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Closedness Properties and Decision Problems for Finite Multi-Tape Automata*

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The present paper is devoted to an investigation of closedness properties of classes of relations which are representable by different kinds of multi-tape automata without using endmarkers and of the (classical) decision problems of finite multi-tape automata. A similiar investigation has been made by Fischer and Rosenberg [1] for the case when an endmarker is used.

The paper is not self-contained, the reader is assumed to be familiar at least with the paper [9].

1. CLOSEDNESS PROPERTIES

Let be X a finite nonempty alphabet and $n \geqq 2$ a natural number. By $\mathcal{N}_n$ resp. $\mathcal{I}\mathcal{N}_n$ resp. $\mathcal{D}_n$ we denote the sets of all n-ary relations over $W(X)$ representable by a weakly initial resp. initial resp. deterministic *ND-n-TA*** and by $\mathcal{F}\mathcal{N}_n$ resp. $\mathcal{F}\mathcal{I}\mathcal{N}_n$ resp. $\mathcal{F}\mathcal{D}_n$ we denote the sets of n-ary relations representable by finite weakly initial resp. finite initial resp. finite deterministic *ND-n-TA*. Then the following proper inclusions and equalities hold:

$$\mathfrak{P}(\mathop{\times}_{n} W(X)) = \mathcal{N}_n \supset \mathcal{I}\mathcal{N}_n \supset \mathcal{D}_n \supset \mathcal{F}\mathcal{D}_n\,, \tag{1}$$

$$\mathcal{N}_n \supset \mathcal{F}\mathcal{N}_n \supset \mathcal{F}\mathcal{I}\mathcal{N}_n \supset \mathcal{F}\mathcal{D}_n\,, \tag{2}$$

$$\mathcal{F}\mathcal{I}\mathcal{N}_n \cap \mathcal{D}_n \supset \mathcal{F}\mathcal{D}_n\,, \tag{3}$$

$$\mathcal{I}\mathcal{N}_n \cap \mathcal{F}\mathcal{N}_n = \mathcal{F}\mathcal{I}\mathcal{N}_n\,. \tag{4}$$

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** Nondeterministic n-tape automaton.

Trivially the set $\mathcal{N}_n$ is closed under union, intersection, complementation, catenation and catenation closure. We shall investigate the working of these operations within the remaining five classes of relations after some preparations.

Let us first recall the definition of representability using an endmarker (cf. e.g. [1]).

Definition. Let be $\varepsilon \notin X$ and $\mathfrak{B} = [n, X \cup \{\varepsilon\}, Z, \tau, f, Z_1, M]$ a *ND-n-TA* with the input alphabet $X \cup \{\varepsilon\}$. Then

$$R_\varepsilon(\mathfrak{B}) = \{\mathfrak{p} \mid \mathfrak{p} \in \underset{n}{\times} W(X) \wedge \bar{f}(Z_1, \mathfrak{p}\varepsilon^{\rightarrow}) \cap M \neq \emptyset\}$$

(where $\varepsilon^{\rightarrow} = [\varepsilon, \varepsilon, \ldots, \varepsilon]$) is the n-ary relation represented by $\mathfrak{B}$ using the endmarker ε.

One can prove (see e.g. [2])

Lemma 1. There is an algorithm which for every finite resp. finite initial resp. finite deterministic *ND-n-TA* $\mathfrak{B}$ with the input alphabet X constructs a finite resp. finite initial resp. finite deterministic *ND-n-TA* $\mathfrak{B}_\varepsilon$ with the input alphabet $X \cup \{\varepsilon\}$ such that $R_\varepsilon(\mathfrak{B}_\varepsilon) = R(\mathfrak{B})$.

Lemma 2. There is an algorithm which for every finite *ND-n-TA* $\mathfrak{B}_\varepsilon$ with the input alphabet $X \cup \{\varepsilon\}$ constructs a finite *ND-n-TA* $\mathfrak{B}$ with the input alphabet X such that $R(\mathfrak{B}) = R_\varepsilon(\mathfrak{B}_\varepsilon)$.

Remark. In general it is not possible to construct an initial or a deterministic *ND-n-TA* $\mathfrak{B}$ with $R(\mathfrak{B}) = R_\varepsilon(\mathfrak{B}_\varepsilon)$ even in the case that $\mathfrak{B}_\varepsilon$ is deterministic (consider e.g. $R_\varepsilon(\mathfrak{B}_\varepsilon) = \{[a, e], [e, a]\}$).

Next we prove that the set of all n-ary relations representable using an endmarker by finite deterministic *n-TA* is closed under complementation. This assertion is given in [1] and for the proof the reader is refered to the paper [3] although representability in [3] is defined in a different way. Therefore we shall give a proof here.

Lemma 3. For every finite deterministic n-tape automaton $\mathfrak{A} = [n, X \cup \{\varepsilon\}, Z, \tau, \delta, z_1, M]$ one can construct a finite deterministic *n-TA* $\overline{\mathfrak{A}} = [n, X \cup \{\varepsilon\}, Z^*, \tau^*, \delta^*, z_1^*, M^*]$ such that

$$R_\varepsilon(\overline{\mathfrak{A}}) = \overline{R_\varepsilon(\mathfrak{A})} = \underset{n}{\times} W(X) \setminus R_\varepsilon(\mathfrak{A}) .$$

Proof. In the theory of ordinary acceptors (i.e. $n = 1$) one obtains $\overline{\mathfrak{A}}$ by replacing in $\mathfrak{A}$ the final set M by $M^* = Z \setminus M$. This construction works only in the case that

$$R_\varepsilon([n, X \cup \{\varepsilon\}, Z, \tau, \delta, z_1, Z]) = \underset{n}{\times} W(X)$$

which in general is not fulfilled. Therefore our aim is to construct from $\mathfrak{A}$ a finite deterministic *n-TA* $\mathfrak{A}' = [n, X \cup \{\varepsilon\}, Z', \tau', \delta', z_1', M']$ with $R_\varepsilon(\mathfrak{A}) = R_\varepsilon(\mathfrak{A}')$ and

$$R_\varepsilon([n, X \cup \{\varepsilon\}, Z', \tau', \delta', z_1', Z']) = \mathop{\times}_n W(X).$$

