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Fuzzy Topologies

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In the paper [1] there was the fuzzy set approach applied to the notion of the metric space. Here, this approach to the general topological space will be used. Fuzzy topological space is defined and studied in [2] but this conception is quite different from that which is presented in this paper.

We shall start from the definition of the topological space in the usual sense.

Definition 1. Let X be a non-empty set and let $\mathscr{P}(X)$ be the system of all subsets of the set X. We say that the couple $\langle X, u \rangle$ is a *topological space* supposing that u is a mapping of the system $\mathscr{P}(X)$ into itself satisfying the following two axioms:

(I) if $A \subset X$ contains at most one element, then $uA = A$

(II) if $A_1 \subset X$, $A_2 \subset X$, then $u(A_1 \cup A_2) = uA_1 \cup uA_2$.

Definition 2. Let X and $\mathscr{P}(X)$ have the same meaning as above, let $\mathscr{F}$ be the system of all fuzzy sets in X. A pair $\langle X, u \rangle$ is called *fuzzy topological space* supposing that u is a mapping from the system $\mathscr{P}(X)$ into $\mathscr{F}$ satisfying the following three axioms:

(FI) if $A \subset X$, then $uA(x) = 1$ for all $x \in A$

(FII) if $A \subset X$ contains at most one element, then $uA(x) = \psi_A(x)$, where ψ_A is the characteristic function of the set A

(FIII) if $A_1 \subset X$, $A_2 \subset X$, then $u(A_1 \cup A_2)(x) = \max \{uA_1(x), uA_2(x)\}$.

Clearly, any topological space is also fuzzy topological space. The property (FIII) implies, immediately, that if $A \subset B \subset X$, then $uA(x) \leqq uB(x)$ for all $x \in X$. In the following we shall write $f_{uA}(x)$ instead of $uA(x)$. The value $f_{uA}(x)$ can be interpreted as "the degree of membership of the element x in the closure of the set A", as $f_{uA}(x)$ forms, in fact, a fuzzy set in X.

For every set $A \subset X$ and every $\lambda \in \langle 0, 1\rangle$ let us define the set $A^\lambda \subset X$ by the relation:

$$A^\lambda = \{x : x \in X, f_{uA}(x) \geqq \lambda\} .$$

This definition involves a mapping from the Cartesian product $\mathscr{S}(X) \times \langle 0, 1\rangle$ into $\mathscr{P}(X)$ having the following properties:

1) $A^0 = X$, as $f_{uA}(x) \geqq 0$ for all $x \in X$,
2) $\emptyset^\lambda = \emptyset$ for every $\lambda \in (0, 1\rangle$, as $f_{u\emptyset}(x) = 0$,
3) if $A \subset B$, then $A^\lambda \subset B^\lambda$ for every $\lambda \in \langle 0, 1\rangle$,
4) for any $A \subset X$ and any $\lambda \in \langle 0, 1\rangle$ $A \subset A^\lambda$ is valid,
5) if $\lambda, \mu \in \langle 0, 1\rangle$, $\lambda \leqq \mu$, then $A^\lambda \supset A^\mu$ for any $A \subset X$,
6) for any $A \subset X$ the equality $A^\mu = \bigcap_{\lambda<\mu} A^\lambda$ holds, as $A^\mu \subset A^\lambda$ for any $\lambda < \mu$, hence $A^\mu \subset \bigcap_{\lambda<\mu} A^\lambda$. Conversely, let $x \in \bigcap_{\lambda<\mu} A^\lambda$, i.e., $f_{uA}(x) \geqq \lambda$ for all $\lambda < \mu$. If $f_{uA}(x) < \mu$ were valid, then such an $\varepsilon_0 > 0$ would exist, that $f_{uA}(x) < \mu - \varepsilon_0$, hence $x \notin A^{\mu-\varepsilon_0}$, as $\mu - \varepsilon_0 < \mu$. This contradicts to our presumption that $x \in \bigcap_{\lambda<\mu} A^\lambda$, which gives $A^\mu = \bigcap_{\lambda<\mu} A^\lambda$.
7) every λ-section for $\lambda > 0$ defines on the set X a closure operator in the sense of Definition 1, these topologies we shall call λ-topologies,
8) this mapping defines the membership function f_{uA} by the relation $f_{uA}(x) = \sup \{\lambda : x \in A^\lambda\}$.

Definition 3. The fuzzy set $\langle X, f_{A'}\rangle$ where $f_{A'}(x) = f_{u(A-\{x\})}(x)$ is called *fuzzy derivative set* of the set A.

