1}{2} - \lambda)\, x_2 &= 0\,, \\ \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 + (\tfrac{1}{3} - \lambda)\, x_3 &= 0\,, \\ &\vdots \end{aligned} \tag{5}$$

Solving this system with the assumption $x_1 = 1$ gives

$$\begin{aligned} \lambda &= 1\,, \\ x_1 &= x_2 = \ldots = 1\,. \end{aligned} \tag{6}$$

Similarly, putting for $l > 1$ $x_1 = \ldots = x_{l-1} = 0$, $x_l = 1$, one gets

$$\begin{aligned} \lambda &= 1/l\,, \\ x_{l+j} &= \binom{l + j - 1}{l - 1}, \quad j = 1, 2, \ldots \end{aligned} \tag{7}$$

The values of (7) have been found in [2]. It is clear that all solutions in (6), (7) are linearly independent. For $\lambda$ from (6), (7), $A_\lambda$ is not existing, as it is clearly seen from (5), since for a nonzero $\{y_n\}$, no $\{x_n\}$ exists fulfilling (5). Thus these $\lambda$ are contained in the point spectrum of $C_0$. Only the sequence $\{1, 1, \ldots\}$ from (6) is in $l_\infty$ and this is the only eigenvector of $C_0$ in $l_\infty$. $\lambda = 1$ is the corresponding eigenvalue.

Subtracting from the equation in (2) the analogous one with $n$ replaced by $n - 1$, one gets

$$x_n - n(\lambda x_n + y_n) = -(n-1)(\lambda x_{n-1} + y_{n-1}) \tag{8}$$

and substituting with the assumption $\lambda \neq 0$

$$\lambda x_n + y_n = \xi_n, \tag{9}$$

one has

$$(1 - n\lambda)\xi n + (n-1)\lambda\xi_{n-1} = y_n \tag{10}$$

or, with the assumption $\lambda \neq 1, 1/2, 1/3, \ldots$:

$$\xi_n + \frac{(n-1)\lambda}{1 - n\lambda}\xi_{n-1} = \frac{y_n}{1 - n\lambda}. \tag{11}$$

The difference equations (10) or (11) represent the inverse $A_\lambda$ to $C_\lambda$, the transformation

$$x_n = \frac{1}{\lambda}(\xi_n - y_n) \tag{12}$$

being very simple. Especially, for $\{y_n\} \in l_\infty$ and $\lambda$ in the spectrum, but $\lambda \neq 1, 1/2, \ldots$, $\{x_n\}$ and $\{\xi_n\}$ possess the same order of growth.

Now, solving (11) recurrently with the restriction

$$\lambda \neq 0, 1, 1/2, 1/3, \ldots \tag{13}$$

one gets an explicit formula representing the solution there of:

$$\xi_n = \frac{1}{n\lambda\left(1 - \frac{1}{\lambda}\right)\cdots\left(1 - \frac{1}{n\lambda}\right)}. \tag{14}$$

$$\cdot\left(-y_1 + y_2\frac{\frac{1}{\lambda} - 1}{1!} + \ldots + (-1)^n y_n \frac{\left(\frac{1}{\lambda} - 1\right)\cdots\left(\frac{1}{\lambda} - (n-1)\right)}{(n-1)!}\right).$$

Remembering the definition of $A_\lambda$ in [6] as a lower triangular matrix, one gets from (14) for its elements (in [6], the subscripts begin with $n = 0$)

$$a_{nk} = \frac{1}{\lambda(1 - (n+1)\lambda)} \frac{1}{\left(1 - \frac{1}{(k+1)\lambda}\right)\cdots\left(1 - \frac{1}{n\lambda}\right)}. \tag{15}$$

For $\lambda = 0$, one finds separately

$$A_0 = \begin{pmatrix} 1, & 0, & \dots \\ -1, & 2, & 0, & \dots \\ 0, & -2, & 3, & 0, & \dots \\ & & \vdots \end{pmatrix}. \tag{16}$$

This is the matrix in (18), [6]. It was found already by Toeplitz.

## 4. THE SPECTRUM OF $C_0$ ON $l_\infty$

From (15) and the known formula for the function $\Gamma$ ([5], p. 439–440), one gets asymptotically for $n \to \infty$

$$a_{n0} \cong -\frac{1}{\lambda^2}\Gamma\left(1 - \frac{1}{\lambda}\right) n^{(1/\lambda)-1}. \tag{17}$$

Since

$$n^{(1/\lambda)-1} = n^{\mathcal{R}e[(1/\lambda)-1]}\, e^{i\mathcal{I}m[(1/\lambda)-1]\log n}, \tag{18}$$

one gets

$$\lim_{n\to\infty} |a_{n0}| = \infty \tag{19}$$

for

$$\mathcal{R}e\,\frac{1}{\lambda} > 1 \tag{20}$$

and

$$\lim_{n\to\infty} |a_{n0}| = 0 \tag{21}$$

for

$$\mathcal{R}e\,\frac{1}{\lambda} < 1. \tag{22}$$

The points $\lambda$ fulfilling (22) lie in the outside of the disk with the centre $\frac{1}{2}$ and the radius $\frac{1}{2}$.