Without loss of generality we can assume that $\mathfrak{A}$ is of Rabin-Scott type, i.e. $\tau(z)$ is a singleton for all $z \in Z$, and that $R_\varepsilon(\mathfrak{A}) \neq \emptyset$. Let be

$$\begin{aligned}
\mathscr{E}_n^1 &= \{[\varepsilon, e, \ldots, e], [e, \varepsilon, e, \ldots, e], \ldots, [e, \ldots, e, \varepsilon]\}, \\
Z'' &= Z \times (\mathfrak{P}(\{1. \ldots, n\}) \cup \{\{0\}\}), \\
z_1'' &= [z_1, \{1, \ldots, n\}], \\
M'' &= M \times \{\emptyset\},
\end{aligned}$$

and for $[z, N] \in Z''$ we put

$$\tau''([z, N]) = \tau(z)$$

$$\delta''([z, N], \mathfrak{x}) = [\delta(z, \mathfrak{x}), N']$$

where

$$N' = \begin{cases} N \setminus \tau(z), & \text{if } \mathfrak{x} \in \mathscr{E}_n^1 \text{ and } \tau(z) \subseteq N, \\ N, & \text{if } \mathfrak{x} \in X_n^1 \text{ and } \tau(z) \subseteq N, \\ \{0\}, & \text{else,} \end{cases}$$

for all $\mathfrak{x} \in X_n^1 \cup \mathscr{E}_n^1$ with $\nu(\mathfrak{x}) = \tau(z)$, and finally

$$\mathfrak{A}'' = [n, X \cup \{\varepsilon\}, Z'', \tau'', \delta'', z_1'', M''].$$

Obviously for all $\mathfrak{r} \in \mathop{\times}_n W(X)$ we have

$$[z_1, \mathfrak{r}\varepsilon^{\to}] \in D_{\bar{\delta}} \to [[z_1, \{1, \ldots, n\}], \mathfrak{r}\varepsilon^{\to}] \in D_{\bar{\delta}''} \wedge \bar{\delta}''(z_1'', \mathfrak{r}\varepsilon^{\to}) = [\bar{\delta}(z_1, \mathfrak{r}\varepsilon^{\to}), \emptyset],$$

hence $R_\varepsilon(\mathfrak{A}) \subseteq R_\varepsilon(\mathfrak{A}'')$.

Moreover one can show for all $\mathfrak{p} \in \mathop{\times}_n W(X \cup \{\varepsilon\})$

$$\begin{aligned}
&([z_1'', \mathfrak{p}] \in D_{\bar{\delta}''} \wedge \bar{\delta}''(z_1'', \mathfrak{p}) = [z, N] \wedge N \neq \{0\}) \to \\
&\to \exists r_1 \ldots \exists r_n \exists \sigma_1 \ldots \exists \sigma_n (r_1, \ldots, r_n \in W(X) \wedge \mathfrak{p} = [r_1\sigma_1, \ldots, r_n\sigma_n] \wedge \\
&\bigwedge_{i \in N} \sigma_i = e \wedge \bigwedge_{i \notin N} \sigma_i = \varepsilon \wedge z = \bar{\delta}(z_1, \mathfrak{p})),
\end{aligned}$$

from which it follows that $R_\varepsilon(\mathfrak{A}'') \subseteq R_\varepsilon(\mathfrak{A})$.

Now let be

$$Z_0 = \{z'' \mid z'' \in Z'' \wedge \exists \mathfrak{p}\, \exists \mathfrak{r}(\bar{\delta}''(z_1'', \mathfrak{p}) = z'' \wedge \bar{\delta}''(z'', \mathfrak{r}) \in M'')\}.$$

Since $R_\varepsilon(\mathfrak{A}'') = R_\varepsilon(\mathfrak{A}) \neq \emptyset$, $z_1'' \in Z_0$. Moreover let be

$$Z_{00} = \mathfrak{P}^*(\{1, \ldots, n\}),$$

and for $N \in Z_{00}$, $\mathfrak{x} \in X_n^1 \cup \mathscr{E}_n^1$

$$\tau_{00}(N) = \max N,$$

$$\delta_{00}(N, \mathfrak{x}) = \begin{cases} N, & \text{if } \mathfrak{x} \in X_n^1 \text{ and } \nu(\mathfrak{x}) = \tau_{00}(N), \\ N \setminus \nu(\mathfrak{x}), & \text{if } \mathfrak{x} \in \mathscr{E}_n^1 \text{ and } \nu(\mathfrak{x}) = \tau_{00}(N), \end{cases}$$

$$Z' = Z_0 \cup Z_{00},$$

for $z' \in Z'$, $\mathfrak{x} \in X_n^1 \cup \mathscr{E}_n^1$ with $\nu(\mathfrak{x}) = \tau'(z')$

$$\tau'(z') = \begin{cases} \tau''(z'), & \text{if } z' \in Z_0, \\ \tau_{00}(z'), & \text{if } z' \in Z_{00}, \end{cases}$$

$$\delta'(z', \mathfrak{x}) = \begin{cases} \delta''(z', \mathfrak{x}), & \text{if } z' \in Z_0 \text{ and } \delta''(z', \mathfrak{x}) \in Z_0, \\ N \setminus \varepsilon(\mathfrak{x}), & \text{if } z' = [z, N] \in Z_0 \text{ and } \delta''(z', \mathfrak{x}) \notin Z_0, \\ \delta_{00}(z', \mathfrak{x}), & \text{if } z' \in Z_{00}, \end{cases}$$

where $\varepsilon(\mathfrak{x}) = \{i \mid (\mathfrak{x})_i = \varepsilon\}$, and finally

$$\mathfrak{A}' = [n, X \cup \{\varepsilon\}, Z', \tau', \delta', z_1'', M'' \cap Z'].$$

One proves without difficulties that

$$R_\varepsilon(\mathfrak{A}') = R_\varepsilon(\mathfrak{A}'')\,(= R_\varepsilon(\mathfrak{A})).$$

