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On Linear Inversion of Moving Averages, Discrete Equalizers and "Whitening" Filters, and the Related Difference Equations and Infinite Systems of Linear Equations

LUDVÍK PROUZA

Connections of difference equations methods of solution of some inversion problems with the theory of infinite systems of linear equations are investigated.

1. INTRODUCTION

In this article, some generalization of the results of [1] are presented. In chap. 2, the problem formulation of inversion with a finite weighting sequence filter from [1] is extended to colored input sequences and the relation to the Kolmogorov-Wiener problem of interpolation is shown.

In chap. 3, a difference equations method, used in special cases in [2] and [3], of solving the inversion problem with a white input sequence, is investigated in detail for the finite length weighting sequence inversion filter. In chap. 4, the difference equations method is extended to the case of the length of the inversion filter weighting sequence tending to infinity, assuming that $B(z)$ from (13) possesses at most simple roots on the unit circle C_1.

In chap. 5, the relation of the results of chap. 4 to [4; 5] is briefly examined. It may be shown that the general theory of [4; 5] is easily applicable to the problems of [6; 7] in the case of $B(z) \neq 0$ on C_1. For $B(z) = 0$ on C_1, recent results exist [8; 9], but they cannot be applied directly to the respective Theorems of chap. 4. The important conditions $\mathscr{A}(z) \neq 0$ on C_1 and ind $\mathscr{A}(z) = 0$ of the general theory are shown for $\mathscr{A}(z) = B(z)$ to be identical with the Nyquist stability criterion.

2. PROBLEM FORMULATION – FINITE CASE

Let $\{x(t)\}$, $(t = 0 \pm 1, \pm 2, \ldots)$ be a complex random weakly stationary sequence.

Let the sequence $\{\xi(t)\}$ be formed from $\{x(t)\}$ by the finite moving average

$$\xi(t) = b_0\, x(t) + b_1\, x(t-1) + \ldots + b_h\, x(t-h)\,, \tag{1}$$

where $h \geqq 1$ is natural, b_j are complex, $b_0 \neq 0$, $b_h \neq 0$.

Let N be a given natural number, $N \geqq h$. We will form the finite moving average

$$(2) \qquad x_N^*(t-T) = a_{N0}\,\xi(t) + a_{N1}\,\xi(t-1) + \ldots + a_{NN}\,\xi(t-N)\,,$$

where T is natural, $0 \leqq T \leqq N + h$.

We seek $a_{N0}, \ldots, a_{NN}$ so that

$$(3) \qquad \mathrm{E}\{|x_N^*(t-T) - x(t-T)|^2\} = \Phi(a_{N0}, \ldots, a_{NN}) = \min\,,$$

where E denotes the mean value. $\{x_N^*(t)\}$ will be called linear inversion of $\{\xi(t)\}$.

Denoting

$$(4) \qquad \mathrm{E}[x(t)\,\bar{x}(t-l)] = \varrho_l\,, \quad l = 0, \pm 1, \ldots,$$

$$(5) \qquad \mathrm{E}[\xi(t)\,\bar{\xi}(t-l)] = R_l\,, \quad l = 0, \pm 1 \ldots$$

(where $\bar{x}$ means the complex conjugate to x), then the necessary and sufficient conditions for $\{a_j\}$ (we omit the first subscripts for the sake of simplicity) to satisfy (3) are the equations

$$(6) \qquad \begin{aligned} R_0 a_0 + R_{-1} a_1 + \ldots + R_{-N} a_N &= \mathrm{E}[\bar{\xi}(t)\,x(t-T)]\,, \\ &\vdots \\ R_N a_0 + R_{N-1} a_1 + \ldots + R_0 a_N &= \mathrm{E}[\bar{\xi}(t-N)\,x(t-T)]\,. \end{aligned}$$