Lemma 1. In every fuzzy topological space the following assertions are valid:

1) if $A \subset X$ contains at most one element, then $f_{A'}(x) = 0$,
2) if $A \subset X$, $B \subset X$, then $f_{(A\cup B)'}(x) = \max \{f_{A'}(x), f_{B'}(x)\}$,
3) if $A \subset X$, then $f_{uA}(x) = \max \{\psi_A(x), f_{A'}(x)\}$,
4) if $A \subset B \subset X$, then $f_{A'}(x) \leqq f_{B'}(x)$ for all $x \in X$.

Proof. If $A = \emptyset$, then $f_{uA}(x) = 0$ and the assertion is valid. If $A = \{x_0\}$, then $f_{u(\{x_0\}-\{x\})}(x) = f_{A'}(x)$ which gives $f_{\{x_0\}'}(x) = 0$, as $f_{u\{x_0\}}(x) = \psi_{\{x_0\}}(x)$.

2) $f_{(A\cup B)'}(x) = f_{u((A\cup B)-\{x\})}(x) = f_{u((A-\{x\})\cup(B-\{x\}))}(x) = \max \{f_{u(A-\{x\})}(x), f_{u(B-\{x\})}(x)\} = \max \{f_{A'}(x), f_{B'}(x)\}$.
3) $\max \{\psi_A(x), f_{u(A-\{x\})}(x)\} = 1$ for $x \in A$. If $x \notin A$, then clearly $\max \{\psi_A(x), f_{u(A-\{x\})}(x)\} = f_{uA}(x)$, as $A - \{x\} = A$ in this case.

4) if $A \subset B$, then $f_{uA}(x) \leqq f_{uB}(x)$ implies $f_{u(A-\{x\})}(x) \leqq f_{u(B-\{x\})}(x)$, which completes the proof.

Lemma 2. For any finite subset $A \subset X$ the equalities $f_{uA}(x) = \psi_A(x)$ and $f_{A'}(x) = 0$ hold for every $x \in X$.

Proof. Immediately from the foregoing lemma.

Definition 4. Let $A \subset X$, $A^c = X - A$, then the fuzzy set $\langle X, f_{A^\circ} \rangle$ where $f_{A^\circ}(x) = 1 - f_{uA^c}(x)$ is called *fuzzy interior set of* the set A.

Lemma 3. In every fuzzy topological space the following assertions are valid:

1) if A contains at most one element, then $f_{A^\circ}(x) = \psi_A(x)$,
2) if $A \subset X$, $B \subset X$, then $f_{(A \cap B)^\circ}(x) = \min \{f_{A^\circ}(x), f_{B^\circ}(x)\}$,
3) for every $A \subset X$ the inequality $f_{A^\circ}(x) \leqq f_{uA}(x)$ holds.

Proof. Follows immediately from Definitions 2 and 4.

Definition 5. Let $A \subset X$, then the fuzzy set $\langle X, f_{\partial A} \rangle$ where $f_{\partial A}(x) = \min \{f_{uA}(x), f_{uA^c}(x)\}$ is called *fuzzy boundary set* of the set A.

Lemma 4. For every $A \subset X$ $f_{\partial A}(x) = f_{\partial A^c}(x)$.

Proof. Immediately from the definition.

Definition 6. A subset $A \subset X$ is called to be *fuzzy closed* if

$$\psi_A(x) \geqq \min \{f_{uA}(x), f_{uA^c}(x)\} .$$

Theorem 1. A subset $A \subset X$ is fuzzy closed if and only if $f_{uA}(x) = \psi_A(x)$ for all $x \in X$.

Proof. If A is fuzzy closed, then for any $x \in A^c$ necessarily $0 = \min \{f_{uA}(x), f_{uA^c}(x)\}$, as $f_{uA^c}(x) = 1$ for $x \in A^c$. From this fact the validity of $f_{uA}(x) = \psi_A(x)$ for every $x \in X$ immediately follows. Conversely, if $f_{uA}(x) = \psi_A(x)$, then $\min \{f_{uA}(x), f_{uA^c}(x)\} = \min \{\psi_A(x), f_{uA^c}(x)\}$. This value is equal to $f_{uA^c}(x)$ for $x \in A$ and is equal to $\psi_A(x)$ for $x \in A$. However, if $x \in A$, then $\psi_A(x) = 1$, hence $\psi_A(x) \geqq \min \{f_{uA}(x), f_{uA^c}(x)\}$ and the theorem is proved.

Definition 7. A subset $A \subset X$ is called to be *fuzzy open* if for every $x \in X$ $\min \{\psi_A(x), f_{\partial A}(x)\} = 0$.