We shall show now that

$$R_0 =_{\text{df}} R_\varepsilon([n, X \cup \{\varepsilon\}, Z', \tau', \delta', z_1'', Z']) = \mathsf{X}_n W(X).$$

Let be $\mathfrak{p} \in \mathsf{X}_n W(X)$ and let be $\mathfrak{r}$ the longest initial segment of $\mathfrak{p}\varepsilon^{\rightarrow}$ such that $[z_1'', \mathfrak{r}] \in D_{\bar{\delta}'}$ and $\bar{\delta}'(z_1'', \mathfrak{r}) \in Z_0$. An initial segment of that kind exists since $z_1'' \in Z_0$. Let be

$$\bar{\delta}'(z_1'', \mathfrak{r}) = [z, N] \in Z_0$$

and $\mathfrak{q}$ that n-word with $\mathfrak{r}\mathfrak{q} = \mathfrak{p}\varepsilon^{\rightarrow}$. We have to show that

$$[[z, N], \mathfrak{q}] \in D_{\tilde{\delta}'} .$$

Let be i_0 such that $\tau'([z, N]) = \{i_0\}$.

Case I. $(\mathfrak{q})_{i_0} = e$.

This implies $(\mathfrak{r})_{i_0} = (\mathfrak{p})_{i_0}\,\varepsilon$, hence $i_0 \notin N$ and $\tau(z) \nsubseteq N$. Therefore we have $\delta''([z, N], \mathfrak{x}) = [\delta(z, \mathfrak{x}), \{0\}]$ for all $\mathfrak{x} \in X_n^1 \cup \mathscr{E}_n^1$ with $\nu(\mathfrak{x}) = \tau(z)$. But, by $[z, N] \in Z_0$, there exists a $\mathfrak{r}'$ with $\tilde{\delta}''([z, N], \mathfrak{r}') \in M \times \{\emptyset\}$. Now, from a state of the kind $[z_0, \{0\}]$ no state from the set $M \times \{\emptyset\}$ is reachable in $\mathfrak{A}''$. This implies that $\mathfrak{r}' = \mathfrak{e}$, hence $N = \emptyset$ and $\mathfrak{r} = \mathfrak{p}\varepsilon^{\rightarrow}$, thus $\mathfrak{p} \in R_0$.

Case II. $(\mathfrak{q})_{i_0} \neq e$.

We choose an $\mathfrak{x}$ and a $\mathfrak{q}'$ with

$$\mathfrak{q} = \mathfrak{x}\mathfrak{q}' \quad \text{and} \quad \nu(\mathfrak{x}) = \{i_0\} .$$

Then $\delta'([z, N], \mathfrak{x}) = N' =_{\mathrm{df}} N \setminus \varepsilon(\mathfrak{x})$ and it holds

$$i \in N' \leftrightarrow (\mathfrak{q}')_i \neq e .$$

Therefore $\tilde{\delta}_{00}(N', \mathfrak{q}')$ is defined and consequently $\tilde{\delta}'(z_1'', \mathfrak{p}\varepsilon^{\rightarrow}) = \tilde{\delta}'(z_1'', \mathfrak{r}\mathfrak{x}\mathfrak{q}')$ is defined which implies $\mathfrak{p} \in R_0$.

Now it is obvious that for $\overline{\mathfrak{A}} = [n, X \cup \{\varepsilon\}, Z', \tau', \delta', z_1'', Z' \setminus M'']$ it holds:

$$R_\varepsilon(\overline{\mathfrak{A}}) = \mathop{\times}_{n} W(X) \setminus R_\varepsilon(\mathfrak{A}) .$$

The following theorem states by a table our results on the working of some of the usual operations on the classes of n-ary relations mentioned above. In that table the entry in the column corresponding to the class $\mathscr{R}$ of relations and in the line corresponding to the operation $\mathscr{op}$ is the least class $\mathscr{R}' \in \{\mathscr{N}_n, \mathscr{I}\mathscr{N}_n, \mathscr{D}_n, \mathscr{F}\mathscr{N}_n, \mathscr{F}\mathscr{I}\mathscr{N}_n, \mathscr{F}\mathscr{D}_n\}$ such that the operation $\mathscr{op}$ executed with elements from $\mathscr{R}$ will always yield an element of $\mathscr{R}'$.

Theorem 1.

	$\mathscr{I}\mathscr{N}_n$	$\mathscr{D}_n$	$\mathscr{F}\mathscr{N}_n$	$\mathscr{F}\mathscr{I}\mathscr{N}_n$	$\mathscr{F}\mathscr{D}_n$
union	$\mathscr{N}_n$	$\mathscr{N}_n$	$\mathscr{F}\mathscr{N}_n$	$\mathscr{F}\mathscr{N}_n$	$\mathscr{F}\mathscr{N}_n$
intersection	$\mathscr{I}\mathscr{N}_n$	$\mathscr{D}_n$	$\mathscr{N}_n$	$\mathscr{I}\mathscr{N}_n$	$\mathscr{D}_n$
complementation	$\mathscr{N}_n$	$\mathscr{N}_n$	$\mathscr{N}_n$	$\mathscr{N}_n$	$\mathscr{F}\mathscr{N}_n$
catenation	$\mathscr{N}_n$	$\mathscr{N}_n$	$\mathscr{F}\mathscr{N}_n$	$\mathscr{F}\mathscr{N}_n$	$\mathscr{F}\mathscr{N}_n$
catenation closure	$\mathscr{I}\mathscr{N}_n$	$\mathscr{I}\mathscr{N}_n$	$\mathscr{F}\mathscr{N}_n$	$\mathscr{F}\mathscr{I}\mathscr{N}_n$	$\mathscr{F}\mathscr{I}\mathscr{N}_n$

Proof. In some cases we give the proof only for $n = 2$. It is easy to see that the same methods apply in case $n > 2$.