With the notations

$$(7) \qquad \begin{aligned} \mu_j &= b_j \bar{b}_0 + \ldots + b_h \bar{b}_{h-j}\,, \quad j = 0, 1, \ldots, h\,, \\ \mu_{-j} &= \bar{\mu}_j\,, \end{aligned}$$

ones obtains

$$(8) \qquad R_j = \varrho_{-h+j}\mu_h + \ldots + \varrho_j \mu_0 + \ldots + \varrho_{h+j}\mu_{-h}$$

and

$$(9) \qquad \mathrm{E}[\bar{\xi}(t-j)\,x(t-T)] = \bar{b}_0 \varrho_{-T+j} + \ldots + \bar{b}_h \varrho_{-T+h+j}\,.$$

Thus especially if $\{x(t)\}$ is white and normed ($\varrho_0 = 1$) then instead of (6) one has

$$(10) \qquad \begin{aligned} \mu_0 a_0 + \mu_{-1} a_1 + \ldots + \mu_{-N} a_N &= \bar{b}_T\,, \\ &\vdots \\ \mu_N a_0 + \mu_{N-1} a_1 + \ldots + \mu_0 a_N &= \bar{b}_{T-N} \end{aligned}$$

with $\mu_j = 0$ for $j > h$, $b_j = 0$ for $j > h$ and for $j < 0$.

The spectral form of the left side of (3) is

$$\Phi(a_0, \ldots, a_N) = \frac{1}{2\pi i} \int_{C_1} \left|z^{-T} - A(z)\, B(z)\right|^2 f(z) \frac{dz}{z}, \tag{11}$$

where

$$A(z) = a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}, \tag{12}$$

$$B(z) = b_0 + b_1 z^{-1} + \ldots + b_h z^{-h}, \tag{13}$$

$$f(z) = \sum_{j=-\infty}^{\infty} \varrho_j z^{-j} \tag{14}$$

and C_1 is the unit circle.

From (11), the relation of the inversion problem to the Kolmogorov-Wiener interpolation problem

$$\frac{1}{2\pi i} \int_{C_1} \left|z^{-T} - \mathscr{A}(z)\right|^2 f(z) \frac{dz}{z} = \min \tag{15}$$

is apparent. In the later, one seeks $\mathscr{A}(z)$ with the supplementary condition that the coefficient of z^{-T} in the Laurent expansion of $\mathscr{A}(z)$ is zero. In the former, the supplementary conditions is that $\mathscr{A}(z) = A(z)\, B(z)$ possesses the factor $B(z)$ or, by other words, $\mathscr{A}(z)$ possesses all roots of $B(z)$.

3. DIFFERENCE EQUATION METHOD OF SOLUTION OF THE INVERSION PROBLEM FOR FINITE N

In what follows, we will be interested mainly in the system (10). The determinant of (10) (and also of (6)) is positive, being the principal minor of a (Hermite) positive definite correlation matrix. Thus (10) has a unique solution.

For N substantially greater than h, this solution may be found with advantage by a finite difference equation method. As it will be shown, instead of a system of N linear equations, only a linear homogeneous difference equation of the order $2h$ with $2h$ boundary conditions is to be solved.

Case 1. Let $T < h$. Then, we write instead of (10)

$$\begin{array}{lllll}
\mu_h a_{-h} & + \ldots + \mu_0 a_0 & & + \ldots + \mu_{-h} a_h & = \bar{b}_T, \\
& \vdots & & & \\
\mu_h a_{-h+T} & + \ldots + \mu_T a_0 & & + \ldots + \mu_{-h} a_{h+T} & = \bar{b}_0, \\
\mu_h a_{-h+T+1} & + \ldots + \mu_{T+1} a_0 & & + \ldots + \mu_{-h} a_{h+T+1} & = 0, \\
& \vdots & & & \\
\mu_h a_{N-h} & + \ldots + \mu_0 a_N & + \mu_{-1} a_{N+1} & + \ldots + \mu_{-h} a_{N+h} & = 0.
\end{array} \tag{16}$$

Comparing (16) with (10), one sees that in (16) in the last h equations, the terms $a_{N+1}, \ldots, a_{N+h}$ appear and to be compatible with (10) the conditions

$$a_{N+1} = a_{N+2} = \ldots = a_{N+h} = 0 \tag{17}$$

must be added. Furthermore, one must define

$$a_{-1} = a_{-2} = \ldots = a_{-h+T+1} = 0\,, \tag{18}$$

as is seen from the equations following in (16) that with $\bar{b}_0$ on the right side. The set in (18) may be empty.

