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Optimization of Linear Systems with Input Time-Delay

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A McLaurin's series expansion is used to obtain a near-optimum controller for linear systems with input time-delay. The control has an exact feedback portion and a truncated series open loop gain. All the series coefficients are obtained by linear non-delay system computations. The proposed method is simulated on the digital computer and is applied to a numerical example.

1. INTRODUCTION

The optimal control of time-delay systems by applying the maximum principle [1] involves the solution of a set of $2n$ two-point boundary-value problem in which terms with both delay and advance terms are present. The solution of such problems is impossible computationally or otherwise. Therefore, the main object of all computational aspects of optimal time-delay systems [2–10] has been to device a methodology to avoid the solution of the mentioned 2-point boundary-value problem.

Eller et al. [2] have proposed a method which involves the solution of a set of successively coupled partial differential equations. This is a refinement of the method by Krasovkii [3] which obtains an analytic expression for the optimal control in terms of a set of three Riccati-type equations. Aggarwal [4] has proposed a computational procedure, without any numerical example, for the solution of the partial differential equations which appeared in reference [2]. Although these methods are exact, but their solution is tedious and cumbersome. Another class of methods of solution is the sensitivity approach [5–9] in which the general approach is to expand a system variable, i.e. costate [5], control [6] or state [7] in a series of some plant or imbedding parameter. Inoue et al. [5] approximates the costate in series expansion in a small delay, present in the system. It proposes an alternative method in which the delay is divided into a number of equally spaced subintervals. Then the time-delay problem is reformulated as a non-delay problem that is of singularly perturbed type [5]. The methods presented by Jamshidi and Malek-Zavarei [6] and

Chan and Perkins [7] are also sensitivity approaches in which the system equation [6] or the two-point boundary value problem [7] are expanded in terms of an imbedding parameter. Another approach is considered by Jamshidi [8] in which a nonlinear time-delay system is expanded about a nominal state-control pair and the resulting linear time-delay system with time-varying coefficients is changed to a non-delay non-homogeneous system using the transformation developed by Bate [9]. The concept of coupling is expanded to time-delay systems by Jamshidi [10] in which the near-optimum control is obtained using the computations of decoupled, non-delay systems only.

In this paper, the classical sensitivity method of reference [6] is expanded to systems with input time-delay. The imbedding parameter is introduced in the system equation so that the delay term is eliminated when the parameter is set to zero. The near-optimum control thus obtained turns out to have an exact feedback term and a truncated forward term. The method is applied to a second-order system with input time-delay.

2. STATEMENT OF THE PROBLEM

Consider the following class of linear systems with input time-delay;

$$\begin{aligned} \dot{x} &= Ax + Bu + Cu(t - T), \\ u(t) &= \alpha(t), \quad t_0 - T \leqq t \leqq t_0, \end{aligned} \tag{1}$$

where $x \in R_n$, $u \in R_m$ are the state and control vectors, A, B and C are constant matrices of appropriate dimensions, $\alpha(t)$ is the control's initial function, t_0 is the initial process time and T is the time delay, assumed to be constant, but not necessarily small. A control vector $u(t)$ should be obtained which would minimize a quadratic cost functional,

$$J = \frac{1}{2}\int_{t_0}^{t_f} (x'Qx + u'Ru)\,\mathrm{d}t \tag{2}$$

and satisfy the constraints of equation (1).

In equation (2), the matrix Q and matrix R are positive semi-definite and positive-definite, t_f is the final process time assumed to be finite and a prime denotes vector transposition.

Consider the linear system with time delay,

$$\mathrm{d}x(t, \varepsilon)/\mathrm{d}t = Ax(t, \varepsilon) + Bu(t, \varepsilon) + \varepsilon Cu(t - T, \varepsilon) \tag{3}$$

with the initial function $\alpha(t)$ given in (1) and ε, $0 \leqq \varepsilon \leqq 1$, is a scalar imbedding parameter. Note that for $\varepsilon \to 0$, the system becomes non-delay and when $\varepsilon = 1$, it represents the original plant given by equation (1). The problem is to find a near-optimum control as an expansion in ε so that it satisfies (1) and approximately minimizes the cost functional (2). The infinite McLaurin's series expansion of the control function u is

$$u = \sum_{i=0}^{\infty} \varepsilon^i u^{(i)}/i! \tag{4}$$

where $u^{(i)} = \lim_{\varepsilon \to 0} \partial^i u/\partial \varepsilon^i$ for $i = 0, 1, 2, \ldots$ These coefficients are referred to as control sensitivity functions.