(1) *Union*

Let be L a nonregular language over X. Then the relations $R_1 = L \times \{e\}$, $R_2 = \{e\} \times L$ are elements of $\mathcal{D}_2 \subset \mathcal{IN}_2$ and $R_1 \cup R_2$ is not representable by an initial *ND-n-TA* since $\bigcap\{\nu(\mathfrak{r}) \mid \mathfrak{r} \in R_1 \cup R_2 \setminus \{\mathfrak{e}\}\} = \emptyset$. The relation $R_1 \cup R_2$ is not 2-regular since in that case (from the so-called projection theorem) it would follow that $P_1(R_1 \cup R_2) = L$ (the projection on the first coordinate) is regular. This proves the assertions on $\mathcal{IN}_n$ and $\mathcal{D}_n$. From the fact that $\mathcal{FN}_n$ is closed under union and $\emptyset \in \mathcal{FN}_n$ we obtain the assertion on $\mathcal{FN}_n$. The same argument as above applied to a regular language L shows that initiality and determinism are lost under union. Thus the assertions on $\mathcal{FIN}_n$ and $\mathcal{FD}_n$ hold.

(2) *Intersection.*

The closedness of $\mathcal{IN}_n$ and $\mathcal{D}_n$ under intersection is a direct consequence from the fact that for all relations $R_1, R_2 \in \mathcal{IN}_n$ we have

$$\bigcap\{\nu(\mathfrak{r}) \mid \mathfrak{r} \in (R_1 \cap R_2) \setminus \{\mathfrak{e}\}\} \supseteq \bigcap\{\nu(\mathfrak{r}) \mid \mathfrak{r} \in R_1 \setminus \{\mathfrak{e}\}\} \neq \emptyset ,$$

and from the fact that the intersection of strongly mesh-free sets is strongly mesh-free again.

To prove the assertion on $\mathcal{FN}_n$ consider the relations

$$R_1 = (W(X) \times \{e\}) \cup \{[e, x]\} , \quad R_2 = (\{e\} \times W(X)) \cup \{[x, e]\}$$

for a fixed $x \in X$. We have $R_1, R_2 \in \mathcal{FN}_2$, but $R_1 \cap R_2 = \{[e, e], [e, x], [x, e]\}$ is not representable by an initial *2-TA*.

The intersection of two relations representable even by deterministic *n-TA* is in general not n-regular. Consider e.g. the relations

$$R_1 = \{[a^k b a^l, a^m b a^k] \mid k, l, m \geqq 0\}$$
$$R_2 = \{[p, p] \mid p \in W(\{a, b\})\}$$

which are described by the admissible regular expressions

$$T_1 = \langle[a, 2]\rangle . [b, 2] . \langle[a, 1] . [a, 2]\rangle . [b, 1] . \langle[a, 1]\rangle ,$$
$$T_2 = \langle[a, 1] . [a, 2] \vee [b, 1] . [b, 2]\rangle$$

and thus are representable by deterministic *2-TA*. The relation

$$R_1 \cap R_2 = \{[a^k b a^k, a^k b a^k] \mid k \geqq 0\}$$

is not 2-regular, since the language $\{a^k b a^k \mid k \geqq 0\}$ is not regular. This proves the assertions on $\mathscr{F}\mathscr{N}_n$, $\mathscr{F}\mathscr{I}\mathscr{N}_n$ and $\mathscr{F}\mathscr{D}_n$.

Remark. It is essential here that X contains at least two letters. In case that X is a singleton the intersection of two n-ary relations representable by finite deterministic n-*TA* with the input alphabet X is again representable by a finite deterministic n-*TA* (cf. [6]).

(3) *Complementation*

It suffices to show that the assertions on $\mathscr{D}_n$, $\mathscr{F}\mathscr{I}\mathscr{N}_n$ and $\mathscr{F}\mathscr{D}_n$ hold. Obviously there are relations $R \in \mathscr{D}_n$ such that $R \cap X_n = \emptyset$ thus $X_n \subseteq \bar{R} = \mathsf{X}_n W(X) \setminus R$ and therefore $\bar{R}$ is not representable by an initial *ND*-n-*TA*.

For an arbitrary language L over $W(X)$ the relation $R_L = \{[p, \ldots, p] \mid p \in L\}$ is an element of $\mathscr{D}_n$. By a theorem in [8] from the assumption $\bar{R}_L \in \mathscr{F}\mathscr{N}_n$ it follows that the diagonal $D(\bar{R}_L) = \{q \mid [q, \ldots, q] \in \bar{R}_L\} = \{q \mid q \notin L\}$ is a context-sensitive language. Since there are languages the complement of which is not context-sensitive, there is a relation R_L such that $\bar{R}_L$ is not n-regular. Thus the assertion on $\mathscr{D}_n$ is proved.

Now we consider $\mathscr{F}\mathscr{I}\mathscr{N}_n$. The same argument as above shows that the complement of a relation $R \in \mathscr{F}\mathscr{I}\mathscr{N}_n$ need not be representable by an initial n-*TA*. We shall show that the assumption

$$R \in \mathscr{F}\mathscr{I}\mathscr{N}_n \rightarrow \bar{R} \in \mathscr{F}\mathscr{N}_n$$

will lead to a contradiction.

By the de Morgan laws $\mathscr{F}\mathscr{N}_n$ is not closed under complementation, let be $R_0 \in \mathscr{F}\mathscr{N}_n$ such that $\bar{R}_0 \notin \mathscr{F}\mathscr{N}_n$. Then, for a fixed $x \in X$, we consider the relation

$$R_0' = \{[x, e, \ldots, e]\} \,.\, R_0 \,.$$

From $R_0 \in \mathscr{F}\mathscr{N}_n$ we obtain $R_0' \in \mathscr{F}\mathscr{I}\mathscr{N}_n$. By our assumption we have $\bar{R}_0' \in \mathscr{F}\mathscr{N}_n$. On the other hand

$$\bar{R}_0 = \partial_{[x,e,\ldots,e]}(\bar{R}_0') \,,$$

since for $\mathfrak{p} \in \mathsf{X}_n W(X)$ we have

$$\begin{aligned} \mathfrak{p} \in \bar{R}_0 &\leftrightarrow \mathfrak{p} \notin R_0 \\ &\leftrightarrow [x, e, \ldots, e]\, \mathfrak{p} \notin R_0' \\ &\leftrightarrow \mathfrak{p} \in \partial_{[x,e,\ldots,e]}(\bar{R}_0') \,. \end{aligned}$$

The following lemma asserts that for $R \in \mathscr{F}\mathscr{N}_n$ the derivative $\partial_{[x,e,\ldots,e]}(R)$ is an element of $\mathscr{F}\mathscr{N}_n$ again. Thus from $\bar{R}_0' \in \mathscr{F}\mathscr{N}_n$ we obtain $\bar{R}_0 = \partial_{[x,e,\ldots,e]}(\bar{R}_0') \in \mathscr{F}\mathscr{N}_n$ in contradiction with the choice of R_0.