Comparing (15) for $k = 0$ and for an arbitrary $k$ with (17), one sees that (19) and (21) hold also for $k > 0$.

Further, we will estimate the sums $\sum_{k=0}^{n} |a_{nk}|$ supposing (22) to hold. Using the inequality

$$|1 - \mu| \geqq |1 - \mathcal{R}e\,\mu| \tag{23}$$

 and putting $\mathscr{Re}\ 1/\lambda = \nu$, one gets recurrently

$$(24)\qquad \sum_{k=0}^{n} |a_{nk}| \leqq \frac{n|\lambda|}{|\lambda|\,|n\lambda - 1|} + \frac{1}{|\lambda|\,|n\lambda - 1|}\,\frac{n-1}{1-\nu}\,.$$

But

$$(25)\qquad \lim_{n\to\infty}\left(\frac{n|\lambda|}{|\lambda|\,|n\lambda - 1|} + \frac{n-1}{|\lambda|\,|n\lambda - 1|}\,\frac{1}{1-\nu}\right) = \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\,\frac{1}{1-\nu}\,.$$

Considering (6), (7), (19), (20), (21), (22), (25) and the known Toeplitz-Schur conditions for the matrix $A_\lambda$ one sees that the following theorem holds.

**Theorem 1.** On $l_\infty$, $C_0$ is a bounded linear operator. Its spectrum is the closed disk with the centre $\frac{1}{2}$ and the radius $\frac{1}{2}$. The points $1, \frac{1}{2}, \frac{1}{3}, \ldots$ lie in the point spectrum, $\lambda = 1$ is the only eigenvalue, its multiplicity is 1, and the sequence $\{1, 1, \ldots\}$ is the only corresponding eigenvector.

The first part of this theorem is merely a special case of a theorem from [2], but our proof is direct and not depending on the results for the continuous operator $C_1$ from (4). Moreover, we will be able using (14) to prove the following theorem.

**Theorem 2.** Then open disk with the centre $\frac{1}{2}$ and the radius $\frac{1}{2}$ is, with exception of the points $\frac{1}{2}, \frac{1}{3}, \ldots$ contained in the residual spectrum of $C_0$ on $l_\infty$. The exception points are the only points of the point spectrum in the open disk.

In [1], [2], there has been shown that this open disk represents the point spectrum of $C_1$ on $L^\infty(0, 1)$, each point $\lambda$ there of being an eigenvalue, the function $t^{1/\lambda - 1}$ being the corresponding eigenfunction. Thus, although the spectrum of $C_0$ and of $C_1$ is the same disk, their finer properties are quite different.

For the first term on the right side of (14), there is ([5], p. 439 – 440) for $n \to \infty$

$$(26)\qquad \frac{1}{n\lambda\left(1 - \dfrac{1}{\lambda}\right)\ldots\left(1 - \dfrac{1}{n\lambda}\right)} \cong \frac{1}{\lambda}\,\Gamma\left(1 - \frac{1}{\lambda}\right) n^{(1/\lambda) - 1}$$

supposing that (13) holds. Moreover, in the open disk, there holds (20). The second term on the right side of (14) is a partial sum of a Newton series in the variable $1/\lambda$ ([4], p. 141 – 163).

Let $\{y_n\} \in l_\infty$ and is such that

$$(27)\qquad \sum_{k=1}^{n} |y_k| \to \infty \quad \text{for} \quad n \to \infty\,.$$

Then, since there exists a $M$ such that

$$(28)\qquad |y_k| < M \quad (M \text{ independent on } k)$$

one has 

$$\frac{\log \sum_1^n |y_k|}{\log n} < 1 + \frac{\log M}{\log n}, \tag{29}$$

thus the abscissa of absolute convergence of the Newton series with coefficients $\{y_n\}$ is ([4], p. 153)

$$\frac{1}{\lambda_0} = \overline{\lim_{n\to\infty}} \frac{\log \sum_1^n |y_k|}{\log n} \leqq 1 . \tag{30}$$

Thus for $\{y_n\} \in l_\infty$ and fulfilling (27) and for $\lambda$ fulfilling (20), the Newton series on the right side of (14) converges absolutely. Suppose $\lambda$ to be given and consider the sequences

$$\{y_1, y_2, \ldots\} = \left\langle \begin{matrix} \{2, 1, 1, \ldots\} \\ \{3, 1, 1, \ldots\} \end{matrix} \right. . \tag{31}$$

At least for one there from the Newton series has for the given $\lambda$ the sum different from zero ([4], p. 163). Choose that sequence and denote the sum of the Newton series $K(\lambda)$. But then, from (14) and (26), for $n \to \infty$

$$\xi_n \sim n^{(1/\lambda)-1} \tag{32}$$

($\sim$ means that $\xi_n$ grows asymptotically as $n^{(1/\lambda)-1}$ for $n \to \infty$).