3. NEAR-OPTIMUM CONTROL

For the optimal-control problem considered above, the theory of the maximum principle for time-delay systems will be used. The necessary conditions for optimality are:

$$\dot{x} = H_p = Ax + Bu + \varepsilon Cu(t - T), \tag{5}$$

$$\dot{p} = -H_x, \tag{6}$$

$$\begin{aligned} 0 &= H_u + H_v(s)|_{s=t+T}, \quad t_0 \leqq t \leqq t_f - T \\ &= H_u, \quad t_f - T \leqq t \leqq t_f \end{aligned} \tag{7}$$

where $v = u(t - T)$ and subscript denote gradients. $H = \frac{1}{2}(x'Qx + u'Ru) + p'\{Ax + Bu + \varepsilon Cu(t - T)\}$ is the Hamiltonian function and p is the costate vector. The boundary conditions are given by (1) and $P(t_f) = 0$. Performing the necessary gradients (5) will be reduced to:

$$\dot{x} = Ax + Bu + \varepsilon Cu(t - T), \tag{8}$$

$$\dot{p} = Qx - A'p, \tag{9}$$

$$0 = -Ru + B'p + \varepsilon C'p(t + T), \quad t_0 \leqq t \leqq t_f - T, \tag{10}$$

$$= -Ru + B'p, \quad t_f - T \leqq t \leqq t_f. \tag{11}$$

Calculating u from equations (10) and (11), and substituting in (8), results in:

$$\begin{aligned} \dot{x} &= Ax + (S_1 + \varepsilon^2 S_2)\, p(t) + \varepsilon S_{12} p(t + T) + \varepsilon S_{21} p(t - T), \\ &\quad t_0 \leqq t \leqq t_f - T, \end{aligned} \tag{12}$$

$$= Ax + S_1 p + \varepsilon S_{21} p(t - T), \quad t_f - T \leqq t \leqq t_f, \tag{13}$$

$$\dot{p} = Qx - A'p \tag{14}$$

where

$$S_1 = BR^{-1}B', \quad S_{12} = BR^{-1}C', \quad S_{21} = CR^{-1}B', \quad S_2 = CR^{-1}C'.$$

The system of equations (12)–(14) represents a linear 2-point boundary value problem involving both delay and advance terms.

Clearly, the coupling that exists between x and p and the fact that variables with t, $t - T$ and $t + T$ arguments are involved, make the solution of such problems impossible, both analytically and numerically. Hence the infinite-series expansion of the control given in (4) will be truncated up to the Nth term. The positive integer N is the order of approximation of the resulting near-optimum control.

To find the series coefficients of (4), differentiate equations (12)–(14) with respect to ε and let $\varepsilon \to 0$, i.e.

$$\dot{x}^{(1)} = Ax^{(1)} + S_1 p^{(1)}(t) + S_{12} p^{(0)}(t + T) + S_{21} p^{(0)}(t - T), \quad t_0 \leqq t \leqq t_f - T, \tag{15}$$

$$= Ax^{(1)} + S_1 p^{(1)}(t) + S_{21} p^{(0)}(t - T), \quad t_f - T \leqq t \leqq t_f, \tag{16}$$

$$\dot{p}^{(1)} = Qx^{(1)} - A'p^{(1)}, \quad p^{(1)}(t_f) = 0, \tag{17}$$

where the superscript i denotes ith-order differentiation with respect to ε, evaluated as ε approaches zero. The zeroth order, i.e. $\varepsilon \to 0$ corresponds to a nonretarded system whose solution can be easily found.