Lemma 4. For every $R \in \mathscr{F}\mathscr{N}_n$, $x_0 \in X$ it holds $\partial_{[x_0,e,\ldots,e]}(R) \in \mathscr{F}\mathscr{N}_n$.

By $\partial_{[x_0,e,\ldots,e]}(R_1 \cup R_2) = \partial_{[x_0,e,\ldots,e]}(R_1) \cup \partial_{[x_0,e,\ldots,e]}(R_2)$ and the closedness of $\mathscr{F}\mathscr{N}_n$ under union it is sufficient to prove the assertion for relations R which are representable by a finite initial ND-n-TA $\mathfrak{B} = [n, X, Z, \tau, f, z_1, M]$ of Rabin-Scott type. If $\tau(z_1) = \{1\}$ then obviously

$$\partial_{[x_0,e,\ldots,e]}(R(\mathfrak{B})) = R([n, X, Z, \tau, f, f(z_1, [x_0, e, \ldots, e]), M]) .$$

If $\tau(z_1) \neq \{1\}$ we consider the ND-n-TA $\mathfrak{B}^* = [n, X, Z^*, \tau^*, f^*, z_1^*, M^*]$ with

$$Z^* = Z \cup (Z \times \{e\}) \cup \{[z, x] \mid z \in Z \wedge x \in X \wedge \tau(z) \neq \{1\}\} ,$$

$$\tau^*(z) = \tau(z) \quad \text{for} \quad z \in Z ,$$

$$\tau^*([z, \sigma]) = \tau(z) \quad \text{for} \quad z \in Z, \sigma \in X \cup \{e\} ,$$

$$f^*(z, \mathfrak{x}) = f(z, \mathfrak{x}) \quad \text{for} \quad z \in Z \text{ with } \tau(z) \neq \{1\} , \quad \mathfrak{x} \in X_n^1 ,$$

$$f^*(z, \mathfrak{x}) = \{[z', \sigma] \mid [\mathfrak{x} = [\sigma, e, \ldots, e] \wedge \sigma \in X \wedge z' \in f(z, \mathfrak{x}_0) \wedge \tau(z') \neq \{1\}] \vee$$

$$\vee [\sigma = e \wedge \exists z''(z'' \in f(z, \mathfrak{x}_0) \wedge \tau(z'') = \{1\} \wedge z' \in f(z'', \mathfrak{x}))]\} ,$$

$$\text{for} \quad z \in Z \text{ with } \tau(z) = \{1\}, \ \mathfrak{x} \in X_n^1 \ (\text{where } \mathfrak{x}_0 = [x_0, e, \ldots, e]) ,$$

$$f^*([z, x], \mathfrak{x}) = \{[z', \sigma] \mid (\sigma = x \wedge z' \in f(z, \mathfrak{x}) \wedge \tau(z') \neq \{1\}) \vee$$

$$\vee (\sigma = e \wedge \exists z''(z'' \in f(z, \mathfrak{x}) \wedge \tau(z'') = \{1\} \wedge z' \in f(z'', [x, e, \ldots, e])))\}$$

$$\text{for} \quad z \in Z \text{ with } \tau(z) \neq \{1\}, \ x \in X, \ \mathfrak{x} \in X_n^1 ,$$

$$f^*([z, e], \mathfrak{x}) = f(z, \mathfrak{x}) \times \{e\} , \quad \text{for all} \quad z \in Z, \ \mathfrak{x} \in X_n^1 ,$$

$$z_1^* = z_1 ,$$

$$M^* = M \times \{e\} .$$

One shows easily that $R(\mathfrak{B}^*) = \partial_{[x_0,e,\ldots,e]}(R(\mathfrak{B}))$.

Now we prove, that the complements of relations from $\mathscr{F}\mathscr{D}_n$ are always in $\mathscr{F}\mathscr{N}_n$. Let be $R \in \mathscr{F}\mathscr{D}_n$ and $\mathfrak{A}$ a finite deterministic n-TA representing R. Then, by Lemma 1, there exists a finite deterministic n-TA $\mathfrak{A}'$ with $R_\varepsilon(\mathfrak{A}') = R$. Hence, by Lemma 3, there exists a finite deterministic n-TA $\mathfrak{A}''$ with $R_\varepsilon(\mathfrak{A}'') = \bar{R}$. Now, by Lemma 2, we obtain that $\bar{R} \in \mathscr{F}\mathscr{N}_n$.

(4) *Catenation*

Let L be a non-regular language over X, then $\{e\} \times L$, $L \times \{e\} \in \mathscr{D}_2 \subset \mathscr{I}\mathscr{N}_2$ and $(\{e\} \times L) . (L \times \{e\}) = L \times L$. The relation $L \times L$ is an element of $\mathscr{N}_2$ but not an element of $\mathscr{F}\mathscr{N}_2$ (by the projection theorem). If $e \in L$ then $L \times L \notin \mathscr{I}\mathscr{N}_2$. Thus the assertions on $\mathscr{I}\mathscr{N}_n$, $\mathscr{D}_n$ are proved. In the same way one using $L = W(X)$ one can verify the assertions on $\mathscr{F}\mathscr{I}\mathscr{D}_n$ and $\mathscr{F}\mathscr{D}_n$.

(5) *Catenation closure*

The assertion on $\mathscr{F}\mathscr{N}_n$ is trivial, the remaining assertions are a consequence of the following construction.