Suppose now a sequence of sequences $\{y_{nm}\}$, $m = 1, 2, \ldots$ converging in $l_\infty$ to the chosen sequence from (31). Since the convergence in $l_\infty$ is uniform with respect to the subscript $n$, we may suppose that a $m_2$ exists so that $|y_{nm}| < 2$ independently on $n$ for every $m > m_2$. Thus to every $\varepsilon > 0$ there exists a $n_\varepsilon$ so that for every $n > n_\varepsilon$ and every $m > m_2$

$$|y_{nm}| \frac{\left|\frac{1}{\lambda} - 1\right| \ldots \left|\frac{1}{\lambda} - (n-1)\right|}{(n-1)!} + |y_{n+1,m}| \frac{\left|\frac{1}{\lambda} - 1\right| \ldots \left|\frac{1}{\lambda} - n\right|}{n!} + \ldots < \varepsilon \tag{33}$$

$\lambda$ being given and the convergence of the Newton series being absolute. Considering the terms $y_k$, $y_{km}$, $k = 1, \ldots, n-1$ one sees that to the given $\varepsilon$ an $m_\varepsilon$ exists such that for $m > \max(m_2, m_\varepsilon)$ the difference of both Newton series for $\{y_n\}$ and $\{y_{nm}\}$ is absolutely smaller than $3\varepsilon$. Thus for $m$ sufficiently large the sum of the Newton series with coefficients $y_{nm}$ is different from zero, moreover, denoting this sum with $K(m, \lambda)$, there is for $m \to \infty$ $K(m, \lambda) \to K(\lambda)$, and for $n \to \infty$

$$\xi_{nm} \sim n^{(1/\lambda)-1} . \tag{34}$$

Thus with exception of the points from (13), for every $\lambda$ fulfilling (20) we have found a sequence $\{y_n\}$ in $l_\infty$ such that the sequence $\{x_n\}$ formed therefrom by $A_\lambda$ is divergent and the same is true for the sequences $\{y_{nm}\}$ in a sufficiently small neighbourhood of $\{y_n\}$ in $l_\infty$. Thus the mapping of $l_\infty$ by $C_\lambda$ into $l_\infty$ is not dense in $l_\infty$ and the theorem 2 is proved.

For the circle enclosing the open disk, the problem of the characterization of the spectrum seems to be difficult. As one has seen, $\lambda = 1$ is an eigenvalue. Now, we will show that $\lambda = 0$ lies in the residual spectrum. From (8), one obtains for $\lambda = 0$

$$\frac{x_n - y_{n-1}}{n} = y_n - y_{n-1}\,, \tag{35}$$

thus for $\{x_n\} \in l_\infty$ not only $\{y_n\} \in l_\infty$, but also

$$\lim_{n\to\infty} (y_n - y_{n-1}) = 0\,. \tag{36}$$

For $\{y_n\} \in l_\infty$ but not fulfilling (36), $\{x_n\}$ cannot be bounded. The same is true for $\{y_{nm}\}$ in some small neighbourhood of $\{y_n\}$, the convergence in $l_\infty$ being uniform with respect to the subscript $n$. Thus $\lambda = 0$ lies in the residual spectrum of $C_0$ on $l_\infty$.

## 5. CONCLUDING REMARKS

Expressing the difference equation (11) as the respective system of linear equations for $n = 1, 2, \dots$ and supposing that for sufficiently small $|z|$ the series

$$\xi_1 z + \xi_2 z^2 + \dots = X\,, \tag{37}$$

$$y_1 z + y_2 z^2 + \dots = Y \tag{38}$$

are convergent and that (37) may be differentiated term by term (these suppositions are true for the sequences from $l_\infty$ or fulfilling (32)), one obtains

$$\lambda z(z-1)\frac{\mathrm{d}X}{\mathrm{d}z} + X = Y\,. \tag{39}$$

For the problem of the spectrum, this replacement seems to be of little advantage, perhaps with exception of the eigenvalues and eigenvectors. Substituting in (39) $Y = 0$ in accordance with (4), one finds the solution

$$X = \left(\frac{z}{1-z}\right)^{1/\lambda} \tag{40}$$

and this function may be expanded in power series precisely for $\lambda$ fulfilling (6), (7), the series with coefficients from (6), (7) resulting.

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