Theorem 2. A subset $A \subset X$ is fuzzy open if and only if $f_{A^\circ}(x) = \psi_A(x)$ for every $x \in X$.

Proof. Let A be fuzzy open, i.e., $\min\{\psi_A(x), f_{\partial A}(x)\} = 0$ for all $x \in X$. This gives that $\min\{f_{uA}(x), f_{uA^c}(x), \psi_A(x)\} = 0$ for all $x \in X$. This implies that if $x \in A$, then $f_{uA^c}(x) = 0$, i.e., $f_{uA^c}(x) = \psi_{A^c}(x)$, hence, $1 - \psi_{A^c}(x) = f_{A^o}(x) = \psi_A(x)$ for every $x \in X$. Conversely, if $f_{A^o}(x) = \psi_A(x)$, then $f_{uA^c}(x) = \psi_{A^c}(x)$, from which $\min\{\psi_A, f_{uA}, f_{uA^c}\}$ vanishes for every $x \in X$, i.e., the set A is fuzzy open. The theorem is proved.

Similarily, it can be proved that a set $A \subset X$ is fuzzy open if and only if the set A^c is fuzzy closed.

Definition 8. A subset $U \subset X$ is called to be *fuzzy neighbourhood* of an element $a \in X$ if $f_{u\{a\}}(x) \leqq f_{U^o}(x)$ for every $x \in X$.

Lemma 5. A set $U \subset X$ is a fuzzy neighbourhood of an element $a \in X$ if and only if $f_{uU^c}(a) = 0$.

Proof. If U is a fuzzy neighbourhood of an element $a \in X$, then Definition 8 gives that $f_{u\{a\}}(x) = \psi_{\{a\}}(x) \leqq f_{U^o}(x) = 1 - f_{uU^c}(x)$. As $\psi_{\{a\}}(x) = 0$ for $x \neq a$, the equality $f_{U^o}(a) = 1 - f_{uU^c}(a) = 1$ must hold, i.e., $f_{uU^c}(a) = 0$. The opposite inclusion can be proved quite analogously.

Lemma 6. If a set $U \subset X$ is a fuzzy neighbourhood of an element $a \in X$, then $\psi_{\{a\}}(x) \leqq f_{uU}(x)$ for every $x \in X$.

Proof follows immediately from the inequallity $f_{U^o}(x) \leqq f_{uU}(x)$.

Lemma 7. The intersection of any finite system of fuzzy neighbourhoods of an element $a \in X$ is again a fuzzy neighbourhood of a.

Proof. Let $U_1, U_2, \ldots, U_n$ be fuzzy neighbourhoods of a. This means that for every $i \leqq n$

$$\psi_{\{a\}}(x) \leqq f_{U^o_i}(x) = 1 - f_{uU^c_i}(x) .$$

Denote $U = U_1 \cap U_2 \cap \ldots \cap U_n$. Then $f_{U^o}(x) = \min\{f_{U^o_i}(x), i \leqq n\}$, hence $\psi_{\{a\}}(x) \leqq \leqq f_{U^o}(x)$ which proves the lemma.

Lemma 8. If a set U is a fuzzy neighbourhood of an element $a \in X$ and $x_0 \neq a$, then $U - \{x_0\}$ again is a fuzzy neighbourhood of a.

Proof. Lemma 5 gives that $f_{uU^c}(a) = 0$. We can write

$$f_{u(U-\{x_0\})^c}(x) = f_{u(X-(U-\{x_0\}))}(x) ,$$

if $x_0 \notin U$, then $U - \{x_0\} = U$ and lemma is valid, if $x_0 \in U$, then $X - (U - \{x_0\}) = = (X - U) \cup \{x_0\}$, hence $f_{u((X-U)\cup\{x_0\})}(x) = \max\{f_{uU^c}(x), f_{u\{x_0\}}(x)\}$. So if $x = a$, then $f_{u(U-\{x_0\})^c}(a) = 0$ and the lemma is proved.

The lemmas 6, 7, 8 characterize certain system $\Sigma(a)$ of subsets in X for every element $a \in X$, $\Sigma(a) = \{U \subset X : f_{uU^c}(a) = 0\}$. The properties of the system $\Sigma(a)$ following from the mentioned lemmas are:

1) the system $\Sigma(a)$ is non-empty,
2) $a \in U$ for every $U \in \Sigma(a)$,
3) if $x \neq a$, then there exists such a $U \in \Sigma(a)$ that $x \in X - U$,
4) if $U \in \Sigma(a)$, $V \in \Sigma(a)$, then there exists $W \in \Sigma(a)$ that $W \subset U \cap V$.