Finally, one adds $-\bar{b}_T, \ldots, -\bar{b}_0$ to the first, second, ..., equations in (16) getting zeros on the right sides and one defines $a_{-h+T}, \ldots, a_{-h}$ with the aid of the equations

$$\begin{aligned}
&\mu_h a_{-h+T} && = -\bar{b}_0\,, \\
&\mu_h a_{-h+T-1} + \mu_{h-1} a_{-h+T} && = -\bar{b}_1\,, \\
&\qquad\vdots && \\
&\mu_h a_{-h} \quad + \mu_{h-1} a_{-h+1} + \ldots + \mu_{h-T} a_{-h+T} && = -\bar{b}_T\,.
\end{aligned} \tag{19}$$

By these definitions one adds $-\bar{b}_T, \ldots, -\bar{b}_0$ on the left side also, moreover, with the aid of boundary conditions.

Since $\mu_h \neq 0$, these equations possess unique solution.

Thus, one has replaced the system (10) via a one-to-one correspondence by the homogeneous difference equation

$$\mu_h a_n + \mu_{h-1} a_{n+1} + \ldots + \mu_{-h} a_{n+2h} = 0 \tag{20}$$

of the order $2h$ and the boundary conditions (17), (18) and those resulting from (19).

Now, since (10) has solution, this solution satisfies (20) with the above boundary conditions. Thus (20) with the given boundary conditions has at least one solution.

Supposing that there were two distinct solutions in the range $a_0, \ldots, a_N$ and remembering the one-to-one correspondence of (10) and (20), (17), (18), (19), these two solutions were also solution of (10), which is impossible.

Thus $a_0, \ldots, a_N$ from (20) with the given boundary conditions form the unique solution of (10). Since $N > h$, one has at least $2h$ successive values "on the left" available as initial conditions to (20). Thus it is obvious that (20) with the given boundary conditions has a unique solution.

Case 2. Let $h \leqq T < 3h$-1. Then (10) is

$$\begin{aligned}
(21)\qquad & \mu_0 a_0 + \mu_{-1} a_1 + \ldots + \mu_{-h} a_h = \bar{b}_T \begin{cases} = \bar{b}_h & \text{for} \quad T = h\,, \\ = 0 & \text{for} \quad T > h\,, \end{cases} \\
& \vdots \\
& \mu_{T-h} a_0 + \mu_{T-h-1} a_1 + \ldots + \mu_{-h} a_T = \bar{b}_h\,, \\
& \vdots \\
& \mu_h a_{T-h} + \ldots + \mu_{-h} a_{T+h} = \bar{b}_0\,, \\
& \mu_h a_{T-h+1} + \ldots + \mu_{-h} a_{T+h+1} = 0\,, \\
& \vdots \\
& \mu_h a_{N-h} + \ldots + \mu_{-h} a_{N+h} = 0\,.
\end{aligned}$$

Now, we will consider the equations beginning with the first one after that with $\bar{b}_0$ on the right side as the homogeneous difference equation (20) and we attach h boundary conditions (17) to it.

Since a_{T-h+1} is the first term expressible with the aid of the characteristic roots of (20), we consider $a_0, \ldots, a_{T-h}$ as unknowns. Thus, one has $2h + T - h + 1 = T + h + 1$ unknowns and with the first $T + 1$ equations in (21) $T + h + 1$ condition equations.

It can be shown by similar reasoning as in case 1 that this system has the same unique solution as (10).

Note that since $T < 3h - 1$, one has to solve less than $4h$ equations. The case 2 may be, alternatively, treated similarly to the following case 3, but then $4h$ equations were to be solved. This seems not to be advantageous.

