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On the Synthesis of Adaptive Multiparameter Control Systems by the Lyapunov Method

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A general form of adaptive algorithm is formulated. The derived algorithm is more general and simpler than algorithms previously published in the cited papers. Also an acceptable choice of measurable state variables is recommended which often facilitates to realize the adaptive algorithms without derivative terms of higher order.

1. INTRODUCTION

Adaptive automatic control systems become very frequently of use in technical practice. They are especially applied in automatic control systems of the flying bodies despite of higher expenses like the variable aerodynamic configuration of an aircraft and the application of a computer in autopilot circuits.

Model reference adaptive control systems constitute an important group of adaptive systems. At present, the synthesis of model reference adaptive control systems is made more perfect by the Lyapunov stability theory. The design by means of Lyapunov method simultaneously guarantees the stability of the whole adaptive system.

The published papers [1; 2; 3] deal only with the Lyapunov design of adaptive algorithms for one-parameter control systems. The objective of this paper is to formulate adaptive algorithms in more general and simpler form than in the cited papers. The derived relations are suitable for multiparameter control systems, too. In addition, it is recommended to avoid the use of derivative members of higher order by a suitable choice of state variables and of a reference model.

2. THE FORMULATION OF THE ADAPTIVE ALGORITHMS

A general multiparameter control system consists of a plant and a controller. Let us consider a plant whose characteristics vary in time. The operation of such a plant is expressed by differential equations with time-varying coefficients. Purposeful changes of controller coefficients are acting against the natural changes of plant coefficients so that the whole control system is time invariant.

The corresponding controller must satisfy two basic conditions. At first it must have a convenient structure in order that each plant parameter change may be possibly compensated. This first condition is not always simple to realize. It usually requires to use the auxiliary manipulated variables and the corresponding servo-drives. The second condition consists in the design of suitable adaptive algorithms so that the adaptive system will be stable as a whole. The contents of this paper is focused only on the second condition.

The reference model is chosen so that its properties are identical with the required properties of the control system. Let the reference model equation be of the linear form

$$\frac{d}{dt} \boldsymbol{x}_m = \boldsymbol{A}_m \boldsymbol{x}_m + \boldsymbol{B}_m \boldsymbol{u}, \tag{1}$$

where $\boldsymbol{x}_m$ is the model state vector of dimension n, $\boldsymbol{u}$ is the control vector of dimension r, $\boldsymbol{A}_m$ is the $n \times n$ model coefficient matrix and $\boldsymbol{B}_m$ is the model $n \times r$ exciting matrix. The controlled variables or outputs are included into the state vector and create r components of this vector.

Papers [1; 2; 3] consider the transformation of the coefficient matrix to the well-known row form. This transformation, however, leads to the state variables which require the utilization of derivative terms of higher order. Such a structure is not usually simple to realize. It is more convenient to choose the measurable state variables despite of more complicated coefficient matrix. Such a choice of state variables enables to avoid the utilization of derivative terms of higher order.

The system, consisting of the plant and the adaptive controller, is expressed by the system state equation

$$\frac{d}{dt} \boldsymbol{x} = \boldsymbol{A}_s(t)\, \boldsymbol{x} + \boldsymbol{B}_s(t)\, \boldsymbol{u}, \tag{2}$$

where $\boldsymbol{x}$ is an $n \times 1$ system state vector, $\boldsymbol{A}_s$ ins an $n \times n$ system coeficient matrix and $\boldsymbol{B}_s$ is an $n \times r$ system exciting matrix.

The task of the adaptive loop is to stabilize the system matrix elements to be the same with the model matrix elements, i.e. the equations (1) and (2) to be identical. The stabilization of system matrix elements is reached by varying the controller parameters, which compensate the effects of the plant parameter variations.

It is convenient to introduce the adaptive error vector, i.e. the difference

$$\boldsymbol{e} = \boldsymbol{x}_m - \boldsymbol{x} \tag{3}$$

and then the system state equation (2) can obtain the form

(4) $$\frac{d}{dt} e = A_m e + [A_m - A_s(t)]\, x + [B_m - B_s(t)]\, u\,.$$

The matrix differences may be expressed by means of the error matrices or through corresponding error matrix elements and it holds

(5) $$A(t) = A_m - A_s(t) = [a_{mij} - a_{sij}(t)] = [a_{ij}(t)]\,,$$

(6) $$B(t) = B_m - B_s(t) = [b_{mik} - b_{sik}(t)] = [b_{ik}(t)]\,,$$

$$i = 1, 2, \dots, n\,; \quad j = 1, 2, \dots, n\,; \quad k = 1, 2, \dots, r\,.$$

Substituting (5), (6) into (4), the system state equation will take on the simple form