These foregoing properties of the system $\Sigma(a)$ enable to introduce certain topology into the set X. Let us denote this topology by v. It is familiarly known that the closure vA of a set $A \subset X$ in a topology is the set $vA = \{x \in X : U \in \Sigma(x) \Rightarrow U \cap A \neq \emptyset\}$. Using this property we can prove immediately the following theorem.

Theorem 3. The pair $\langle X, v\rangle$ is a topological space where

$$vA = \{x \in X : f_{uA}(x) > 0\}$$

for every $A \subset X$.

Proof. Let $x \in X$ belong to the set vA where A is an arbitrary subset in X. It means that for every $U \in \Sigma(x)$ the intersection $U \cap A$ is non-empty. The set U belongs to the system $\Sigma(x)$ of fuzzy neighbourhoods if and only if $f_{uU^c}(x) = 0$. If were $A \cap U = \emptyset$ then $A \subset U^c$ would hold. This implies that also $f_{uA}(x) = 0$. Conversely, if $f_{uA}(x) = 0$ then the set $U = X - A$ is a fuzzy neighbourhood of x because $f_{(X-A)^o}(x) = 1 - f_{u(X-A)^c}(x) = 1 - f_{uA}(x) = 1$. But in this case x cannot belong to the closure vA because $A \cap U = A \cap (X - A) = \emptyset$. This completes the proof.

Remark. It is clear, immediately, that also the closure vA can be written as $vA = \bigcup_{\lambda > 0} A^\lambda$ where A^λ is the closure of the set A in λ-topology. It implies that the topology v is coarser than all λ-topologies because $vA \supset A^\lambda$ for every $\lambda > 0$. The set $A \subset X$ is closed in the topology v if and only if A is closed in every λ-topology because if $A = vA = \bigcup_{\lambda > 0} A^\lambda$ then $A = A^\lambda$ must hold. In this case $f_{uA}(x) = \psi_A(x)$ for every $x \in X$ must hold, hence the set A is closed in topology v if and only if A is fuzzy closed. From this point of view a fuzzy topology in the set X can be understood as the one-parametric system of the due λ-topologies if we put the topology v in the case $\lambda = 0$.

A very important notion in the theory of topological spaces is that of *F-topology*. For the sake of completness let us introduce the definition.

Definition 9. Let (X, u) be a topological space. The topology u is called *F-topology* if the closure of any subset of X is a closed set, in symbols: $u^2A = uA$ for any $A \subset X$.

The following definition introduces an analogous notion into the fuzzy topological spaces.

Definition 10. A fuzzy topological space (X, u) is called *F-fuzzy topological space*, if for any $A \subset X$ and any ordered pair $\langle \lambda, \mu \rangle \in \langle 0, 1 \rangle \times \langle 0, 1 \rangle$ there exists such a $\varrho = \varrho_A(\lambda, \mu)$, that $(A^\lambda)^\mu \subset A^\varrho$; at the same time such a pair $\langle \lambda_0, \mu_0 \rangle$, $\lambda_0 > 0$, $\mu_0 > 0$ must exist, that $\varrho_A(\lambda_0, \mu_0) > 0$, as $A^0 = X$ for any $A \subset X$ (said in other words, $\varrho_A(\lambda, \mu)$ is not allowed to be an identical zero on $(0, 1\rangle \times (0, 1\rangle)$. For any F-fuzzy topological space (X, u) and any $A \subset X$ we can define

$$T_A(\lambda, \mu) = \min \{\sup \{\varrho : (A^\lambda)^\mu \subset A^\varrho\}, \sup \{\varrho : (A^\mu)^\lambda \subset A^\varrho\}\} .$$

Theorem 4. The following assertions concerning the function T_A are valid:

(1) T_A is non-decreasing in $\langle 0, 1 \rangle \times \langle 0, 1 \rangle$, i.e. $T_A(\lambda, \mu) \leqq T_A(\lambda', \mu')$ for $\lambda \leqq \lambda'$, $\mu \leqq \mu'$,

(2) $T_A(\lambda, 0) = T_A(0, \mu) = 0$ for all $\lambda, \mu \in \langle 0, 1 \rangle$,

(3) $T_A(\lambda, \mu) = T_A(\mu, \lambda)$ for all $\lambda, \mu \in \langle 0, 1 \rangle$,

(4) $t(\lambda, \mu) \leqq T_A(\lambda, \mu) \leqq \min(\lambda, \mu)$, where
$t(\lambda, \mu) = \lambda$ if $\mu = 1$, $t(\lambda, \mu) = \mu$ if $\lambda = 1$, $t(\lambda, \mu) = 0$ else.