The system of equations (15)–(17) also represents a linear 2-point boundary value problem, with the difference that the delay and advance terms appear not as dependent variables, but rather as forcing functions resulting from zeroth-order terms $x^{(0)}$ and $p^{(0)}$. The zeroth-order 2-point boundary-value problem is obtained by letting $\varepsilon \to 0$ in (12)–(14), i.e.

$$\dot{x}^{(0)} = Ax^{(0)} + S_1 p^{(0)}, \quad x^{(0)}(t_0) = \alpha(t_0), \tag{18}$$

$$\dot{p}^{(0)} = Qx^{(0)} - A'p^{(0)}, \quad p^{(0)}(t_f) = 0. \tag{19}$$

The system of equations (18)–(19) is the well known state-regulator problem whose solution is [11]

$$\dot{x}^{(0)} = (A - S_1 K)\, x^{(0)}, \quad x^{(0)}(t_0) = x_0, \tag{20}$$

$$p^{(0)} = -Kx^{(0)}, \tag{21}$$

$$u^{(0)} = -R^{-1}B'Kx^{(0)}, \tag{22}$$

where matrix K is the symmetric positive-semidefinite solution of the Riccati equation

$$-A'K - KA + KSK - Q = \dot{K}, \quad K(t_f) = 0. \tag{23}$$

Therefore, the solution of equation (23) will result in the zeroth-order terms which are the forcing functions of the 1st-order 2-point boundary-value problem equations (15)–(17).

The system of equations (15)–(17) can be transformed into an uncoupled system by letting

$$p^{(1)} = -Kx^{(1)} + g_1\,, \tag{24}$$

where K is the solution of equation (23) and g_1 is an adjoint vector obtained by substituting equation (24) and its time derivative in equations (15)–(17), and equating the coefficients of powers of $x^{(1)}$, i.e.

$$\dot{g}_1 = -(A - S_1K)'\,g_1 - K\delta_1(t) - K\sigma_1(t)\,, \quad t_0 \leqq t \leqq t_f - T\,, \tag{25}$$

$$= -(A - S_1K)'\,g_1 + K\sigma_1(t)\,, \qquad t_f - T \leqq t \leqq t_f\,, \tag{26}$$

where

$$g_1(t_f) = 0\,, \quad \delta_1(t) = S_{12}p^{(0)}(t + T)\,, \quad \sigma_1(t) = S_{21}p^{(0)}(t - T)\,.$$

By comparing equations (21) and (24), it follows that $g_0 = 0$. The coefficient $u^{(1)}$ is then easily obtained by (10) and as:

$$u^{(1)} = -R^{-1}B'(Kx^{(1)} - g_1) + R^{-1}C'p^{(0)}(t + T) = \tag{27}$$

$$= -R^{-1}B'(Kx^{(1)} - g_1)\,, \tag{28}$$

where $x^{(1)}$ is obtained from

$$x^{(1)} = Ax^{(1)} - S_1Kx^{(1)} + \delta_1(t) + \sigma_1(t)\,, \quad t_0 \leqq t \leqq t_f - T\,, \tag{29}$$

$$= Ax^{(1)} - S_1Kx^{(1)} + S_1g_1 + \delta_1(t)\,, \quad t_f - T \leqq t \leqq T\,. \tag{30}$$

Similarly for the ith-order sensitivity terms, the following equations can be obtained:

$$\dot{x}^{(i)} = (A - S_1K)\,x^{(i)} + S_1g_i + \delta_i(t) + \sigma_i(t)\,, \quad t_0 \leqq t \leqq t_f - T\,, \tag{31}$$

$$= (A - S_1K)\,x^{(i)} + S_1g_i + \sigma_i(t)\,, \qquad t_f - T \leqq t \leqq t_f\,, \tag{32}$$

and

$$u^{(i)} = -R^{-1}B'(Kx^{(i)} - g_i) - R^{-1}C'(Kx^{(i-1)} - g_{i-1})\big|_{t+T}\,, \qquad t_0 \leqq t \leqq t_f - T\,, \tag{33}$$

$$= -R^{-1}B'(Kx^{(i)} - g_i)\,, \qquad t_f - T \leqq t \leqq t_f\,, \tag{34}$$

where

$$\dot{g}_i = -(A - S_1K)'\,g_i - K\delta_i(t) - K\sigma_i(t)\,, \quad t_0 \leqq t \leqq t_f - T\,, \tag{35}$$

$$= -(A - SK)'\,g_i + K\sigma_i(t)\,, \qquad t_f - T \leqq t \leqq t_f\,, \tag{36}$$

and

$$\delta_i(t) = S_{12}p^{(i-1)}(t + T)\,, \qquad \sigma_i(t) = S_{21}p^{(i-1)}(t - T)\,.$$

