KYBERNETIKA — VOLUME 12 (1976), NUMBER 1

On the Weight Matrices of Linear Difference Equations

LUDVÍK PROUZA

Necessary and sufficient conditions fulfilled by the weight matrix of a linear difference equation are derived.

1. INTRODUCTION

It is known [2] that a discrete linear physically realizable system is uniquely determined by its (infinite lower triangular) weight matrix

$$\mathbf{W} = \begin{pmatrix} w_{00} & 0 & \ldots & & \\ w_{10} & w_{11} & 0 & \ldots & \\ w_{20} & w_{21} & w_{22} & 0 & \ldots \\ & & \vdots & & \end{pmatrix}. \tag{1}$$

The columns of $\mathbf{W}$ are responses of the system to the sequence of discrete unit impulses $(1, 0, \ldots)$, $(0, 1, 0, \ldots)$, $(0, 0, 1, 0, \ldots)$, $\ldots$

The system's bounded input- bounded output stability is simply expressed by the famous Toeplitz necessary and sufficient condition

$$\sum_{j=0}^{i} |w_{ij}| < K < \infty, K \text{ independent on } i\,. \tag{2}$$

Since for systems with infinite $\mathbf{W}$ only realizations with the aid of linear difference equations are of interest, there may be asked conditions for $\mathbf{W}$ to belong to a difference equation.

2. LINEAR DIFFERENCE EQUATION, INFINITE SYSTEM OF LINEAR EQUATIONS, AND THE WEIGHT MATRIX

Let

$$a_{tt}x_t = a_{t,t-1}x_{t-1} + \ldots + a_{t,t-n}x_{t-n} + y_t \tag{3}$$

with $\{a_{ts}\}$ and $\{y_t\}$ complex, $t = 0, \pm 1, \pm 2, \ldots$ and

$$a_{tt} \neq 0\,, \quad a_{t+n,t} \neq 0 \quad \text{for} \quad t \geqq 0 \tag{4}$$

be the given linear difference equation of the n-th order with the "input" sequence $\{y_t\}$ and the "output" sequence $\{x_t\}$ and let (as usual in the control theory)

$$\begin{aligned} x_{-1} &= x_{-2} = \ldots = x_{-n} = 0\,, \\ y_{-1} &= y_{-2} = \ldots = 0\,. \end{aligned} \tag{5}$$

The equation (3) with (4), (5) is clearly equivalent to the following infinite system of linear equations:

$$\begin{array}{lr} a_{00}x_0 & = y_0\,, \\ -a_{10}x_0 + a_{11}x_1 & = y_1\,, \\ \vdots & \\ -a_{n0}x_0 - a_{n1}x_1 - \ldots - a_{n,n-1}x_{n-1} + a_{nn}x_n & = y_n\,, \\ \quad - a_{n+1,1}x_1 - \ldots - a_{n+1,n-1}x_{n-1} - a_{n+1,n}x_n + a_{n+1,n+1}x_{n+1} & = y_{n+1}\,, \\ \vdots & \end{array} \tag{6}$$

The sequence $\{x_t\}$, $(t = 0, 1, \ldots)$ is recurrently uniquely determined and clearly

$$x_t = f(y_t, y_{t-1}, \ldots, y_0)\,. \tag{7}$$

With the aid of the theory of infinite matrices [1], (6) is

$$\mathbf{A}\mathbf{x} = \mathbf{y} \tag{8}$$

The meaning of the symbols is clear. Let us consider the infinite system of systems

$$\begin{array}{lr} a_{00}w_{00} & = 1\,, \\ -a_{10}w_{00} + a_{11}w_{10} & = 0\,, \\ -a_{20}w_{00} - a_{21}w_{10} + a_{22}w_{20} & = 0\,, \\ \vdots & \\ a_{00}w_{01} & = 0\,, \\ -a_{10}w_{01} + a_{11}w_{11} & = 1\,, \\ -a_{20}w_{01} - a_{21}w_{11} + a_{22}w_{21} & = 0\,, \\ \vdots & \end{array} \tag{9}$$

or, in the matrix notation

$$\mathbf{AW} = \mathbf{I} \tag{10}$$

where $\mathbf{I}$ is the infinite unit matrix. Remembering the definition of the weight matrix one sees that $\mathbf{W}$ in (10) is the weight matrix belonging to the linear discrete system defined by the equation (3). Thus, $\mathbf{W}$ is a right inverse of the matrix $\mathbf{A}$ and is unique and lower triangular. From (4) and (9) there follows

$$w_{tt} \neq 0 \quad \text{for} \quad t \geqq 0\,. \tag{11}$$

From (10), one gets

$$\mathbf{AWy} = \mathbf{Iy} = \mathbf{y} \tag{12}$$

and comparing with (8)

$$\mathbf{x} = \mathbf{Wy} \tag{13}$$

and this solution is unique.