Case 3. Let $T \geqq 3h - 1$. From (10), we pass to the difference equation (20) with boundary conditions (17) and

$$(22)\qquad a_{-1} = a_{-2} = \ldots = a_{-h} = 0\,.$$

Showing only the "midle part" of the sequence of equations (10) representing (20), one has

$$\begin{aligned}
(23)\qquad & \vdots \\
& \mu_h a_{T-2h-1} + \ldots + \mu_{-h} a_{T-1} = 0\,, \\
& \mu_h a_{T-2h} + \ldots + \mu_{-h} a_T = \bar{b}_h \neq 0\,, \\
& \vdots \\
& \mu_h a_{T-h} + \ldots + \mu_{-h} a_{T+h} = \bar{b}_0 \neq 0\,, \\
& \mu_h a_{T-h+1} + \ldots + \mu_{-h} a_{T+h+1} = 0\,, \\
& \vdots \\
& \mu_h a_{T-1} + \ldots + \mu_{-h} a_{T+2h-1} = 0\,, \\
& \mu_h a_T + \ldots + \mu_{-h} a_{T+2h} = 0\,, \\
& \vdots
\end{aligned}$$

Now, we will express the term a_{T-1} and the preceding ones with the aid of the characteristic roots and $2h$ unknown constants (coefficients). The term a_T and the following ones will be expressed with the aid of the characteristic roots and other $2h$ constants. Thus, one has $4h$ unknowns and $2h$ boundary conditions (17) and (22). The remaining $2h$ conditions will be taken from (23) beginning with the equation with $\bar{b}_h$ on the right (since the preceding equation will be satisfied with arbitrary constants) and ending with the last but one equation (since the following one will be satisfied with arbitrary constants).

Now, the fact that every solution of the difference equation with the given boundary conditions is simultaneously solution of (10) is obvious. Thus the system has no more than one solution. But taking $a_{T-1}, \ldots, a_{T-2h}$ from the "middle" of (23) (i.e. of (10)) as initial conditions, one sees that (since $\mu_h \neq 0$) a_{T-2h-1} is uniquely defined from the difference equation and thus must be the same as that given by (10). In this way, one comes back to a_0 and, by similar reasoning with $a_T, \ldots, a_{T+2h-1}$ as initial conditions, to a_N. Thus the difference equation problem has the same solution as (10).

4. DIFFERENCE EQUATIONS METHOD FOR $N \to \infty$

The characteristic equation to (20) is

$$M(z) = \mu_{-h} z^{2h} + \ldots + \mu_h = B(z)\, \bar{B}(z^{-1})\, z^h = 0\,, \tag{24}$$

where

$$\bar{B}(z^{-1}) = \bar{b}_0 + \bar{b}_1 z + \ldots + \bar{b}_h z^h\,. \tag{25}$$

If (24) has a root ζ, it has also the root $\bar{\zeta}^{-1}$, thus the roots of (24) occur in Fejér-Riesz pairs.

The solution of (20) is a linear combination of the $2h$ particular solutions of the form

$$\zeta^n,\ n\zeta^n,\ n^2\zeta^n, \ldots, n^l\zeta^n\,, \tag{26}$$

ζ being a $l + 1$-tuple root of (24).

Let (24) posses on C_1 at most double roots. Let us arrange the summands in the linear combination representing the solution of (20) so that firstly the summands formed from (26) pertaining to the roots lying inside of C_1 in some fixed sequence occur, then the summands pertaining to the roots on C_1 follows so that firstly all terms the form ζ^n and then the terms of the form $n\zeta^n$ occur. Finally, there follow the summands pertaining to the roots outside of C_1.

Thus one has $2h$ summands in two groups each containing h terms. The unknown coefficients of the summands of the first group will be denoted $A_1, A_2, \ldots, A_h$, those of the summands of the second group $B_1, B_2, \ldots, B_h$.

In what follows only the case 1 of the chapter 3 will be considered in detail, the other cases will be leaved to the reader.

Also, since the general formulas in the case of multiple roots of (24) are quite intractable, the reader is for insight in what follows referred to [1, formulas (31), (32)].

The system of equations for A_j, B_j is formed by the boundary conditions a_{-h}, $a_{-h+1}, \ldots, a_{-1}, a_{N+1}, \ldots, a_{N+h}$.

One knows that the system possesses a unique solution and thus also a nonzero determinant. We will express this determinant with the aid of the Laplace expansion similarly as in [1] with minors from the first h rows and the remaining h rows.