The substitution of $U^{(i)}$, obtained in (33), into (4) results in

$$(37)\qquad u = \sum_{i=0}^{\infty} -R^{-1}B'\frac{\varepsilon^i}{i!}(Kx^{(i)} - g_i) + \sum_{i=1}^{\infty} -R^{-1}C'\frac{\varepsilon^i}{i!}(Kx^{(i-1)} - g_{i-1})\big|_{t+T},$$

$$t_0 \leqq t \leqq t_f - T,$$

$$(38)\qquad = \sum_{i=0}^{\infty} -R^{-1}B'\frac{\varepsilon^i}{i!}(Kx^{(i)} - g_i), \quad t_f - T \leqq t \leqq t_f.$$

Note that for the interval $t_0 \leqq t \leqq t_f - T$ the second summation evaluated at advanced time has been obtained previously and first series of (37)–(38) is nothing but the infinite-series expansion of x. Thus equations (37)–(38) reduced to

$$(39)\qquad u = -R^{-1}B'Kx + \sum_{i=1}^{\infty} R^{-1}B'\frac{\varepsilon^i}{i!}g_i - \sum_{i=1}^{\infty} R^{-1}C'\frac{\varepsilon^i}{i!}(Kx^{(i-1)}(t+T) -$$

$$- g_{i-1}(t+T)), \quad t_0 \leqq t_f - T,$$

$$(40)\qquad = -R^{-1}B'Kx + R^{-1}B'\sum_{i=1}^{\infty}\frac{\varepsilon^i}{i!}g_i, \quad t_f - T \leqq t \leqq t_f.$$

The control u as represented by (39)–(40) is exact; however, in actual design situations, infinite sums cannot be obtained. A near-optimum control is obtained by truncating the series in (39)–(40) after Nth terms,

$$(41)\qquad u \simeq -R^{-1}B'Kx + \sum_{i=1}^{N}\frac{\varepsilon^i}{i!}\gamma_i(t)$$

$$(42)\qquad \simeq -R^{-1}B'Kx + \sum_{i=1}^{N}\frac{\varepsilon^i}{i!}R^{-1}B'g_i(t),$$

where $\gamma_i(t)$ follows from (39).

Fig. 1 shows a flow chart for the computational procedure of the method.

4. NUMERICAL EXAMPLE

The proposed method is applied to the following linear second-order time-delay system,

$$(43)\qquad \dot{x} = Ax + Bu + Cu(t - 0.1)$$

with initial conditions,

$$(44)\qquad x(0) = x_0 = [2\ 1],$$

$$u(t) = -1.0, \quad -0.1 \leqq t \leqq 0,$$

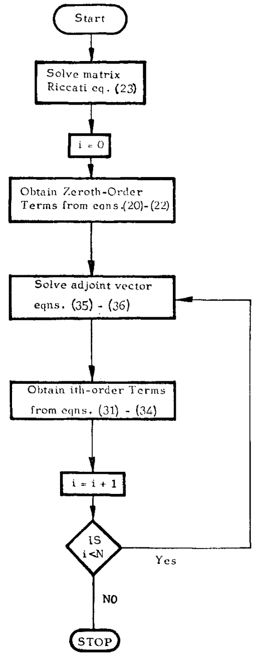

Fig. 1.

where the matrices are:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is required to find an optimal control $u(t)$, $0 \leqq t \leqq 1$, satisfying (43)–(44) and minimizing a quadratic cost function,

$$J = \frac{1}{2} \int_0^1 (x'Qx + u^2) \, dt \tag{45}$$

where $Q = 2I_2$. The application of the method proposed in Fig. 1, calls for the optimal solution of the zeroth-order system, i.e. when $\varepsilon \to 0$ for