Let $\mathfrak{B} = [n, X, Z, \tau, f, z_1, M]$ be an initial *ND-n-TA* of Rabin-Scott-type and let be z', z'', z_1' pairwise different elements not contained in Z. We consider the *ND-n-TA* $\mathfrak{B}' = [n, X, Z \cup \{z', z'', z_1'\}, \tau', f', z_1', M']$ with

$$\tau'(z_1') = \tau(z_1)\,, \quad \tau'(z') = \tau'(z'') = \{1\}\,,$$

$$\tau'(z) = \tau(z) \quad \text{for all} \quad z \in Z\,,$$

$$f'(z_1', \mathfrak{x}) = f(z_1, \mathfrak{x})\,, \quad f'(z', \mathfrak{x}) = f'(z'', \mathfrak{x}) = \{z''\}\,,$$

$$f'(z, \mathfrak{x}) = f(z, \mathfrak{x}) \cup \begin{cases} \{z'\}, & \text{if} \ f(z, \mathfrak{x}) \cap M \neq \emptyset\,, \\ \emptyset, & \text{else,} \end{cases}$$

$$M' = \{z'\} \cup \begin{cases} \{z_1'\}, & \text{if} \ z_1 \in M\,, \\ \emptyset, & \text{else.} \end{cases}$$

One can see without difficulties that $R(\mathfrak{B}') = R(\mathfrak{B})$. Now we put for all $z \in Z' = Z \cup \{z', z'', z_1'\}$

$$f''(z, \mathfrak{x}) = f'(z, \mathfrak{x}) \cup \begin{cases} \{z_1'\}, & \text{if} \ z' \in f'(z, \mathfrak{x})\,, \\ \emptyset, & \text{else,} \end{cases}$$

and obtain

$$R([n, X, Z', \tau', f'', z_1', M']) = \langle R(\mathfrak{B}')\rangle = \langle R(\mathfrak{B})\rangle\,.$$

Since the catenation closure of a strongly mesh-free set is in general not strongly mesh-free (consider e.g. $R = \{[e, xx], [x, xx], [xx, xx]\}$) the remaining assertions are proved.

2. Decision problems

In this section we investigate the solvability of 10 decision problems. Most of them are known from the theory of finite ordinary (one-tape) acceptors and they all are solvable or trivial in that case. Again we give the results by a table. Thereby e.g. the entry "solvable" in the first line and first column indicates the assertion that there exists an algorithm which decides for finite deterministic *n-TA* $\mathfrak{B}$ whether or not $R(\mathfrak{B})$ is empty, and the entry "unsolvable" in the last line and last column indicates that it is undecidable whether a relation represented by a finite *ND-n-TA* is representable by a deterministic finite *n-TA*. Thus the corresponding problem for regular expressions pointed out in [5] and [9] is undecidable. In the proof of the theorem we mainly follow ideas developed by Fischer and Rosenberg [1] for the case when an endmarker is used.

Theorem 2.

$\mathfrak{B}, \mathfrak{B}'$ finite *ND-n-TA*		$\mathfrak{B}, \mathfrak{B}'$ deterministic	nondeterministic
Emptiness	$R(\mathfrak{B}) = \emptyset$?	solvable	solvable
Finiteness	$R(\mathfrak{B})$ finite?	solvable	solvable
Initiality	$R(\mathfrak{B}) \in \mathscr{I}\mathscr{N}_n$?	always	solvable
Universe	$R(\mathfrak{B}) = \underset{n}{\mathrm{X}} W(X)$?	never	unsolvable
Co-finiteness	$\underset{n}{\mathrm{X}} W(X) \setminus R(\mathfrak{B})$ finite?	never	unsolvable
Disjointness	$R(\mathfrak{B}) \cap R(\mathfrak{B}') = \emptyset$?	unsolvable	unsolvable
	$R(\mathfrak{B}) \cap R(\mathfrak{B}') \in \mathscr{F}\mathscr{N}_n$?	unsolvable	unsolvable
Containment	$R(\mathfrak{B}) \subseteq R(\mathfrak{B}')$?	unsolvable	unsolvable
Equivalence	$R(\mathfrak{B}) = R(\mathfrak{B}')$?	?	unsolvable
Determinism	$R(\mathfrak{B}) \in \mathscr{F}\mathscr{D}_n$?	always	unsolvable

Proof. 1. *Emptiness*

By the projection theorem for every finite *ND-n-TA* $\mathfrak{B}$ one can construct an ordinary finite acceptor $\mathfrak{A}$ such that $L(\mathfrak{A}) = P_1(R(\mathfrak{B}))$. Since $R(\mathfrak{B})$ is empty if and only if $L(\mathfrak{A})$ is empty from the decidability of the latter the assertion follows.

2. *Finiteness*

Since $R(\mathfrak{B})$ is finite if and only if the regular language $\bigcup_{i=1}^{n} P_i(R(\mathfrak{B}))$ is finite an argument similar to that one used above applies.

3. *Initiality*

Here the solvability follows from the

Lemma 5. If $\mathfrak{B} = [n, X, Z, f, Z_1, M]$ is a finite *ND-n-TA* of Rabin-Scott type then for every $\mathfrak{r} \in R(\mathfrak{B})$ there exists a $\mathfrak{r}' \in R(\mathfrak{B})$ such that $v(\mathfrak{r}') = v(\mathfrak{r})$ and $l(\mathfrak{r}') < (n + 1)\, card(Z)$.

Let be $\mathfrak{r} \in R(\mathfrak{B})$ and $l(\mathfrak{r}) \geqq (n + 1)\, card(Z)$. Since $\mathfrak{B}$ is of Rabin-Scott type the sequence of states $\mathfrak{B}$ runs through accepting $\mathfrak{r}$ has the length $l(\mathfrak{r}) + 1 > (n + 1) \cdot card(Z)$. Therefore in that sequence at least one state z^* appears at least $n + 2$ times. Hence, there exist states $z_1 \in Z_1$, $z^* \in Z$, $z' \in M$ and n-words $\mathfrak{r}_0, \ldots, \mathfrak{r}_{n+2}$ such that

$$\mathfrak{r} = \mathfrak{r}_0 \mathfrak{r}_1 \ldots \mathfrak{r}_{n+2}\,, \quad \mathfrak{r}_1, \mathfrak{r}_2, \ldots, r_{n+1} \neq \mathfrak{e}\,,$$