Lemma 1. Let $B(z)$ possess on C_1 at most simple roots. Then in the Laplace expansion of the determinant of the equations system for A_j, B_j the term d given by the product of the first minor from first h rows and the last minor Δ from the last h rows is dominant in the sense that

$$\lim_{N\to\infty} d_j/d = 0\,, \tag{27}$$

where d_j is any other term of the Laplace expansion.

Proof. The first minor from the first h rows is independent from N and distinct from 0, since analogously as in [1] taking out proper factors a determinant is obtained known from the theory of linear difference equations to be distinct from 0 (see [10, p. 335 ff.]).

The last minor Δ from the last h rows has for a $(l+1)$-tuple root ζ outside C_1 the columns

$$\begin{gathered}\zeta^{N+1}, (N+1)\,\zeta^{N+1}, (N+1)^2\,\zeta^{N+1}, \ldots, (N+1)^l\,\zeta^{N+1}\,,\\ \zeta^{N+2}, (N+2)\,\zeta^{N+2}, (N+2)^2\,\zeta^{N+2}, \ldots, (N+2)^l\,\zeta^{N+2}\,,\\ \vdots\\ \zeta^{N+h}, (N+h)\,\zeta^{N+h}, (N+h)^2\,\zeta^{N+h}, \ldots, (N+h)^l\,\zeta^{N+h}\,.\end{gathered} \tag{28}$$

Taking out from each column the common factor ζ^{N+1}, there remain the columns

$$\begin{gathered}1,\quad (N+1)\,.\,1,\quad (N^2+2N\,.\,1+1)\,.\,1,\quad \ldots, (N+1)^l\,.\,1\,,\\ \zeta,\quad (N+2)\,.\,\zeta,\quad (N^2+2N\,.\,2+2^2)\,.\,\zeta,\quad \ldots, (N+2)^l\,.\,\zeta\,,\\ \vdots\\ \zeta^{h-1}, (N+h)\,.\,\zeta^{h-1}, (N^2+2N\,.\,h+h^2)\,.\,\zeta^{h-1}, \ldots, (N+h)^l\,.\,\zeta^{h-1}\,.\end{gathered} \tag{29}$$

Now, from the known rules on determinants, there is clear that one can omit N in the brackets in the second column, in the brackets in the third column one can omit $N^2 + 2Nj$ ($j = 1, \ldots, h$), etc. Thus in the minor there remain the columns

$$\begin{array}{lllll} 1, & 1 \, . \, 1, & 1^2 \, . \, 1, & \ldots, & 1^l \, . \, 1 \, , \\ \zeta, & 2 \, . \, \zeta, & 2^2 \, . \, \zeta, & \ldots, & 2^l \, . \, \zeta \, , \\ & & \vdots & & \\ \zeta^{h-1}, & h \, . \, \zeta^{h-1}, & h^2 \, , \zeta^{h-1}, & \ldots, & h^l \, . \, \zeta^{h-1} \, . \end{array} \tag{30}$$

For ζ on C_1, which is double in (24), the respective column in Δ has the terms

$$(N + 1) \zeta^{N+1}, (N + 2) \zeta^{N+2}, \ldots, (N + h) \zeta^{N+h} \, . \tag{31}$$

Taking out the factor $(N + 1) \zeta^{N+1}$ there remains

$$1, \frac{N + 2}{N + 1} \, . \, \zeta, \ldots, \frac{N + h}{N + 1} \, . \, \zeta^{h-1} \, , \tag{32}$$

From (30) and (32) there is seen that after taking out the respective factors there remains from Δ a determinant having for $N \to \infty$ a known nonzero determinant as the limit (see [10]).

Thus for N sufficiently great this determinant is certainly distinct from 0 and the respective term of the Laplace expansion is also distinct from 0.

Let us consider other terms of the Laplace expansion. For these terms, at least one column of Δ must be replaced by a column choosen from the first h columns. Let us denote the respective root of (24) by η.