Further, one may find a lower triangular solution $\mathbf{W}_1$ of $\mathbf{W}_1\mathbf{A} = \mathbf{I}$. Since $\mathbf{W}$ is unique, there follows easily $\mathbf{W}_1 = \mathbf{W}$, thus

$$\mathbf{WA} = \mathbf{I} \tag{14}$$

and $\mathbf{W}$ (with respect to (10) is the inverse of $\mathbf{A}$.

Now, having a lower triangular $\mathbf{W}$ satisfying (11), it is possible to compute a lower triangular $\mathbf{A}$ from (14) and this $\mathbf{A}$ is unique. If $\mathbf{A}$ is of the form given in (6), it belongs to the difference equation (3).

Theorem. *For a lower triangular* $\mathbf{W}$ *to be the weight matrix to the difference equation* (3) *satisfying* (4), *there is necessary and sufficient that for every* $k \geqq 0$

(I) *the determinants formed from* w_{il}, $i = j, \ldots, j + n - 1$, $l = k, \ldots, k + n - 1$, $j \geqq k$, *are distinct from zero,*

(II) *the determinants formed from* w_{il}, $i = j, \ldots, j + n$, $l = k, \ldots, k + n$, $j \geqq k + 1$, *are zero.*

Proof. Suppose that the equation (3) holds. The columns of the matrix $\mathbf{W}$ computed from (10) are

$$\begin{array}{lllll}
w_{kk}, & 0, & \ldots & & \\
w_{k+1,k}, & w_{k+1,k+1}, & 0, & & \ldots \\
w_{k+2,k}, & w_{k+2,k+1}, & w_{k+2,k+2}, & & 0, \ldots \\
\vdots & \vdots & \vdots & & \\
w_{k+n-1,k}, & w_{k+n-1,k+1}, & w_{k+n-1,k+2}, & \ldots, & w_{k+n-1,k+n-1} \\
\vdots & \vdots & \vdots & & \vdots
\end{array} \tag{15}$$

Every column is solution of the homogeneous difference equation belonging to (3), uniquely determined by the n initial terms. The determinant formed from the above terms is distinet from zero, as is seen from (11). Now, suppose that some other determinant formed in accordance with (I) is zero. Then some column thereof is linear combination of the others. According to (4), one may compute the terms in each column upward and the linear dependence remains valid, thus the first determinant is zero, which is impossible. Thus (I) holds.

Now, consider the further column with the column subscript $k + n$ in (15) and the weight sequences beginning with the row subscript $k + 1$. The first n columns are linearly independent solutions and thus the last column is linear combination thereof and II) holds.

Suppose further that $\mathbf{W}$ is given and (I), (II) hold. Then, a lower triangular $\boldsymbol{A}$ may be computed from (14), especially

$$
\begin{aligned}
(16)\qquad & w_{kk}a_{kk} && = 1\,,\\
& w_{k+1,k}a_{kk} + w_{k+1,k+1}a_{k+1,k} && = 0\,,\\
& \vdots && \\
& w_{k+n,k}a_{kk} + w_{k+n,k+1}a_{k+1,k} + \ldots + w_{k+n,k+n}a_{k+n,k} && = 0\,,\\
& w_{k+n+1,k}a_{kk} + w_{k+n+1,k+1}a_{k+1,k} + \ldots + w_{k+n+1,k+n}a_{k+n,k} + && \\
& \vdots \qquad\qquad + w_{k+n+1,k+n+1}a_{k+n+1,k} && = 0\,,
\end{aligned}
$$

With respect to (11), there is $a_{kk} \neq 0$. Suppose that $a_{k+n,k} = 0$. Then, the n equations beginning with the second posses a nonzero solution. The determinant there of is 0 in contradiction with (I). Thus $a_{k+n,k} \neq 0$.