$$\dot{x} = Ax + Bu + \varepsilon Cu(t - 0.1) \tag{46}$$

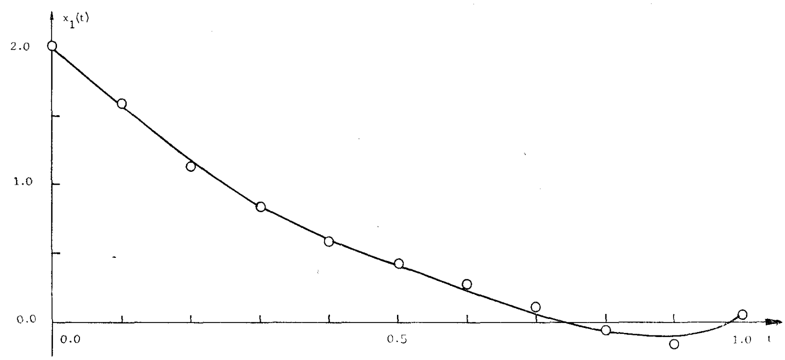

Fig. 2.

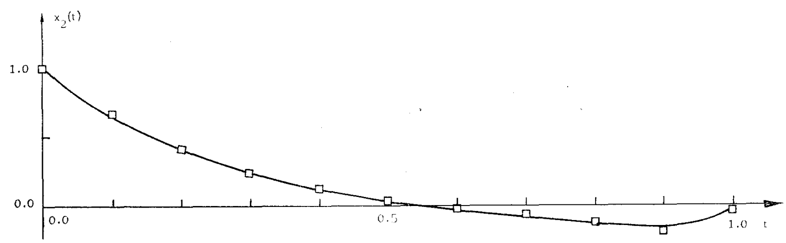

Fig. 3.

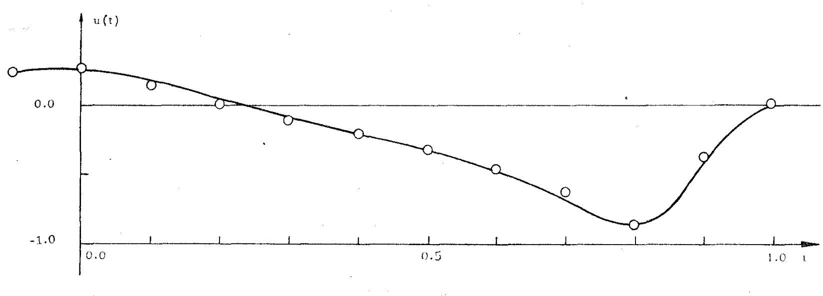

Fig. 4.

or

$$\dot{x}^{(0)} = Ax^{(0)} + Bu^{(0)}. \tag{47}$$

The optimal control of (45) and (47) is the well known linear state regulator [11] which calls for the solution of the matrix Riccati equation (23). The solution of this equation is obtained using the method proposed by Razzaghi and Flower [12] in which the Riccati matrix is obtained by solving differential equations in terms of the partitioned transition matrix of system

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & S_1 \\ -Q & -A' \end{bmatrix} \cdot \begin{bmatrix} x \\ p \end{bmatrix}. \tag{48}$$

The resulting Riccati matrix can analytically be obtained:

$$K(t) = \begin{bmatrix} \dfrac{1-t}{2-t} & 0 \\ 0 & \frac{1}{4}(1 - e^{4(t-1)}) \end{bmatrix}. \tag{49}$$

Hence the zeroth-order term state, costate and control are given below,

$$x^{(0)}(t) = \begin{bmatrix} \dfrac{-4}{t-2} \\ e^{-2t} \end{bmatrix}, \quad p^{(0)}(t) = \begin{bmatrix} \dfrac{4(t-1)}{(t-2)^2} \\ \frac{1}{4}(e^{-2t} - e^{2(t-2)}) \end{bmatrix} \tag{50}$$

and

$$u^{(0)}(t) = 4(t-1)/(t-2)^2.$$

The remaining steps of the computation as outlined by the flow-chart in Fig. 1 are done by the digital computer (IBM 370/135). The states and control as functions of time are shown in Figures 2 through 4.

5. CONCLUSIONS

A near-optimum control for linear systems with input time delay is obtained in this paper. The control has an exact feedback and approximate forward portion. For all orders of approximations, only one Riccati equation must be solved and the new approximation needs only the previous terms, rather than all the previous history as in some near-optimization techniques such as differential dynamic programming. The method seems to be attractive computationally and can be easily extended to nonlinear and time-varying systems.

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