(7) $$\frac{d}{dt} e = A_m e + A(t)\, x + B(t)\, u\,.$$

The adaptive couplings are designated for the elimination of the error state vector $e(t)$ and the error matrix elements $a_{ij}(t)$, $b_{ik}(t)$, i.e. the steady state must be provided as follows

(8) $$\lim_{t\to\infty} e(t) = 0\,, \quad \lim_{t\to\infty} A(t) = 0\,, \quad \lim_{t\to\infty} B(t) = 0\,.$$

It we take the steady-state conditions (8) into consideration, it is convenient to choose the Lyapunov function as a positive definite function of the error state vector and the error matrix elements

(9) $$V(e, A, B) = e^T P e + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 + \sum_{i=1}^{n} \sum_{k=1}^{r} b_{ik}^2\,,$$

where P is a positive definite matrix. The argument t is and will be omitted for the sake of simplicity of mathematical expressions.

The adaptive control system will be asymptotically stable if the first time derivative of the Lyapunov function is negative definite. The time derivative of the Lyapunov function

$$\frac{dV}{dt} = \left(\frac{d}{dt} e\right)^T P e + e^T P \left(\frac{d}{dt} e\right) + 2\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \left(\frac{d}{dt} a_{ij}\right) + 2\sum_{i=1}^{n} \sum_{k=1}^{r} b_{ik} \left(\frac{d}{dt} b_{ik}\right)$$

(10)

may be arranged step by step in the forms (10a) and (10b)

(10a) $$\frac{dV}{dt} = (A_m e + A x + B u)^T P e + e^T P (A_m e + A x + B u) + $$
$$+ 2\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \left(\frac{d}{dt} a_{ij}\right) + 2\sum_{i=1}^{n} \sum_{k=1}^{r} b_{ik} \left(\frac{d}{dt} b_{ik}\right),$$

$$\text{(10b)} \qquad \frac{dV}{dt} = e^T(A_m^T P + P A_m)\, e + x^T A^T (P + P^T)\, e + u^T B^T (P + P^T)\, e + $$

$$+ 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \left(\frac{d}{dt} a_{ij} \right) + 2 \sum_{i=1}^{n} \sum_{k=1}^{r} b_{ik} \left(\frac{d}{dt} b_{ik} \right).$$

The first term on the right hand side of the expression (10b) is always negative definite for the chosen stable reference model. Respecting the further design procedure it is necessary to determine the positive definite matrix $\boldsymbol{P}$ which satisfies the Lyapunov equation

$$\text{(11)} \qquad A_m^T P + P A_m = -Q\,,$$

where $\boldsymbol{Q}$ is a positive definite matrix. The solution of the matrix $\boldsymbol{P}$ is not an unambignous problem because the matrix $\boldsymbol{Q}$ is not given in advance. For the simplicity of the adaptive algorithms it is convenient to proceed in opposite way and to choose at first the matrix $\boldsymbol{P}$ in a simple form. The matrix $\boldsymbol{Q}$ need not be determined because a stable reference model was chosen. This basic idea will significantly simplify the unknown adaptive algorithms.

The second and other terms on the right-hand side of the equation (10b) are not definite. For the system to be stable the sum of these four terms must be zero. The second and the fourth terms contain the matrix $\boldsymbol{A}$ and therefore, it is of advantage if their sum is zeroed separately. Similarly, the sum of the third and fifth terms will also be set to zero.

Thus the additional conditions of system stability can be written as follows

$$\text{(12)} \qquad x^T A^T (P + P^T)\, e + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \left(\frac{d}{dt} a_{ij} \right) = 0\,,$$

$$\text{(13)} \qquad u^T B^T (P + P^T)\, e + 2 \sum_{i=1}^{n} \sum_{k=1}^{r} b_{ik} \left(\frac{d}{dt} b_{ik} \right) = 0\,.$$

The condition (12) can be expressed as the sum of components and it holds

$$\text{(14)} \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} \left[x_j a_{ij} \sum_{q=1}^{n} (p_{iq} + p_{qi})\, e_q + 2a_{ij} \left(\frac{d}{dt} a_{ij} \right) \right] = 0\,.$$

The adaptive system will be stable if the adaptive algorithm of the coefficient matrix element satisfies the condition

$$\text{(15)} \qquad \frac{d}{dt} a_{ij} = -\frac{1}{2} x_j \sum_{q=1}^{n} (p_{iq} + p_{qi})\, e_q\,.$$

If we take into consideration the relation (5), the relation (15) can be simplified to the following form

(16) $$\frac{\mathrm{d}}{\mathrm{d}t}\, a_{sij} = \varkappa_a \sigma_i x_j\,,$$

$$i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, n\,,$$

where $\varkappa_a$ is an arbitrary positive constant and

(17) $$\sigma_i = \sum_{q=1}^{n} (p_{iq} + p_{qi})\, e_q$$

is the linear combination of error state variable components

(18) $$e_q = x_{mq} - x_q\,, \quad q = 1, 2, \ldots, n\,.$$

The magnitude of the constant $\varkappa_a$ affects the rate of the adaptive process and it can be determined by modelling on an analog computer.