Suppose further that $a_{k+n+1,k} \neq 0$. Put the nonzero term $w_{k+n+1,k+n+1}\ a_{k+n+1,k}$ to the right side. Then, the $n + 1$ equations beginning with the second posses zero determinant in accordance with (II). But the determinant from the first n rows and columns is nonzero in accordance with (I) and thus the rank of the matrix of the left side is n. Since the rank of the augmented matrix containing the right side is $n + 1$, the n equations posses only zero solution, which is impossible, since $a_{kk} \neq 0$. Thus $a_{k+n+1,k} = 0$. Similarly will be proved $a_{k+n+2,k} = a_{k+n+3,k} = \ldots = 0$. Thus the matrix $\boldsymbol{A}$ defines the system (6), or, what is the same, (3), (4).

3. APPLICATIONS

Let the matrix $\mathbf{W}$ be the Cesàro matrix

$$
(17)\qquad \mathbf{W} = \begin{pmatrix} 1, & 0, & \ldots & & \\ 1/2, & 1/2, & 0, & \ldots & \\ 1/3, & 1/3, & 1/3, & 0, & \ldots \\ & & \vdots & & \end{pmatrix}.
$$

From (I), (II) there is clear that one may find a linear difference equation of the first order to this **W**. From (14), one gets

(18)
$$\begin{aligned} x_0 \qquad\qquad &= y_0\,, \\ -x_0 + 2x_1 \qquad &= y_1\,, \\ -2x_1 + 3x_2 &= y_2\,, \\ &\vdots \end{aligned}$$

or

(19)
$$(n+1)\,x_n = nx_{n-1} + y_n\,, \qquad n \geqq 0\,, \quad x_{-1} = 0\,.$$

Dividing by $n + 1$ and denoting $z_n = y_n/(n + 1)$, one gets

(20)
$$x_n = \frac{n}{n+1}\,x_{n-1} + z_n\,.$$

Equations with $a_{tt} = 1$ and all other coefficients bounded have been extensively treated in the literature (see [2]). From (2), (17), the bounded input- bounded output stability of (19) is apparent. For (20), however, it follows therefrom that $\{x_n\}$ is bounded for $\{z_n\}$ fulfilling $|z_n| < K/(n + 1)$, $K < \infty$ and independent of n. For (20), naturally, **W** is not the same as that in (17). Thus, although (20) is important for implementation, (19) is to be preferred for general considerations.

Let further the matrix **W** be the Euler $\mathscr{E}_1$ transform matrix, [3]:

(21)
$$\mathbf{W} = \begin{pmatrix} 1\,, & 0\,, & \ldots & & & \\ 1/2\,, & 1/2\,, & 0\,, & \ldots & & \\ 1/4\,, & 1/2\,, & 1/4\,, & 0\,, & \ldots & \\ \dfrac{\binom{N}{0}}{2^N}\,, & \dfrac{\binom{N}{1}}{2^N}\,, & \ldots, & \dfrac{\binom{N}{N}}{2^N}\,, & 0\,, & \ldots \\ & & \vdots & & & \end{pmatrix}.$$

Suppose that an N-th order linear difference equation belongs to this **W**. Then from (21) and (II), there should be

(22)
$$\begin{vmatrix} \binom{1}{0}, & \binom{1}{1}, & 0\,, & \ldots & & \\ \binom{2}{0}, & \binom{2}{1}, & \binom{2}{2}, & 0\,, & \ldots & \\ \binom{N+1}{0}, & \binom{N+1}{1}, & \binom{N+1}{2}, & \ldots, & \binom{N+1}{N} & \end{vmatrix} = 0\,.$$

But, subtracting from each row beginning with the second one the preceding one, the degree of this determinant is reduced, its appearance being the same with $N + 1$ replaced by N. Repeating the procedure one finds that the determinant in (22) is 1.

Thus, the Euler transformation $\mathscr{E}_1$ cannot be realized by feedback and a linear difference equation of finite order.

(Received April 1, 1975.)

REFERENCES

[1] R. G. Cooke: Infinite Matrices and Sequence Spaces. Macmillan, London 1950.

[2] L. Prouza: Zur Stabilität der linearen Impulsfilter. Kybernetika *3* (1967), 6, 587–599.

[3] L. Prouza: On the Inversion of Moving Averages, Linear Discrete Equalizers and "Whitening" Filters, and Series Summability. Kybernetika *6* (1970), 3, 225–240.

Dr. Ludvík Prouza, CSc.; Tesla - Ústav pro výzkum radiotechniky (Research Institute for Radio Engineering), Opočínek, p. Lány na Důlku. Czechoslovakia.