Proof. (1) Let $\lambda \leqq \lambda'$, $\mu \leqq \mu'$, then $A^\lambda \supset A^{\lambda'}$, hence $(A^\lambda)^\mu \supset (A^{\lambda'})^{\mu'}$, analogously $A^\mu \supset A^{\mu'}$, hence $(A^\mu)^\lambda \supset (A^{\mu'})^{\lambda'}$. If $(A^\lambda)^\mu \subset A^\varrho$, then $(A^{\lambda'})^{\mu'} \subset A^\varrho$, so the function $\sup \{\varrho : (A^\lambda)^\mu \subset A^\varrho\}$ is non-decreasing, this gives that also T_A is non-decreasing.

(2) $T_A(\lambda, 0) = 0$, as $(A^\lambda)^0 = X$ for any $\lambda \in \langle 0, 1 \rangle$, analogously $T_A(0, \mu) = 0$ for any $\mu \in \langle 0, 1 \rangle$.

(3) Symmetry of T_A follows immediately from the definition.

(4) If $\mu = 1$, then $(A^\lambda)^1 = \bigcap_{\mu<1} (A^\lambda)^\mu \subset \bigcap_{\mu<1} (A^\lambda)^{T_A(\lambda,\mu)} = \lim_{\mu \uparrow 1} (A^\lambda)^{T_A(\lambda,\mu)}$. For any $\mu \in \langle 0, 1 \rangle$ is $A^\lambda \subset (A^\lambda)^\mu$, hence $A^\lambda \subset (A^\lambda)^1$, which gives $T_A(\lambda, 1) \geqq \lambda$. Symmetry of T_A gives that $T_A(1, \mu) \geqq \mu$. Combining these results we have that the inequality $T_A(\lambda, \mu) \geqq t(\lambda, \mu)$ must hold. As $A \subset A^\lambda \subset (A^\lambda)^\mu \subset A^{T_A(\lambda,\mu)} = \bigcap \{A^\varrho : (A^\lambda)^\mu \subset \subset A^\varrho\}$, another saying $A^{T_A(\lambda,\mu)} = \{x : f_{uA}(x) \geqq T_A(\lambda, \mu)\} \supset \{x : f_{uA}(x) \geqq \lambda\} = A^\lambda$, necessarily $T_A(\lambda, \mu) \leqq \lambda$ and symmetry gives also $T_A(\lambda, \mu) \leqq \mu$. This implies that $T_A(\lambda, \mu) \leqq \min(\lambda, \mu)$ and, also, that $(A^\lambda)^\lambda = A^\lambda$ for any $\lambda \in \langle 0, 1 \rangle$. The theorem is proved.

Theorem 5. For any fuzzy-topological space (X, u) there exists a probability space (Ω, Σ, P) such that Ω is a set of certain topologies defined in X, σ-algebra Σ contains all the sets of the type $\{\tau_\lambda : f_{uA}(x) \geqq \lambda\}$ and $P(\{\tau_\lambda : \tau_\lambda \in \Omega, f_{uA}(x) \geqq \lambda\}) = f_{uA}(x)$.

Proof. Let $\lambda \in (0, 1\rangle$, let us ascribe, to any $A \subset X$, a subset $A^\lambda = \{x : f_{uA}(x) \geqq \lambda\}$. We shall prove that the mapping $\theta_\lambda : A \to A^\lambda$ defines a topology in the space X. If $A = \emptyset$, then $A^\lambda = \{x : f_{uA}(x) \geqq \lambda > 0\} = \{x : 0 \geqq \lambda > 0\} = \emptyset$. If the set A

contains just one element, i.e. $A = \{a\}$, $a \in X$, then $A_\lambda = \{x : f_{u\{a\}}(x) \geqq \lambda\} = \{x : \psi_{\{a\}}(x) \geqq \lambda\} = \{a\}$, as $\lambda > 0$ and $\psi_{\{a\}}(x) = 0$ foı $x \neq a$, $\psi_{\{a\}}(a) = 1$. The only thing which rests to be proved is the equality $(A \cup B)^\lambda = A^\lambda \cup B^\lambda$. However, $(A \cup B)^\lambda = \{x : f_{u(A \cup B)}(x) \geqq \lambda\} = \{x : \max\{f_{uA}(x), f_{uB}(x)\} \geqq \lambda\} = \{x : f_{uA}(x) \geqq \lambda\} \cup \{x : f_{uB}(x) \geqq \lambda\} = A^\lambda \cup B^\lambda$.