Clearly for all ζ from Δ

$$|\eta| < |\zeta| \tag{33}$$

or

$$|\eta| = |\zeta| \tag{34}$$

but in this last case $|\eta| = 1$ and the respective column has the terms

$$\eta^{N+1}, \eta^{N+2}, \ldots, \eta^{N+h} \tag{35}$$

according to our arrangement of columns.

Now, with (33) one gets for each fixed k

$$\lim_{N \to \infty} \frac{|\eta|^N}{|\zeta|^N} \, . \, N^k = 0 \, . \tag{36}$$

Moreover, it is seen from (31) and (35) that in the case (34) the factor $N + 1$ is lost. This completes the proof.

Theorem 1. Let $B(z)$ possess on C_1 at most simple roots. Then

$$\lim_{N\to\infty} B_j = 0\,, \quad (j = 1, 2, \ldots, h) \tag{37}$$

and

$$\lim_{N\to\infty} A_j = A_j^*\,, \quad (j = 1, 2, \ldots, h) \tag{38}$$

where A_j^* are solution of the equations system

$$\begin{aligned} \eta_1^{-h} A_1^* + \ldots + \eta_l^{-h} A_h^* &= a_{-h}\,, \\ \eta_1^{-1} A_1^* + \ldots + \eta_l^{-1} A_h^* &= a_{-1}\,, \end{aligned} \tag{39}$$

where η_i are the roots of (24) inside and on C_1, arranged according to what follows after (26) and $a_{-1}, \ldots, a_{-h}$ are defined by (18), (19).

Proof. To compute B_j according to Cramer's rule, the respective numerator determinant has all terms of the Laplace expansion of lower order than the "dominant" term of Lemma 1. This gives (37). Further, we pose (37) in the first h equations for computing A_j, B_j. From this substitution, (39) follows at once.

Theorem 2. Let $B(z)$ posses on C_1 at most simple roots. Then the "limit" weighting sequence $\{a_n\}$ formed with

$$a_n = A_1^* \eta_1^n + \ldots + A_h^* \eta_l^n \tag{40}$$

is solution of the infinite equations system

$$\begin{aligned} \mu_0 a_0 &+ \mu_{-1} a_1 + \ldots = \bar{b}_T\,, \\ \mu_1 a_0 &+ \mu_0 a_1 \quad + \ldots = \bar{b}_{T-1}\,, \\ &\vdots \\ \mu_T a_0 &+ \mu_{T-1} a_1 + \ldots = \bar{b}_0\,, \\ \mu_{T+1} a_0 &+ \mu_T a_1 \quad + \ldots = 0\,, \\ &\vdots \end{aligned} \tag{41}$$

and is bounded.

Proof. For the infinite system (41) the conditions (17) vanish and from (18) with $B_j^* = 0$ $(j = 1, \ldots, h)$ one gets (39) for A_j^*. The boundedness of the solution is obvious.

Theorem 3. Let $B(z)$ posses all roots only inside C_1 or on C_1, the last ones being simple. Then the formal inversion

(42) $$A(z) = \frac{z^{-T}}{B(z)}$$

gives the same sequence $\{a_n\}$ as the system (41).

Proof. Forming from (42) the infinite equations system one finds it equivalent with the difference equation

(43) $$b_0 a_n + b_1 a_{n-1} + \ldots + b_h a_{n-h} = 0$$

with the characteristic equation

(44) $$B(z)\, z^h = 0$$

and with the initial conditions $a_{-1}, \ldots, a_{-h}$ given by (18), (19). Since the roots of (44) lie only inside or on C_1, the last ones being simple, the h equations to find the coefficients G_j of

(45) $$a_n = G_1 \eta_1^n + \ldots + G_h \eta_l^n$$

are the same as in (39). From this and Theorem 2 the proof is completed.