The adaptive loops are connected to the stable control system and, therefore, it is possible to choose the matrix $\boldsymbol{P}$ in the simplest diagonal form

(19) $$\boldsymbol{P} = \tfrac{1}{2} \begin{bmatrix} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ 0 & 0 & . & . & 1 \end{bmatrix}.$$

Due to these simplifications the adaptive algorithm (17) will have the resultant simple form

(20) $$\frac{\mathrm{d}}{\mathrm{d}t}\, a_{sij} = \varkappa_a e_i x_j\,,$$

$$i = 1, 2, \ldots, n\,; \quad j = 1, 2, \ldots, n\,.$$

It follows from the algorithm that the rate of adjusting a_{sij} is to be proportional to the product of the i-th error state component and the j-th system state component.

The second adaptive algorithm can be obtain in a similar way, i.e. by expanding the stability condition (13) into the component form

(21) $$\sum_{i=1}^{n} \sum_{k=1}^{r} \left[u_k b_{ik} \sum_{q=1}^{n} (p_{iq} + p_{qi})\, e_q + 2 b_{ik} \left(\frac{\mathrm{d}}{\mathrm{d}t}\, b_{ik} \right) \right] = 0\,,$$

and by other simplifications we obtain the adaptive algorithm of the elements of the exciting matrix in the resultant form

(22) $$\frac{\mathrm{d}}{\mathrm{d}t}\, b_{sik} = \varkappa_b e_i u_k\,,$$

$$i = 1, 2, \ldots, n\,; \quad k = 1, 2, \ldots, r\,.$$

It follows from (22) that the rate of adjusting b_{sik} is to be proportional to the product of the i-th error state component and the j-th control component.

A simplified scheme of the adaptive system is shown in Fig. 1. It follows from the figure that each adaptive loop requires a multiplier. The scheme does not contain the derivative terms because the measurable state components have been considered for

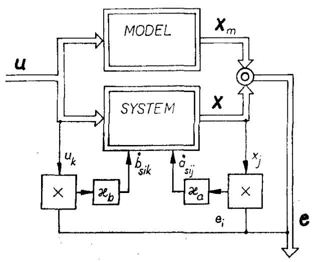

Fig. 1.

simplicity. The derived structure of adaptive system is, therefore, relatively simple for realization.

Proportional linear adaptive loops can be replaced by relay loops

$$\frac{\mathrm{d}}{\mathrm{d}t} a_{sij} = \varkappa_a \operatorname{sign}(e_i x_j), \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} b_{sik} = \varkappa_b \operatorname{sign}(e_i u_k) \tag{24}$$

whose properties are analyzed in detail in paper [3].

3. EXAMPLES

Example 1. A two-parameter control system consists of a plant and controller. The properties of the system are expressed by the system state equation

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} x_1 \\ \frac{\mathrm{d}}{\mathrm{d}t} x_2 \end{bmatrix} = \begin{bmatrix} a_{s11}(t) & a_{s12}(t) \\ a_{s21}(t) & a_{s22}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_{s11}(t) & b_{s12}(t) \\ b_{s21}(t) & b_{s22}(t) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

At a chosen nominal operating point of the variable system coefficients have the same values as the coefficients of reference model

$$\begin{bmatrix} \frac{d}{dt} x_{m1} \\ \frac{d}{dt} x_{m2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x_{m1} \\ x_{m2} \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The reference model is stable and, therefore, the adaptive algorithms of eight variable coefficients have the following form

$$\frac{d}{dt} a_{s11} = \varkappa_a e_1 x_1 , \qquad \frac{d}{dt} b_{s11} = \varkappa_b e_1 u_1 ,$$

$$\frac{d}{dt} a_{s12} = \varkappa_a e_1 x_2 , \qquad \frac{d}{dt} b_{s12} = \varkappa_b e_1 u_2 ,$$

$$\frac{d}{dt} a_{s21} = \varkappa_a e_2 x_1 , \qquad \frac{d}{dt} b_{s21} = \varkappa_b e_2 u_1 ,$$

$$\frac{d}{dt} a_{s22} = \varkappa_a e_2 x_2 , \qquad \frac{d}{dt} b_{s22} = \varkappa_b e_2 u_2 ,$$

by relations (20), (22) where

$$e_1 = x_{m1} - x_1 \quad \text{and} \quad e_2 = x_{m2} - x_2 .$$

Example 2. The plant is an aerodynamic controlled flying body and its properties in vertical plane are expressed by the state equations

$$\frac{d}{dt} \omega = -a_\omega \omega - a_\alpha \alpha - a_\delta \delta ,$$

$$\frac{d}{dt} \alpha = \omega - A_\alpha \alpha - A_\delta \delta + g \frac{\sin \theta}{v} ,$$

where

ω — is the angular speed of flying body,
α — the angle of attack,
δ_v — the deflection angle of the rudder,
δ — the deflection angle of wings,
g — the gravity acceleration,
v — the speed of center of gravity and
θ — the angle of longitudinal inclination of the speed.