Denote $\Omega = \{\tau_\lambda : 0 < \lambda \leqq 1\}$, where τ_λ is the topology defined by the closure operator θ_λ. Denote also, for the sake of lucidity, the sets of the type $\{\{\tau_\lambda \in \Omega : \varrho_1 < \lambda \leqq \varrho_2\}, \varrho_1, \varrho_2 \in \langle 0, 1 \rangle\}$ by $\{\{(\varrho_1, \varrho_2\rangle\} : \varrho_1, \varrho_2 \in \langle 0, 1 \rangle\}$. These sets can be easily proved to form a semi-ring in Ω, at the same time $\Omega = (0, 1\rangle$. Clearly, a set $(\varrho_1, \varrho_2\rangle$ cannot be, in general, expressed in the form $(\varrho_1, \varrho_2\rangle$ in the unique way. If there exist more such possibilities for one subset of this type, i.e. if, e.g. $(\varrho_1, \varrho_2\rangle = (\varrho_3, \varrho_4\rangle$, $\varrho_1 \neq \varrho_3, \varrho_2 \neq \varrho_4$, then such an ambiguousity can be eliminated by expressing $(\varrho_1, \varrho_2\rangle$ in its "maximal" form, i.e. in the form $(\inf \varrho_1, \sup \varrho_2\rangle$. In this way the sets of the type $(\varrho_1, \varrho_2\rangle$ are defined unambigously and we can ascribe to them a probabilistic measure $P\{(\varrho_1, \varrho_2\rangle\} = \varrho_2 - \varrho_1$. Clearly $P\{\Omega\} = 1$ and P can be easily proved to be a measure on the semi-ring mentioned above. Let Σ be the minimal σ-algebra over this semi-ring, let us extend P to this Σ-algebra (it is a well-known fact that such an extension is defined unambigouosly). As can be easily seen, $P\{(\tau_\lambda : f_{uA}(x) \geqq \lambda)\} = P\{(\tau_\lambda : x \in A^\lambda)\} = f_{uA}(x)$, i.e. the value $f_{uA}(x)$ can be understood as the probability, that an element x belongs to the closure of the set A. Q. E. D.

New, we shall study probabilistic metric spaces as a special case of fuzzy topological spaces. Let $\langle X, F \rangle$ be a probabilistic metric space, let A be a subset in X and let us define the function $f_{uA}(.)$ by the relation $f_{uA}(x) = \inf_{t>0} \sup_{y \in A} F_{xy}(t)$. If the set A contains only one point, say, $A = \{a\}$, then $f_{u\{a\}}(x) = \inf_{t>0} F_{xa}(t)$. Then there are the two posibilities:

1) $\inf_{t>0} F_{xa}(t) = 0$,
2) $\inf_{t>0} F_{xa}(t) > 0$.

If the first case occurs for every pair (x, a), $x \neq a$ in the set X, then the function $f_{u\{a\}}(.)$ is just the indicator of the set $\{a\}$. If the other case occurs, at least for one pair (x, a), $x \neq a$, then the considered probabilistic metric space cannot be a fuzzy topological space in the sense of Definition 2. This fact leads us to the modification of Definition 2 as follows.

Definition 11. Let X be a non-empty set, let $\mathscr{P}(X)$ be the system of all subsets in X and let $\mathscr{F}$ be the system of all fuzzy sets in X. We shall say that the pair $\langle X, u \rangle$ is a *generalized fuzzy topological space* supposing that u is a mapping of the system $\mathscr{P}(X)$ into $\mathscr{F}$ satisfying the following properties:

1) if $A \subset X$ then $uA(x) = 1$ for all $x \in A$
2) $A \subset X, B \subset X$ then $u(A \cup B)(x) = \max\{uA(x), uB(x)\}$ for every $x \in X$.