Example 1. Let

(46) $$\xi(t) = x(t) - 2x(t-1)\,,$$

thus $b_0 = 1$, $b_1 = -2$, $h = 1$, $T = 0$, $\mu_0 = 5$, $\mu_1 = -2$, $\mu_j = 0$ for $j > 1$. The system (41) is

(47) $$\begin{aligned} 5a_0 - 2a_1 \qquad\qquad &= 1\,, \\ -2a_0 + 5a_1 - 2a_2 \qquad &= 0\,, \\ -2a_1 + 5a_2 - 2a_3 &= 0\,, \\ &\vdots \end{aligned}$$

From (18), (19) there is $a_{-1} = 1/2$ and since the roots of the characteristic equation are $\eta = 1/2$, $\zeta = 2$, one gets from (39) $A_1^* = 1/4$. Thus from (40)

(48) $$a_n = (1/2)^{n+2}\,, \quad n = 0, 1, \ldots$$

In [1, p. 232]. Example 2, we have found for finite N

$$a_n = \frac{2^{2(N+1)-n} - 2^n}{2^{2(N+2)} - 1} \tag{49}$$

and thus

$$\lim_{N\to\infty} a_n = (1/2)^{n+2}\,. \tag{50}$$

Example 2. Let

$$\xi(t) = x(t) - x(t-1)\,, \tag{51}$$

thus $b_0 = 1$, $b_1 = -1$, $h = 1$, $T = 0$, $\mu_0 = 2$, $\mu_1 = -1$, $\mu_j = 0$ for $j > 1$. The system (41) is

$$\begin{aligned} 2a_0 - a_1 \qquad\qquad &= 1\,,\\ -a_0 + 2a_1 - a_2 \qquad &= 0\,,\\ -a_1 + 2a_2 - a_3 &= 0\,,\\ \vdots \qquad \end{aligned} \tag{52}$$

From (18), (19), there is $a_{-1} = 1$. The characteristic equation has double root $\eta = 1$. From (39), one gets $A_1^* = 1$. Thus from (40)

$$a_n = 1\,, \quad n = 0, 1, \ldots \tag{53}$$

In [1, p. 232]. Example 3, we have found for finite N

$$a_n = \frac{N + 1 - n}{N + 2} \tag{54}$$

and thus

$$\lim_{N\to\infty} a_n = 1\,. \tag{55}$$

Note that in [1], the relations (50) and (55) have been shown without connection to the solution of infinite systems of equations.

The example 1 is covered by a Theorem of Walsh (see [1] p. 238, formula (84), p. 237, formula (77)). To the example 2, this Theorem cannot be applied without generalization since it supposes no roots of the characteristic equation lying on C_1.

5. CONNECTION WITH THE THEORY OF INFINITE SYSTEMS OF LINEAR EQUATIONS

A general theory of infinite systems of linear equations with a Toeplitz matrix can be found in [4; 5], covering the situation with no roots of characteristic equation on C_1.

For this situation, a theory of approximation of the solution of infinite system with solutions of "truncated" systems for $N \to \infty$ is constructed in [5].

For the system (41) with arbitrary T and no roots of $M(z)$ in (24) on C_1, the two fundamental conditions of solvability are

$$M(z) \neq 0 \quad \text{on} \quad C_1\,, \tag{56}$$

$$\operatorname{ind} M(z) = [\arg M(e^{i\theta})]_{\theta=-\pi}^{\pi} = 0\,. \tag{57}$$

Both are fulfilled, the first one by supposition and the second one because $M(\exp i\theta)$ is in this case a positive Fejér trigonometric polynomial. Thus the general theory can be easily applied to our infinite system.

For the infinite analog of (6), to remain in the frame of [4], there may be shown easily with the aid of (8), (9) that the conditions

$$\sum_{j=-\infty}^{\infty} |\varrho_j| < \infty\,, \tag{58}$$

$$f(z) \neq 0 \quad \text{on} \quad C_1(f(z) \text{ defined in } (14))\,, \tag{59}$$

$$M(z) \neq 0 \quad \text{on} \quad C_1 \tag{60}$$

are sufficient. However, to solve the equations system by the simple methods of preceding chapters, $f(\exp i\theta)$ has to be a positive Fejér trigonometric polynomial.

The case of roots of the characteristic equation lying on C_1 has been treated in [8; 9], but the results there of cannot be applied directly to our case.

Let us now investigate in some detail the role of the conditions the specialized version of which are (56), (57).

