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The Number of Automata, Boundedly Determined Functions and Hereditary Properties of Automata*

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Asymptotic formulas for the number of strongly connected, sourced and initially connected automata with labelled states are given. Further, the concept of hereditary properties of automata is introduced and theorems concerning these properties are established. Finally, these theorems are used to obtain asymptotic formulas for the number of boundedly determined functions and some other assertions.

1. INTRODUCTION

In the present report an automaton will be understood to be a completely defined noninitial Mealy automaton (all pertinent definitions may be found in [1]). The set of all automata with input alphabet $X = \{x_1, \ldots, x_m\}$, output alphabet $Y = \{y_1, \ldots, y_n\}$ and set of internal states $S = \{s_1, \ldots, s_k\}$ will be denoted by $A(m, n, k)$.

The problem of finding the number of automata $A \in A(m, n, k)$ having a particular property is a natural problem of the theory of automata. These quantities may also be exploited when solving certain noncounting problems of the theory of automata (for example estimating some important parameters of automata).

This problem was considered first by E. Moore in the classical paper [2]. Later E. Livshic, M. Harrison and other authors established the exact formulas for the number of classes of isomorphic automata and for the cardinality of other particular subclasses in $A(m, n, k)$. A survey of these results is contained in [3].

In the present report we pursue a few purposes. First we obtain asymptotic formulas for the number of strongly connected, sourced and initially connected automata with

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 labelled states. Further we introduce the concept of hereditary properties of automata and establish theorems concerning these properties. Finally, these theorems are used in obtaining asymptotic formulas for the number of boundedly determined functions and some other assertions.

2. STRONGLY CONNECTED AUTOMATA

The set of all strongly connected automata $A \in A(m, n, k)$ will be denoted by $A_1(m, n, k)$. Depending on the relations between the values m and k the asymptotic formulas for cardinality of the set $A_1(m, n, k)$ have different form. These formulas are simple at $m = m(k) \geqq \ln k$ and the most complex at $m = \text{const} = 2$.

Theorem 1. *For any* $m = m(k) \geqq 2 \ln k$, $n \geqq 1$ *and* $k \to \infty$

$$|A_1(m, n, k)| \sim |A(m, n, k)| .$$

Theorem 2. *For any* $m \in [\frac{2}{3} \ln k, 2 \ln k]$, $n \geqq 1$ *and* $k \to \infty$

$$|A_1(m, n, k)| \sim |A(m, n, k)| \exp(-k e^{-m}) .$$

Theorem 3. *For any* $m \in [2, \frac{2}{3} \ln k]$, $n \geqq 1$ *and* $k \to \infty$

$$|A_1(m, n, k)| \sim |A(m, n, k)| \cdot c_0 \cdot \left(1 - \frac{m a_m}{1 + a_m}\right)^{-1/2} (v(m))^k$$

where $a_m \in [0, 1]$ *is the solution of the equation* $1 + x = x e^{m/(1+x)}$,

$$c_0 = \left(1 + \sum_{r=1}^{\infty} \binom{mr}{r} (e^m v(m))^{-r}\right)^{-1}$$

and

$$v(m) = a_m^{a_m} \cdot (1 + a_m)^{m-1-a_m} .$$

3. SOURCED AUTOMATA

An automaton $A \in A(m, n, k)$ is called *sourced* if $A \notin A_1(m, n, k)$ and there exists at least one state $s \in A$ such that any state $s' \in A$ is accessible from s in A. By $A_2(m, n, k)$ will be denoted the set of all sourced automata in $A(m, n, k)$.

We study the sourced automata from the motive that all initial connected automata are obtained from aütomata of the classes $A_1(m, n, k)$ and $A_2(m, n, k)$ by a suitable choice of initial state.

Theorem 4. *For any* $m \geqq 2 \ln k$, $n \geqq 1$ *and* $k \to \infty$

$$|A_2(m, n, k)| \sim |A(m, n, k)| \,.\, k \,.\, (1 - 1/k)^{m(k-1)} \,.$$

Theorem 5. *For any* $m \in [\frac{2}{3} \ln k, 2 \ln k]$, $n \geqq 1$ *and* $k \to \infty$

$$|A_2(m, n, k)| \sim |A(m, n, k| \,.\, k \,.\, \exp\left(- m - k e^{-m}\right) .$$

Theorem 6. *For any* $m \in [2, \frac{2}{3} \ln k]$, $n \geqq 1$ *and* $k \to \infty$

$$|A_2(m, n, k)| \sim |A(m, n, k)| \, k(v(m))^k \times$$

$$\times \frac{\left(1 - \frac{m a_m}{1 + a_m}\right)^{-1/2} \sum_{r=1}^{\infty} \frac{1}{r} \binom{mr}{r-1} (e^m v(m))^{-r}}{1 + \sum_{r=1}^{\infty} \binom{mr}{r} (e^m v(m))^{-r}} .$$

where a_m *and* $v(m)$ *are defined in Theorem* 3.