It is obvious that every fuzzy topological space is a generalized fuzzy topological space. In the case of a fuzzy topological space we could define for every $\lambda \in (0, 1\rangle$ and every $A \subset X$ the λ-closure $A^\lambda = \{x : f_{uA}(x) \geqq \lambda\}$. We have proved that the λ-closure is a topological closure in the sense of Definition 1. It is not true in the case of a generalized fuzzy topological space because some one-point sets in X need not be λ-closed. Let $\langle X, F\rangle$ be such a probabilistic metric space in which for every pair $x, y \in X$, $x \neq y$ $\lim_{t\downarrow 0} F_{xy}(t) = 0$ holds. Then the requirement (F II) in Definition 2 is fulfilled. If $A \subset X$ then $\sup_{y\in A} F_{xy}(t) = 1$ for every $x \in A$ because $F_{xx}(t) = 1$ for every $x \in X$ and every $t > 0$. Hence, the requirement (F I) in Definition 2 is also fulfilled. Let A, B be arbitrary subsets in X, then $f_{u(A\cup B)}(x) = \inf_{t>0} \sup_{y\in A\cup B} F_{xy}(t) \geqq$ $\geqq \max\{\inf_{t>0}\sup_{y\in A} F_{xy}(t), \inf_{t>0}\sup_{y\in B} F_{xy}(t)\}$. On the other hand, $f_{u(A\cup B)}(x) = \inf_{t>0} \sup_{y\in A\cup B} F_{xy}(t)$ and, hence, for every $\varepsilon > 0$ there exists such a $t_\varepsilon > 0$ that $f_{u(A\cup B)}(x) + \varepsilon > \sup_{y\in A\cup B} F_{xy}(t_\varepsilon)$. Evidently, $\sup_{y\in A\cup B} F_{xy}(t_\varepsilon) \geqq F_{xy}(t_\varepsilon)$ for every $y \in A \cup B$. There exists such a $y_\varepsilon \in A \cup B$ that $\sup_{y\in A\cup B} F_{xy}(t_\varepsilon) - \varepsilon < F_{xy_\varepsilon}(t_\varepsilon) \leqq \max\{\sup_{y\in A} F_{xy}(t_\varepsilon), \sup_{y\in B} F_{xy}(t_\varepsilon)\}$. As the inequality $f_{u(A\cup B)}(x) \leqq \sup_{y\in A\cup B} F_{xy}(t)$ holds, also $f_{u(A\cup B)}(x) - \varepsilon \leqq \sup_{y\in A\cup B} F_{xy}(t_\varepsilon) - \varepsilon \leqq \max\{\sup_{y\in A} F_{xy}(t_\varepsilon),$ $\sup_{y\in B} F_{xy}(t_\varepsilon)\}$ must hold for every $\varepsilon > 0$. From this fact it follows that $f_{u(A\cup B)}(x) =$ $= \max\{f_{uA}(x), f_{uB}(x)\}$ for every $x \in X$. We have just proved the following theorem.

Theorem. 6 Let $\langle X, F\rangle$ be such a probabilistic metric space that $\lim_{t\downarrow 0} F_{xy}(t) = 0$ for every pair $x, y \in X, x \neq y$. Then the mapping $u : \mathscr{P}(X) \to \mathscr{F}$ where $uA(x) =$ $= \inf_{t>0} \sup_{y\in A} F_{xy}(t)$ defines a fuzzy topology in the space $\langle X, F\rangle$.

Further, we shall study the basic properties of this fuzzy topology in these probabilistic metric spaces.

Lemma 9. A set $A \subset X$ is fuzzy closed if and only if for every $x \in A^c$ $\lim_{t\downarrow 0} F_{xy}(t) = 0$ uniformly with respect to $y \in A$.

Proof. This statement follows immediately from that fact that a set A is fuzzy closed if and only if $f_{uA}(x) = \psi_A(x)$ for every $x \in X$.

Similarily, we can prove the following lemma.

Lemma 10. A set $U \subset X$ is a fuzzy neighbourhood of a point $a \in X$ if and only if $\lim_{t\downarrow 0} F_{ay}(t) = 0$ uniformly with respect to $y \in U^c$.

If we denote $\mathscr{O}(a, \varepsilon, t) = \{x \in X : F_{ax}(t) > \varepsilon\}$ and if a set $U \subset X$ is a fuzzy neighbourhood of the point $a \in X$ then using the foregoing lemma we are able to find, for every $\varepsilon > 0$, such a $t_\varepsilon > 0$ that $U^c \cap \mathscr{O}(a, \varepsilon, t_\varepsilon) = \emptyset$, hence $\mathscr{O}(a, \varepsilon, t_\varepsilon) \subset U$. From this fact it follows, immediately, that some fuzzy neighbourhoods of the point a can be expressed in the form $U = \bigcup_{\varepsilon > 0} \mathcal{O}(a, \varepsilon, t_\varepsilon)$ where t_ε is a non-decreasing function at ε.

Now, we shall study in details the λ-topologies defined by the closure operators $u_\lambda : A \to A^\lambda = \{x \in X : \inf_{t > 0} \sup_{y \in A} F_{xy}(t) \geqq \lambda\}$.

Lemma 11. The system of the sets $\{\mathcal{O}(a)\}_\lambda = \{\mathcal{O}(a, \lambda - \delta, \eta),\ \delta \in \langle 0, \lambda),\ \eta > 0\}$ forms the complete system of neighbourhoods for the λ-topology.