Let

$$X(z) = \frac{\dfrac{-1}{\lambda}}{1 + \left[-\dfrac{1}{\lambda}\, B(z)\right]}\,.\,Y(z) = \frac{1}{B(z) - \lambda}\,.\,Y(z) \tag{61}$$

where $\lambda \neq 0$ is a complex number and $B(z)$ is defined in (13). Supposing $Y(z)$ be the Z-transform of an input sequence, $X(z)$ being the same for the output, one sees from the right side of (61) that for $\lambda = 0$ the inversion of the relation $Y(z) = X(z)\,B(z)$ would result. In the middle of (61), an approximate inversion is realized with $|\lambda| \ll 1$.

Comparing the coefficients in the expansion of (61) one gets

$$\begin{aligned} &(b_0 - \lambda)\,x_0 &&= y_0\,,\\ &b_1x_0 + (b_0 - \lambda)\,x_1 &&= y_1\,,\\ &b_2x_0 + b_1x_1 + (b_0 - \lambda)\,x_2 &&= y_2\,,\\ &\vdots \end{aligned} \tag{62}$$

This is an infinite system of equations and its characteristic function in the sense of [4] is

$$\mathscr{B}(\zeta) = (b_0 - \lambda) + b_1\zeta + \ldots + b_h\zeta^h = B(z) - \lambda\,, \tag{63}$$

thus

$$\mathscr{B}(\zeta) = -\lambda\left[1 - \frac{1}{\lambda}\,B(\zeta^{-1})\right]. \tag{64}$$

Now, for (62) to posses an unique solution for arbitrary bounded $\{y_n\}$ the necessary and sufficient conditions are

$$|\mathscr{B}(\zeta)| \neq 0 \quad \text{on} \quad C_1\,, \tag{65}$$

$$\operatorname{ind} \mathscr{B}(\zeta) = [\arg \mathscr{B}(e^{i\theta})]_{\theta=-\pi}^{\pi} = 0\,. \tag{66}$$

From (64), (65) there follows (with $z = \zeta^{-1}$)

$$\left|1 - \frac{1}{\lambda}\,B(z)\right| \neq 0 \quad \text{for} \quad z \quad \text{on} \quad C_1\,. \tag{67}$$

Furthermore, from (64)

$$\arg \mathscr{B}(\zeta) = \arg(-\lambda) + \arg\left[1 - \frac{1}{\lambda}\,B(z)\right]. \tag{68}$$

Moving ζ on C_1 in the positive sense results in moving z in the negative sense. The corresponding vectors from the origin to $\mathscr{B}(\zeta)$ and to $1 - B(z)/\lambda$ are moving in the same sense and differ by a constant angle as is seen from (68).

Let us formulate the analogy of the Nyquist stability criterion to our case (see [11, p. 61, Theorem 5, 14]):

The relation (61) is stable if and only if

a) the vector v from the origin to the point $1 - B(\exp i\theta)/\lambda$ has for every θ a non-zero length and

b) moving θ in $z = \exp i\theta$ trough the interval $\langle -\pi, \pi\rangle$ in negative sense the number of complete rotations of the vector v is zero.

But a) is (67), thus also (65), and b) is (66) (with the aid of (68)).

Thus, in our special case the conditions (65), (66), and the Nyquist stability criterion are identical.

Since in the theory of infinite systems of linear equations more general matrices than lower triangular ones – corresponding to physical realizable relations – are admissible, one may say that the fundamental conditions responding to (65), (66) represent an extension of the Nyquist stability criterion to systems more general than the physically realizable ones.

6. CONCLUDING REMARKS

In the present article, the results of [1] have been extended in various directions and their relations to the general theory of infinite systems of linear equations with Toeplitz matrix [4; 5] have been shortly discussed.

It seems that the extension of the Wiener-Hopf method to the case of the characteristic equation having roots on C_1 will play an important role in better handling various summation and numerical integration methods, resonant discrete filters, and the applications there of.

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Dr. Ludvík Prouza, CSc.; Tesla - Ústav pro výzkum radiotechniky (Research Institute for Radio Engineering), Opočínek, 533 31 p. Lány na Důlku. Czechoslovakia.