$$z^* \in \bar{f}(z_1, \mathfrak{r}_0)\,, \quad z^* \in \bar{f}(z^*, \mathfrak{r}_i) \quad \text{for} \quad i = 1, \ldots, n + 1,$$

and

$$z' \in \bar{f}(z^*, \mathfrak{r}_{n+2}) .$$

Let be $N_j = \nu(\mathfrak{r}_j)$ for $j = 1, \ldots, n + 1$. Then there is a number j such that

$$N_j \subseteq N_1 \cup \ldots \cup N_{j-1} \cup N_{j+1} \cup \ldots \cup N_{n+1} ,$$

since otherwise the set $N_1 \cup \ldots \cup N_{n+1}$ has to contain $n + 1$ pairwise different numbers which is impossible. From this we obtain that $\mathfrak{r}' = \mathfrak{r}_0\mathfrak{r}_1 \ldots r_{j-1}\mathfrak{r}_{j+1} \ldots r_{n+2}$ has the following properties

$$\nu(\mathfrak{r}') = \nu(\mathfrak{r}) , \quad l(\mathfrak{r}') < l(\mathfrak{r}) , \quad \mathfrak{r}' \in R(\mathfrak{B}) .$$

The construction of $\mathfrak{r}'$ from $\mathfrak{r}$ can be repeated with $\mathfrak{r}'$ as long as the length of the result exceeds $(n + 1)\, card(Z) - 1$.

4. *Universe*

For every deterministic *n-TA* $\mathfrak{B}$ the set $R(\mathfrak{B})$ is mesh-free, $\underset{n}{\times} W(X)$ is not mesh-free, therefore the universe problem is trivial for deterministic *n-TA*.

Assume that the universe problem for finite *ND-n-TA* is decidable and let $\mathfrak{B}_\varepsilon$ be an arbitrary finite *ND-n-TA* with the input alphabet $X \cup \{\varepsilon\}$ $(\varepsilon \notin X)$. By Lemma 2 we can construct a finite *ND-n-TA* $\mathfrak{B}'$ with the input alphabet X such that $R(\mathfrak{B}') = R_\varepsilon(\mathfrak{B}_\varepsilon)$. By our assumption we can decide whether $(R_\varepsilon(\mathfrak{B}_\varepsilon) =)\ R(\mathfrak{B}') = \underset{n}{\times} W(X)$. This is in contradiction with Theorem 8 in [1] which states the universe problem to be undecidable when an endmarker is used.

5. *Co-finiteness*

The unsolvability of the co-finiteness problem for *ND-n-TA* is proved in the same way by reduction to the case when an endmarker is used.

Let be $x_0 \in X$ fixed, $m > 0$. Every mesh-free relation over $W(X)$ can contain at most one of the $m + 1$ n-words

$$[e, x_0^m, e, \ldots, e] , \quad [x_0, x_0^{m-1}, e, \ldots, e], \ldots, [x_0^m, e, e, \ldots, e] ,$$

thus there is no finite covering of $\underset{n}{\times} W(X)$ using mesh-free relations as elements. Therefore, if $\mathfrak{B}$ is a (finite) deterministic *n-TA* then $\underset{n}{\times} W(X) \setminus R(\mathfrak{B})$ is infinite, since otherwise

$$\{R(\mathfrak{B})\} \cup \{\{\mathfrak{q}\} \mid \mathfrak{q} \in \underset{n}{\times} W(X) \setminus R(\mathfrak{B})\}$$

would be a finite covering of $\underset{n}{\times} W(X)$ with mesh-free relations.

7. $R(\mathfrak{B}) \cap R(\mathfrak{B}') \in \mathcal{FN}_n$?

Both problems will be shown to be unsolvable by the same construction. Obviously it is sufficient to prove the unsolvability of these problems for finite deterministic *n-TA*.

Let be $[\varphi, \psi]$ a Post Correspondence Problem over $[A, B]$, i.e. A and B are finite nonempty sets and φ, ψ are homomorphisms from A into $W(B)$. Moreover let be $0, 1 \notin A \cup B, X = A \cup B \cup \{0, 1\}$ and

$$R^0_\varphi = \{[a, \varphi(a)] \mid a \in A\}, \quad R^0_\psi = \{[a, \psi(a)] \mid a \in A\},$$
$$R_\varphi = R^0_\varphi . \langle R^0_\varphi \rangle . \{[1, 1]\} . \langle\{[0, e]\}\rangle . \{[1, e]\} . \langle\{[0, 0]\}\rangle . \{[e, 1]\} . \langle\{[e, 0]\}\rangle,$$
$$R_\psi = R^0_\psi . \langle R^0_\psi \rangle . \{[1, 1]\} . \langle\{[0, 0], [1, 1]\}\rangle .$$

Hence, we have

$$R_\varphi = \{[p, \varphi(p)] \mid p \in W(A) \setminus \{e\}\} . \{[1, 1]\} . \{[0^i 10^k, 0^k 10^l] \mid i, j, l \geqq 0\}$$
$$R_\psi = \{[p, \psi(p)] \mid p \in W(A) \setminus \{e\}\} . \{[1, 1]\} . \{[r, r] \mid r \in W(\{0, 1\})\} .$$

One can see easily that

$$R_\varphi, R_\psi \in \mathcal{FD}_2 .$$

Moreover, we obtain

$$R_\varphi \cap R_\psi = \{[p, \varphi(p)] \mid p \in L_{\varphi,\psi}\} . \{[1, 1]\} . \{[0^k 10^k, 0^k 10^k] \mid k \geqq 0\},$$

where

$$L_{\varphi,\psi} = \{p \mid p \in W(A) \setminus \{e\} \wedge \varphi(p) = \psi(p)\} .$$

Therefore it holds

$$P_1(R_\varphi \cap R_\psi) = L_{\varphi,\psi}\{1\} . \{0^k 10^k \mid k \geqq 0\} .$$

It is wellknown that for every regular language E and for every language F the quotient

$$E/F = \bigcup_{u \in F} \partial_u(E)$$

is a regular language since E has only a finite number of derivatives all being regular.