The angular speed ω is the controlled variable, δ_V and δ are manipulated variables and g represents a disturbance. The coefficients a_ω, a_α, a_δ, A_α, A_δ depend on the dynamic presure of the lift and they vary considerably in the course of the flight. Two manipulated variables were chosen to possibly act against all variable coefficients. The corresponding controller, i.e. the autopilot is interpreted by equations

$$\delta_V = k_u u + k_\omega \omega + k_\alpha \alpha ,$$

$$\delta = c_u u + c_\alpha \alpha ,$$

where u is the control variable or input signal. Substituting δ_V, δ into the plant equation we obtain the state equation of the system

$$\begin{bmatrix} \frac{d}{dt}\omega \\ \frac{d}{dt}\alpha \end{bmatrix} = \begin{bmatrix} a_{s11}(t) & a_{s12}(t) \\ 1 & a_{s22}(t) \end{bmatrix} \begin{bmatrix} \omega \\ \alpha \end{bmatrix} + \begin{bmatrix} b_{s1}(t) \\ b_{s2}(t) \end{bmatrix} u + \begin{bmatrix} 0 \\ g\frac{\sin\theta}{v} \end{bmatrix},$$

where

$$a_{s11}(t) = -a_\omega(t) - a_\delta(t)\, k_\omega(t) ,$$

$$a_{s12}(t) = -a_\alpha(t) - a_\delta(t)\, k_\alpha(t) ,$$

$$a_{s22}(t) = -A_\alpha(t) - A_\delta(t)\, c_\alpha(t) ,$$

$$b_{s1}(t) = -a_\delta(t)\, k_u(t) ,$$

$$b_{s2}(t) = -A_\delta(t)\, c_u(t) .$$

Let the reference model have the equation

$$\begin{bmatrix} \frac{d}{dt}\omega_m \\ \frac{d}{dt}\alpha_m \end{bmatrix} = \begin{bmatrix} a_{m11} & a_{m12} \\ 1 & a_{m22} \end{bmatrix} \begin{bmatrix} \omega_m \\ \alpha_m \end{bmatrix} + \begin{bmatrix} b_{m1} \\ b_{m2} \end{bmatrix} u + \begin{bmatrix} 0 \\ g\frac{\sin\theta}{v} \end{bmatrix},$$

which represents the required properties of the system. The effect of the disturbance is realized in the model to eliminate the effect of the disturbance on the adaptive process.

Applying the relations (20) and (22) the following adaptive algorithms are obtained

$$\frac{d}{dt} a_{s11}(t) = \varkappa(\omega_m - \omega)\, \omega ,$$

$$\frac{d}{dt} a_{s12}(t) = \varkappa(\omega_m - \omega)\, \alpha ,$$

$$\frac{d}{dt} a_{s22}(t) = \varkappa(\alpha_m - \alpha)\,\alpha\,,$$

$$\frac{d}{dt} b_{s1}(t) = \varkappa(\omega_m - \omega)\,u\,,$$

$$\frac{d}{dt} b_{s2}(t) = \varkappa(\alpha_m - \alpha)\,u\,.$$

Practically only the coefficients of the autopilot can be changed and the realizable adaptive algorithms have the simplified form

$$\frac{d}{dt} k_\omega(t) = K(\omega - \omega_m)\,\omega\,,$$

$$\frac{d}{dt} k_\alpha(t) = K(\omega - \omega_m)\,\alpha\,,$$

$$\frac{d}{dt} c_\alpha(t) = K(\alpha - \alpha_m)\,\alpha\,,$$

$$\frac{d}{dt} k_u(t) = K(\omega - \omega_m)\,u\,,$$

$$\frac{d}{dt} c_u(t) = K(\alpha - \alpha_m)\,u\,.$$

These algorithms also give the correct steady state value of coefficients. The technical simplification of adaptive algorithms does not disturb the system stability; this conclusion can be drawn from the relations (14) and (21).

4. CONCLUSIONS

The derived adaptive algorithms (20) and (22) are suitable for the synthesis of adaptive multivariable systems. The algorithms are more general and more advantageous for the realization than many other published adaptive algorithms.

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