Proof. The system $\{\mathcal{O}(a)\}_\lambda$ is, evidently, non-empty and every set $U \in \{\mathcal{O}(a)\}_\lambda$ contains the point a because $F_{aa}(t) = 1$ for every $t > 0$. Let $x \neq a$, $x \in X$.We assume that $\lim_{t \downarrow 0} F_{xa}(t) = 0$, i.e., $\forall \varepsilon > 0 \exists\, t_\varepsilon > 0$ such that if $F_{xa}(t_\varepsilon) \leqq \varepsilon$, then $x \notin \mathcal{O}(a, \lambda - (\lambda - \varepsilon), t_\varepsilon)$ if we choose $\varepsilon < \lambda$. When the sets $\mathcal{O}(a, \lambda - \delta_1, \eta_1)$, $\mathcal{O}(a, \lambda - \delta_2, \eta_2)$ belong to the system $\{\mathcal{O}(a)\}_\lambda$ then also the set $\mathcal{O}(a, \lambda - \min(\delta_1, \delta_2), \max(\eta_1, \eta_2))$ belongs to $\{\mathcal{O}(a)\}_\lambda$ and $\mathcal{O}(a, \lambda - \min(\delta_1, \delta_2), \max(\eta_1, \eta_2) \subset \mathcal{O}(a, \lambda - \delta_1, \eta_1) \cap \mathcal{O}(a, \lambda - \delta_2, \eta_2)$. So we have proved that by the using of the system $\{\mathcal{O}(a)\}_\lambda$ one can introduce certain topology into the set X. Let A be any subset in X, then the closure of A in this topology is the set $\bar{A}_\lambda = \{x \in X : U \in \{\mathcal{O}(x)\}_\lambda \Rightarrow U \cap A \neq \emptyset\}$. Let $a \in \bar{A}_\lambda$. It means that for every neighbourhood $\mathcal{O}(a, \lambda - \delta, \eta)$ there exists an $x_{\delta\eta} \in \mathcal{O}(a, \lambda - \delta, \eta) \cap A$. From this fact it follows, immediately, that the inequality $f_{uA}(a) \geqq \lambda$ must hold. On the other hand, if $f_{uA}(a) \geqq \lambda$. i.e., $\inf_{t>0} \sup_{y \in A} F_{ay}(t) \geqq \lambda$, hence, $\sup_{y \in A} F_{ay}(t) \geqq \lambda$ for every $t > 0$ and for every $\varepsilon > 0$ and every $t > 0$ there exists such a $y_{\varepsilon t} \in A$ that $F_{ay_{\varepsilon t}}(t) > \lambda - \varepsilon$, i.e., $\mathcal{O}(a, \lambda - \varepsilon, t) \cap A \neq \emptyset$. This implies that this topology is identical with the λ-topology.

The foregoing two lemmas demonstrate the necessity of the assumption of the continuity of all distribution functions $F_{xy}(.)$ at 0 for every pair $x, y \in X$, $x \neq y$. If this assuption is not fulfilled, then every λ-topology is a so called generalized topology in that sense that some one-point sets in X need not be λ-closed. In this general case only the most gentle λ-topology, i.e., 1-topology, is a topology in the sense of Definition 1. It follows from this assuption that $F_{xy}(t) = 1$ for every $t > 0$ implies $x = y$.

Theorem 7. If $\langle X, F \rangle$ is a Menger space with the t-norm $T = \min$ then every λ-topology is F-topology.

Proof. The proof of this statement is quite based on the strongest possible form of the generalized triangular inequality $F_{xy}(\lambda + \mu) \geqq \min(F_{xz}(\lambda), F_{yz}(\mu))$.

Remark. If the space $\langle X, F \rangle$ is a Menger space with t-norm T which is continuous on $\langle 0, 1\rangle \times \langle 0, 1 \rangle$ then one can prove that 1-topology is an F-topology.

One of the most important problems in the theory of topological spaces is the problem of the metrizability. The similar problem arises also in fuzzy topological spaces.

Definition 12. Let $\langle X, v\rangle$ and $\langle X, u\rangle$ be two fuzzy topological spaces. We say that the fuzzy topologies u and v are *equivalent* if every fuzzy closed set in the fuzzy topology u is also fuzzy closed in fuzzy topology v and vice versa.

Definition 13. A fuzzy topological space $\langle X, u\rangle$ is *fuzzy metrizable* if there exists such a fuzzy metric space $\langle X, F\rangle$ that the fuzzy topology defined by the functions $f_{uA}(x) = \inf_{t>0} \sup_{y\in A} F_{xy}(t)$, $A \subset X$, is equivalent to the fuzzy topology u.

Immediately, the following question arises: Under which conditions a fuzzy topology is fuzzy metrizable?

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