From the fact that if $L_{\varphi,\psi} \neq \emptyset$ then

$$P_1(R_\varphi \cap R_\psi)/W(A) . \{1\} = \{0^k 10^k \mid k \geqq 0\}$$

is a nonregular language, we obtain that $P_1(R_\varphi \cap R_\psi)$ is not regular and, hence, $R_\varphi \cap R_\psi \notin \mathcal{FN}_2$ in that case.

Now we have

$$R_\varphi \cap R_\psi \in \mathscr{F}\mathscr{N}_2 \quad \text{iff} \quad R_\varphi \cap R_\psi = \emptyset\,,$$
$$\text{iff} \quad L_{\varphi.\psi} = \emptyset\,.$$

But whether or not $L_{\varphi,\psi}$ is empty is undecidable. Thus, the disjointness problem and the problem "$R(\mathfrak{B}) \cap R(\mathfrak{B}') \in \mathscr{F}\mathscr{N}_2$?" are unsolvable.

8. *Containment*

As above it is sufficient to prove the unsolvability of the problem in the deterministic case. By the following lemma we reduce our problem to the containment problem in the case when an endmarker is used. The latter has been shown to be unsolvable in the paper [1].

Lemma 6. For every finite deterministic *n-TA* $\mathfrak{A} = [n, X \cup \{\varepsilon\}, Z, \tau, \delta, z_1, M]$ one can construct effectively a finite deterministic *n-TA* $\mathfrak{A}' = [n, X \cup \{\varepsilon\}, Z', \tau', \delta', z_1', M']$ such that $R(\mathfrak{A}') = R_\varepsilon(\mathfrak{A}) . \{\varepsilon^{\rightarrow}\}$.

The construction is as follows (again we assume that $\mathfrak{A}$ is of Rabin-Scott type):

$$Z' = Z \times \mathfrak{P}(\{1, \ldots, n\}) \cup \{z^*\}\,,$$
$$z_1' = [z_1, \{1, \ldots, n\}]\,,$$
$$M' = M \times \{\emptyset\}\,,$$
$$\tau'(z^*) = \{1\}\,,$$
$$\tau'([z, N]) = \tau(z) \quad \text{for all} \quad [z, N] \in Z'\,,$$
$$\delta'([z, N], \mathfrak{x}) = \begin{cases} [\delta(z, \mathfrak{x}), N]\,, & \text{if} \quad \mathfrak{x} \in X_n^1 \wedge \tau(z) \subseteq N\,, \\ [\delta(z, \mathfrak{x}), N \setminus \tau(z)]\,, & \text{if} \quad \mathfrak{x} \in \mathscr{E}_n^1 \wedge \tau(z) \subseteq N\,, \\ z^*\,, & \text{else} \end{cases}$$
$$\delta'(z^*, \mathfrak{x}) = z^*$$

for all $\mathfrak{x} \in X_n^1 \cup \mathscr{E}_n^1$ with $\nu(\mathfrak{x}) = \tau(z)$, $[z, N] \in Z'$. It is easy to see that $R(\mathfrak{A}') = R_\varepsilon(\mathfrak{A}) . \{\varepsilon^{\rightarrow}\}$.

9. *Equivalence*

For deterministic *n-TA* the problem is open, for the nondeterministic case one proof of the result can be found in [5]. On the other hand, the solvability of the equivalence problem for finite *ND-n-TA* would imply the solvability of the containment problem for finite *ND-n-TA* since it holds

$$R(\mathfrak{B}) \subseteq R(\mathfrak{B}') \quad \text{iff} \quad R(\mathfrak{B}) \cup R(\mathfrak{B}') = R(\mathfrak{B}')\,.$$

10. $R(\mathfrak{B}) \in \mathcal{FD}_n$?

We choose φ, ψ, A, B, X, R_φ and R_ψ as above in the proof of the unsolvability of the disjointness problem. Moreover let be $\varepsilon \notin X$ and

$$R_1 = ((W(X) \times W(X) \setminus R_\varphi) \cup (W(X) \times W(X) \setminus R_\psi)) \,.\, \{[\varepsilon, \varepsilon]\} = \\ = (\bar{R}_\varphi \cup \bar{R}_\psi) \,.\, \{[\varepsilon, \varepsilon]\} \,.$$

Since $R_\varphi, R_\psi \in \mathcal{FD}_2$, by Theorem 1 we obtain that

$$\bar{R}_\varphi \cup \bar{R}_\psi \in \mathcal{FN}_2 \,.$$

Therefore, R_1 is representable by a finite ND-2-TA with the input alphabet $X \cup \{\varepsilon\}$, i.e.

$$R_1 \in \mathcal{FN}_2 \,.$$

Moreover, R_1 is mesh-free and strongly mesh-free (since $n = 2$). If $L_{\varphi,\psi} = \emptyset$ then $R_\varphi \cap R_\psi = \emptyset$, hence

$$R_1 = (\bar{R}_\varphi \cup \bar{R}_\psi) \,.\, \{[\varepsilon, \varepsilon]\} = (W(X) \times W(X)) \,.\, \{[\varepsilon, \varepsilon]\} \in \mathcal{FD}_2 \,.$$

Next we show, that

$$R_1 \notin \mathcal{FD}_2$$

if $L_{\varphi,\psi} \neq \emptyset$, so that

$$R_1 \in \mathcal{FD}_2 \quad \text{iff} \quad L_{\varphi,\psi} = \emptyset \,,$$

which gives the desired result.

Assume that $L_{\varphi,\psi} \neq \emptyset$ and $R_1 \in \mathcal{FD}_2$. One shows without difficulties that, if R_1 is represented by a finite deterministic 2-TA $\mathfrak{A}$ (with the input alphabet $X \cup \{\varepsilon\}$), then

$$R_\varepsilon(\mathfrak{A}) = \bar{R}_\varphi \cup \bar{R}_\psi \,.$$

From this using Lemma 2 and Lemma 3 one can conclude that there exists a finite ND-2-TA $\mathfrak{B}$ (with the input alphabet X) such that

$$R(B) = \overline{R_\varepsilon(\mathfrak{A})} = R_\varphi \cap R_\psi$$

which contradicts $L_{\varphi,\psi} \neq \emptyset$.

This completes the proof of Theorem 2.

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