4. INITIALLY CONNECTED AUTOMATA

Let $A_3(m, n, k)$ denote the set of all initially connected automata which can be obtained from automata $A \in A(m, n, k)$ by a suitable choice of initial state.

Theorem 7. *For any* $m \geqq \frac{2}{3} \ln k$, $n \geqq 1$ *and* $k \to \infty$

$$|A_3(m, n, k)| \sim |A(m, n, k)| \, k \,.\, \exp\left(-k e^{-m}\right) .$$

Theorem 8. *For any* $m \in [2, \frac{2}{3} \ln k]$, $n \geqq 1$ *and* $k \to \infty$

$$|A_3(m, n, k)| \sim |A(m, n, k)| \, k(v(m))^k \times$$

$$\times \frac{\left(1 - \frac{m a_m}{1 + a_m}\right)^{-1/2} \left(1 + \sum_{r=1}^{\infty} \frac{1}{r} \binom{mr}{r-1} (e^m v(m))^{-r}\right)}{1 + \sum_{r=1}^{\infty} \binom{mr}{r} (e^m v(m))^{-r}} .$$

5. HEREDITARY PROPERTIES OF AUTOMATA

Some problems of the theory of automata are connected with the study of several characteristics of not all (or not almost all) the automata in $A(m, n, k)$ but only of automata in some special classes. For example, among these subclasses the set of initially connected automata presents a significant interest. At the same time when considering these problems it is convenient to deal with automata in $A(m, n, k)$ rather than with automata in $A_i(m, n, k)$ $(i = 1, 2, 3)$. In this connection we have the following question: What can one say about the value of certain parameters for automata in $A_i(m, n, k)$ $(i = 1, 2, 3)$ if one has information on the value of these parameters for automata in $A(m, n, k)$? More formally this question is formulated in terms of hereditary properties of automata.

Let E be a property of automata. The property E is called *hereditary* if the following requirements are satisfied:

1. If an automaton A has the property E then every automaton isomorphic to A has the property E.

2. An automaton A has the property E if and only if every initial automaton corresponding to the automaton A has the property E.

3. If an automaton A has the property E then every subautomaton of the automaton A has the property E.

The concept of a hereditary property is highly natural since all substantial properties are hereditary. For example, the following properties are hereditary:

a) an automaton is reduced;

b) an automaton is strongly connected;

c) the diameter of an automaton is not larger than r.

The above question consists in the following. Let E be a particular hereditary property of automata and let the fraction of these automata in $A(m, n, k)$ which have the property E be known. We then have the following question: What can one say about the fraction of automata in $A_i(m, n, k)$ $(i = 1, 2, 3)$ having the property E? The following theorems give answer to this question.

Theorem 9. *Let* $m \geqq 2$, $n \geqq 1$, $k \to \infty$ *and let the number of automata in* $A(m, n, k)$ *having a particular hereditary property E be not less than* $|A(m, n, k)| \,.\, (1 - o(\min\{1, (e^m/k)^{1/2}\}))$. *Then almost all* automata in* $A_i(m, n, k)$ $(i = 1, 2, 3)$ *have the property E.*

Theorem 10. *Let* $m \in [2, \frac{2}{3} \ln k]$, $n \geqq 1$, $k \to \infty$ *and let the number of automata in* $A(m, n, k)$ *having a particular hereditary property E be not less than* $|A(m, n, k)| \,.\, (1 - o((e^m/k)^{1/2}))$. *Then almost all automata in* $A_2(m, n, k)$ *have the property E.*

* Let $R(m, n, k)$ be an arbitrary class of automata and E some property of automata in $R(m, n, k)$. We say that almost all automata in $R(m, n, k)$ have the property E if the fraction of the automata in $R(m, n, k)$ having the property E approaches to 1 for $k \to \infty$.

Now we formulate statements which follow from Theorems 9, 10 and some previous assertions.

Corollary 1. *For any* $m \geqq 2$, $n \geqq 2$, $k \to \infty$ *and* $i = 1, 2, 3$

$$|A_i^{\text{red}}(m, n, k)| \sim \alpha(m)\, |A_i(m, n, k)|$$

where $\alpha(2) = \exp\left(-\frac{1}{2}n^2\right)$, $\alpha(m) = 1$ *for* $m \geqq 3$ *and* $A_i^{\text{red}}(m, n, k)$ *denotes the set of all reduced automata in* $A_i(m, n, k)$.

For $m \geqq 3$ this assertion immediately follows from [4] and Theorems 9, 10. In the case $m = 2$ it is necessary to make supplementary investigations.

A mapping f of the set of all words over X in a set of words over Y is a *boundedly determined function* (*b. - d. function*) if f can be realized in an automaton. We say that the b.-d. function f has *the weight* k if f can be realized in an initial automaton with k states and cannot be realized in any initial automaton with the number of states less than k.

The set of all b.-d. functions of weight k which can be realized by automata in $A_3(m, n, k)$ will be denoted by $F(m, n, k)$. It is easy to see that an arbitrary reduced automaton in $A_3(m, n, k)$ realizes a b.-d. function of weight k and distinct functions in $F(m, n, k)$ are realized by nonisomorphic automata.

Using this fact, Theorems 7, 8 and Corollary 1 the following corollary is easily established.

Corollary 2. *For any* $m \geqq 2$, $n \geqq 2$ *and* $k \to \infty$

$$|F(m, n, k)| \sim \alpha(m) \,.\, |A_3(m, n, k)|/k!\,.$$

7. DEGREE OF DISTINGUISHABILITY AND DIAMETER OF AUTOMATA AND B.-D. FUNCTIONS

The degree of distinguishability and the diameter are the most important parameters of automata and of the boundedly determined functions realized by them. An estimate of their magnitude is of great importance for the formulation and solution of many problems in the abstract theory of automata, e.g. synthesis, minimization, experiments and so forth. For distinct net automata with the same number of states the values of each of the above parameters are, in general, distinct. At the very beginning of the theory of automata it was established [2] that the maximum of the degree of distinguishability and of the diameter of an automaton is greater by one than the number of its states. But the question as to the most probable values of these parameters remained open for a long time. It was proved only in [5] and [6] that the diameter and the degree of distinguishability of almost all automata in $A(m, n, k)$ are essentially smaller than the maximum possible. Namely, Ya. Barzdin' and A. Korshunov [5] found that for any $m \geqq 2$, $n \geqq 1$ and $k \to \infty$ not less than $|A(m, n, k)| (1 - 1/k)$ automata $A \in A(m, n, k)$ have the diameter $d(A) < c \log_m k + 1$ where $c \to 1$ for $m \to \infty$.

From this fact and from Theorems 9 and 10 we have the following corollary.

Corollary 3. *If $m \geqq 2$, $n \geqq 1$ and $k \to \infty$ then almost all automata $A \in A_i(m, n, k)$ $(i = 1, 2, 3)$ have the diameter $d(A) < c \log_m k + 1$.*

If in proving Lemmas in [6] we use the more exact bounds on the being investigated values then we obtain the following result.

Corollary 4. *If $m \geqq 2$, $n \geqq 1$ and $k \to \infty$ then for almost all automata $A \in A_i(m, n, k)$ $(i = 1, 2, 3)$ the degree of distinguishability $h(A)$ satisfies the inequalities*

$$[\log_m \log_n k] \leqq h(A) \leqq \log_m \log_n k + 4 .$$

By the degree of distinguishability and the diameter of a boundedly determined function we mean the degree of distinguishability and the diameter of the reduced initially connected automaton realizing the given boundedly determined function.

From Corollaries 2–4 we have the following assertion.

Corollary 5. *If $m \geqq 2$, $n \geqq 2$ and $k \to \infty$ then for almost all boundedly determined functions $f \in F(m, n, k)$ the degree of distinguishability $h(f)$ and the diameter $d(f)$ satisfy the relations:*

$$[\log_m \log_n k] \leqq h(f) \leqq \log_m \log_n k + 4 ,$$
$$d(f) < c \log_m k .$$

8. DECIPHERING OF AUTOMATA AND B.-D. FUNCTIONS

The problem of reconstructing boundedly determined functions occurs in the theory of synthesis or in the theory of experiments in the following way. Let us assume that the "client" has thought of a certain automaton A in $A_3(m, n, k)$. Then the "performer" who knows the parameters m, n and k must discover the boundedly determined function f realized by the automaton A where he has the right to ask his questioner for answers to questions of the type "what does A transform such and such an input word into?".

It is easy to see that if a given automaton A has the degree of distinguishability h and the diameter d then f can be recovered if we know how A transforms input words of length $h + d + 1$ into output words of the same length, i.e. the function f is recovered by means of a multiple experiments of length $h + d + 1$. From this and from Corollaries $3-5$ we have the following assertion.

Corollary 6. *For $m \geqq 2$, $n \geqq 2$ and $k \to \infty$ almost all automata in $A_3(m, n, k)$ and almost all boundedly determined functions in $F(m, n, k)$ can be recovered by means of multiple experiments of length not more than $c_1 \log_m k + 1$, where $c_1 \to 1$ for $m \to \infty$.*

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